Collocation Methods for A Class of Volterra Integral Functional Equations with Multiple Proportional Delays

Kai Zhang* and Jie Li

Department of Mathematics, Jilin University, Changchun, Jilin 130023, China

Received 17 September 2011; Accepted (in revised version) 13 December 2011
Available online 30 July 2012

Abstract. In this paper, we apply the collocation methods to a class of Volterra integral functional equations with multiple proportional delays (VIFEMPDs). We shall present the existence, uniqueness and regularity properties of analytic solutions for this type of equations, and then analyze the convergence orders of the collocation solutions and give corresponding error estimates. The numerical results verify our theoretical analysis.

AMS subject classifications: 65R20, 34K06, 34K28
Key words: Volterra integral functional equation, multiple proportional delays, collocation method.

1 Introduction

The Volterra integral functional equations with proportional delays (VIFEPDs) provide a powerful model of phenomena when processes are modeled evolving in time, where the rate of change of the process is not only determined by its present state but also by a certain past state. VIFEPDs play an important role in explaining many different phenomena in biology, economy, control theory, electrodynamics, demography, viscoelastic materials and insurance. Numerical methods based on finite difference methods, discontinuous Galerkin methods and spectral methods etc., have also been developed for various VIFEPDs and we refer to [2–5, 8, 9, 11–13, 17], and references therein for details about the rich literature.

In this paper, we shall study the collocation method for Volterra integral functional equations (VIFE) with multiple delay (or: lag) functions \( \theta_k = \theta_k(t), k = 1, 2, \ldots, p \) of the form

\[
u(t) = \sum_{k=1}^{p} a_k(t)u(\theta_k(t)) + f(t) + (Vu)(t) + \sum_{k=1}^{p} (V\theta_k u)(t), \quad t \in I := [0, T],
\]

(1.1)

*Corresponding author.
Email: kzhang@jlu.edu.cn (K. Zhang), lijie09@mails.jlu.edu.cn (J. Li)
where $p$ is some positive integer. The Volterra integral operators $V$ and $V_{\theta_k}$ ($k = 1, 2, \cdots, p$) are defined by

\[
(Vu)(t) := \int_{0}^{t} K_0(t, s)u(s)ds, \quad (V_{\theta_k}u)(t) := \int_{0}^{\theta_k(t)} K_k(t, s)u(s)ds,
\]

where $a_k$, $f$, $K_0$ and $K_k$ are given smooth functions. The delay functions $\theta_k(t)$, $k = 1, 2, \cdots, p$ are assumed to have the following properties:

(P1) $\theta_k(0) = 0$, and $\theta_k$ is strictly increasing on $I$;

(P2) $\theta_k(t) \leq \bar{q}_k t$ on $I$ for some $\bar{q}_k \in (0, 1)$;

(P3) $\theta_k \in C^{\nu_k}(I)$ for some integer $\nu_k \geq 0$.

An important special case is the linear vanishing delay or proportional delay, i.e.,

\[
\theta_k(t) = q_k t = t - (1 - q_k)t := t - \tau_k(t) \quad \text{with} \quad 0 < q_k < 1,
\]

which are known as the pantograph delay functions (see [1, 7, 14, 16]). In rest of this paper, we shall concern on the corresponding VIFEMPDs given by

\[
u(t) = \sum_{k=1}^{p} a_k(t)u(q_k t) + f(t) + (Vu)(t) + \sum_{k=1}^{p} (V_{\theta_k}u)(t), \quad t \in I,
\]

(1.2)

where

\[
(V_{\theta_k}u)(t) := \int_{0}^{\theta_k t} K_k(t, s)u(s)ds, \quad k = 1, 2, \cdots, p,
\]

as the multi-pantograph Volterra integral functional equations.

The collocation method for the Volterra integral equation with proportion delay (VIEPD) of the form

\[
u(t) = f(t) + \int_{0}^{t} K_0(t, s)u(s)ds + \int_{0}^{t} K_1(t, s)u(s)ds,
\]

(1.3)

with $t \in [0, T]$ is discussed in [6], and recently Hermann and his collaborators also study the collocation method for functional equation

\[
u(t) = b(t)u(qt) + f(t),
\]

(1.4)

where $b$ and $f$ are given functions (see [10]). To the best of our knowledge, there is few work on collocation method for VIFEMPDs of form (1.2). In order to gain some insight approaches for VIFE of first and second kinds, we present a study of piecewise polynomial collocation solutions for (1.2).

There are two main challenges for these VIFEMPDs:

- the situations for the multiple proportional delays ($V_{\theta_k}u(t)$ in (1.2) are more complicated than single proportional delay;
the $q$-difference terms $u(q_k t)$ in the right hand side of (1.2) make the corresponding numerical schemes more difficult to be solved than schemes for VIFE.

For the former one, we shall present an algorithm to enumerate all possible cases for multiple proportional delays, and we give the numerical schemes for the particular cases of $p = 2$ and $3$; For the latter issue, we shall show the $q$-difference terms $u(q_k t)$ always make the numerical schemes lose superconvergence properties both in theoretically and numerically.

The rest of this paper is organized as follows: in Section 2, the existence, uniqueness and regularity of the analytic solution to (1.2) is proposed. Section 3 is devoted to construct the collocation schemes. The conditions for the uniqueness and the attainable global convergence order of collocation scheme are presented in Section 4, and finally in Section 5, we give some numerical experiments to verify our theoretic results.

## 2 Existence, uniqueness and regularity

For the simplification, we introduce the linear operator $K : L^\infty(I) \to L^\infty(I)$, $I = [0, T]$, by setting

$$(K \varphi)(t) = \sum_{k=1}^{p} a_k(t) \varphi(q_k t) + (V \varphi)(t) + \sum_{k=1}^{p} (V q_k \varphi)(t), \quad t \in I. \quad (2.1)$$

Then the Eq. (1.2) can be rewritten as

$$(I - K)u = f, \quad (2.2)$$

where $I$ denotes the identity operator.

We begin with a result on the existence and uniqueness of the analytic solution of (1.2).

**Theorem 2.1.** Assume that the functions $a_k$, $f$ and $K_k$ in (1.2) satisfy

(i) $a_k, f \in C(I)$, $K_0 \in C(D)$ and $K_k \in C(D_{q_k})$, $k = 1, \cdots, p$, where

$$D = \{(t,s) | 0 \leq s \leq t \leq T\}, \quad D_{q_k} = \{(t,s) | 0 \leq s \leq q_k t\};$$

(ii) $\sum_{k=1}^{p} \|a_k\|_{\infty} < 1$, where $\|v\| := \max_{t \in I} v(t)$.

Then there exists a unique solution $u \in C(I)$ of (1.2).

**Proof.** We shall prove this result by Banach fixed point theorem and mathematical induction. Since the given kernels functions $K_k$ ($k = 0, 1, \cdots, p$) are continuous on their closed domains respectively, there exist positive constants $M_k$ ($k = 0, 1, \cdots, p$) such that $|K_k(t,s)| \leq M_k$. From the condition (ii), we can choose $\delta$ such that

$$0 < \delta < \frac{1 - \sum_{k=1}^{p} \|a_k\|_{\infty}}{M_0 + \sum_{k=1}^{p} q_k M_k}.$$
Denote $S = \{u : u \in C[0, \delta]\}$. Using the inequalities
\[
\|Ku\|_\infty \leq \left( \sum_{k=1}^{p} \|a_k\|_\infty \right) \|u\|_\infty + \delta \left( M_0 + \sum_{k=1}^{p} q_k M_k \right) \|u\|_\infty < \|u\|_\infty,
\]
we know that $K : S \rightarrow S$ is a contraction map. Hence the operator $I - K$ has a bounded inverse, which implies (1.2) has a unique continuous solution on $[0, \delta]$.

Assume (1.2) has a unique continuous solution on $[0, k\delta]$ for some positive integer $k$, we want to prove that it is also true on $[k\delta, (k+1)\delta]$, that is mathematical induction with index $k$. For $t \in [k\delta, (k+1)\delta]$, we have
\[
u(t) = \sum_{k=1}^{p} a_k(t)u(q_k t) + (\nu u)(t) + \sum_{k=1}^{p} (\nu_q u)(t) + \sum_{k=1}^{p} (q_k M_k) \|u\|_\infty 
\]
Eq. (2.4) formally leading to
\[
u(t) = \sum_{k=1}^{p} a_k(t)u(q_k t) + (\nu u)(t) + \sum_{k=1}^{p} (\nu_q u)(t) + \sum_{k=1}^{p} (q_k M_k) \|u\|_\infty.
\]

Before the discussion of the regularity for VIFEMPDs of form (1.2), we first give a regularity result about the corresponding multiple delays functional equation.

**Lemma 2.1.** Consider the multiple delays functional equation
\[
u(t) = \sum_{k=1}^{p} a_k(t)u(q_k t) + (\nu u)(t), \quad t \in I,
\]
if $a_k, f \in C^v(I)$ for some integer $v \geq 1$ and $\sum_{k=1}^{p} \|a_k\|_\infty < 1$, then the solution satisfies $u \in C^v(I)$.

**Proof.** The proof is similar to the proof for the functional equation with proportional delay in [10]. Here for the sake of reader’s convenience, we give a detail proof for functional equation with multiple delays. By Theorem 2.1 with $K_0 = K_1 = \cdots = K_p = 0$, we know that $u \in C(I)$. For $a_k, f \in C^1(I)$, differentiate both sides of the Eq. (2.4) formally leading to
\[
u'(t) = \sum_{k=1}^{p} q_k a_k(t)u'(q_k t) + \tilde{f}(t),
\]
Then we have

\[ \tilde{f}(t) = \sum_{i=1}^{p} a'_i(t)u(q_k t) + f'(t). \]

Since the existence of a differentiable solution of the Eq. (2.4) is still unknown, we consider the equation

\[ \tilde{u}(t) = \sum_{k=1}^{p} q_k a_k(t)\tilde{u}(q_k t) + \tilde{f}(t). \]  

(2.5)

Noting \( a_k, f \in C^1(I) \) and the fact that \( u \in C(I) \), we know \( q_k a_k, \tilde{f} \in C(I) \). It now follows from Theorem 2.1 that there exists a unique solution \( \tilde{u} \in C(I) \) for Eq. (2.5).

Next, we will prove the unique solution \( \tilde{u}(t) \) equals to \( u'(t) \), that is to show

\[ \lim_{h \to 0} \left| \tilde{u}(t) - \frac{u(t+h)-u(t)}{h} \right| = 0. \]  

(2.6)

For any \( t, t+h \in I \) with \( h \neq 0 \), we have

\[
\begin{align*}
\tilde{u}(t) - \frac{u(t+h)-u(t)}{h} &= \sum_{k=1}^{p} q_k a_k(t) \left[ \tilde{u}(q_k t) - \frac{u(q_k(t+h)) - u(q_k t)}{q_k h} \right] \\
&\quad + \sum_{k=1}^{p} \left[ a'_k(t)u(q_k t) - \frac{a_k(t+h) - a_k(t)}{h} u(q_k(t+h)) \right] + f'(t) - \frac{f(t+h) - f(t)}{h}.
\end{align*}
\]

Let

\[
\omega[f, v, s] := \sup_{t \in I, 0 < |s| < h} \left| f(t) - \frac{v(t+s) - v(t)}{s} \right|,
\]

\[
A_1 := \sum_{k=1}^{p} q_k a_k(t) \left[ \tilde{u}(q_k t) - \frac{u(q_k(t+h)) - u(q_k t)}{q_k h} \right],
\]

\[
A_2 := \sum_{k=1}^{p} \left[ a'_k(t)u(q_k t) - \frac{a_k(t+h) - a_k(t)}{h} u(q_k(t+h)) \right],
\]

\[
A_3 := f'(t) - \frac{f(t+h) - f(t)}{h}.
\]

Then we have

\[ \left| \tilde{u}(t) - \frac{u(t+h)-u(t)}{h} \right| \leq |A_1| + |A_2| + |A_3|. \]  

(2.7)

Noting that

\[
|A_1| \leq \sum_{k=1}^{p} q_k a_k(t) \left| \tilde{u}(q_k t) - \frac{u(q_k(t+h)) - u(q_k t)}{q_k h} \right|
\]

\[
\leq \sum_{k=1}^{p} q_k a_k(t) |\omega[\tilde{u}, u, h]| \leq \left( \sum_{k=1}^{p} q_k \|a_k\|_\infty \right) \omega[\tilde{u}, u, h],
\]

\[
|A_2| \leq \sum_{k=1}^{p} q_k |a'_k(t)u(q_k t)| \leq \sum_{k=1}^{p} q_k \|a'_k\|_\infty |u(q_k t)| \leq \left( \sum_{k=1}^{p} q_k \|a'_k\|_\infty \right) \omega[u, q_k t, h],
\]

\[
|A_3| \leq |f'(t)| + \frac{|f(t+h) - f(t)|}{h} \leq \omega[f, t, h] + \omega[f, t, h],
\]

\[
|A_3| \leq \omega[f, t, h].
\]
and taking sup-norm on both sides of (2.7), we obtain
\[
\left(1 - \sum_{k=1}^{p} q_k \|a_k\|_{\infty}\right) \omega[\tilde{u}, u, h] \leq |A_2| + |A_3|.
\]

From the assumptions \(a_k, f \in C^1(I)\) and \(u \in C(I)\), we know that
\[
\lim_{h \to 0} |A_i| = 0, \quad i = 2, 3.
\]

Hence the relation (2.6) holds, which implies \(\tilde{u}(t) = u'(t)\) for any \(t \in I\). Therefore, the solution of (2.4) satisfies \(u \in C^1(I)\).

Furthermore, assume that \(u \in C^\mu(I)\) holds with
\[
0 \leq \mu \leq v - 1,
\]
since \(a_k, f \in C^v(I) \subset C^{\mu+1}(I)\), we can obtain that \(u \in C^{\mu+1}(I)\). By the mathematical induction, we reach the conclusion.

Our regularity result about the solution of (1.2) is given by following theorem.

**Theorem 2.2.** Assume that the functions \(a_k, f\) and \(K_k\) in (1.2) satisfy

(i) \(a_k, f \in C^v(I), K_0 \in C^v(D)\) and \(K_k \in C^v(D_{q_k}), k = 1, \cdots, p\), for some integer \(v \geq 1\);

(ii) \(\sum_{k=1}^{p} \|a_k\|_{\infty} < 1\).

Then the solution of (1.2) satisfies \(u \in C^v(I)\).

**Proof.** By Theorem 2.1, the solution \(u\) is continuous. Since the given functions are smooth, differentiating both sides of the Eq. (1.2) formally and replace \(u'(t)\) by \(\tilde{u}(t)\) yields

\[
\tilde{u}(t) = \sum_{k=1}^{p} q_k a_k(t) \tilde{u}(q_k t) + \tilde{f}(t),
\]

where
\[
\tilde{f}(t) = \sum_{k=1}^{p} a'_k(t) u(q_k t) + f'(t) + K_0(t, t) u(t) + \int_{0}^{t} \frac{\partial K_0}{\partial t}(t, s) u(s) ds
\]
\[
+ \sum_{k=1}^{p} q_k K_k(t, q_k t) u(q_k t) + \sum_{k=1}^{p} \int_{0}^{q_k t} \frac{\partial K_k}{\partial t}(t, s) u(s) ds.
\]

Since \(a_k, f \in C^1(I), K_0 \in C^1(D)\) and \(K_k \in C^1(D_{q_k}), k = 1, 2, \cdots, p\) and \(u \in C(I)\), it follows that \(q_k a_k, \tilde{f} \in C(I)\). Thus Lemma 2.1 implies that Eq. (2.8) has a unique solution \(u \in C^1(I)\).

Furthermore, assume that \(u \in C^\mu(I)\) holds with \(0 \leq \mu \leq v - 1\), since \(a_k, f \in C^{\mu+1}(I), K_0 \in C^{\mu+1}(D)\) and \(K_k \in C^{\mu+1}(D_{q_k}), k = 1, 2, \cdots, p\), we can obtain that \(u \in C^{\mu+1}(I)\) by Lemma 2.1. By the mathematical induction, we reach the conclusion of the theorem. \(\square\)
3 Collocation methods

In this section, it will propose an algorithm to enumerate the all possible cases for multiple proportional delays in (1.2). In particularly for \( p = 2 \) and \( p = 3 \), we give the numerical scheme for each case.

For the simplification, we first introduce some notations. Let

\[
I_h = \{ t_n = nh, \, n = 0, \cdots, N \} \quad \text{with} \quad t_N = Nh = T,
\]

be a given uniform mesh on \( I \) and set \( e_n := (t_n, t_{n+1}], \, n = 0, \cdots, N - 1 \). We shall be concerned with the collocation solution \( u_h \) lying in the piecewise polynomial space

\[
S^{(-1)}_{m-1}(I_h) := \{ v : v|_{e_n} \in \pi_{m-1}, \, 0 \leq n \leq N - 1 \}, \tag{3.1}
\]

where \( \pi_{m-1} \) (\( m \geq 1 \)) denotes the set of polynomials of degree not exceeding \( m - 1 \). The dimension of the space \( S^{(-1)}_{m-1}(I_h) \) equals \( Nm \). Hence, we are natural to choose the set of collocation points to be

\[
X_h := \{ t_{n,i} = t_n + c_i h : 0 < c_1 < \cdots < c_m \leq 1, \, n = 0, \cdots, N - 1 \},
\]

as its cardinality is \( Nm \). Here, \( \{ c_i \}^{m}_{i=1} \) is a given set of collocation parameters in \( (0, 1] \).

Hence, We are looking for \( u_h \in S^{(-1)}_{m-1}(I_h) \) satisfying the collocation equation

\[
u_h(t) = \sum_{k=1}^{p} a_k(t) u_h(q_k t) + f(t) + (V u_h)(t) + \sum_{k=1}^{p} (V q_k u_h)(t), \quad t \in X_h. \tag{3.2}
\]

Setting

\[
U_{n,j} = u_h(t_n + c_j h), \quad j = 1, \cdots, m,
\]

we can express \( u^n_h \) (the restriction of \( u_h \) on interval \( e_n \)) by interpolation

\[
u_h|_{e_n} = u^n_h(t) = u_h(t_n + sh) = \sum_{j=1}^{m} L_j(s) U_{n,j}, \quad 0 < s \leq 1, \tag{3.3}
\]

with Lagrange interpolation polynomials

\[
L_j(s) = \prod_{k=1, k \neq j}^{m} \frac{s - c_k}{c_j - c_k}, \quad 0 < s \leq 1, \quad j = 1, \cdots, m. \tag{3.4}
\]

Therefore, the global collocation solution \( u_h \) on \( I \) is given by

\[
u_h(t) = \sum_{n=0}^{N-1} \chi_n(t) u^n_h(t),
\]

where \( \chi_n(t) \) is the characteristic function on \( e_n \).
3.1 Properties of the images $q_k t$ in the vanishing delays

The main difficulty in the numerical analysis of VIFEMPDs on uniform meshes is the overlap of the images $q_k t$ ($k = 1, \cdots, p$) of the collocation points of the vanishing delays.

To be more precise, for a uniform mesh $I_h$ and $t_{n,i} := t_n + c_i h \in I_h$, we first discuss some properties of $\{q t_{n,i}\}_{i=1}^m$ for each fixed $n$, that is the single proportional delay case for (1.2). Note that

$$q t_{n,i} = q(t_n + c_i h) = q(n h + c_i h) = q(n + c_i) h.$$  

Let

$$q t_{n,i} := (q n, i + \gamma_{ni}) h = q n, i h + \gamma_{ni} h = t q_n, i + \gamma_{ni} h \in (t q_n, i, t q_n, i + 1),$$

with

$$q n, i = |q(n + c_i)|, \quad \gamma_{ni} = q(n + c_i) - q n, i \in (0, 1), \quad q n = \min_{1 \leq i \leq m} \{q n, i\}.$$  

Here for any $x \in \mathbb{R}$, $[x]$ is the greatest integer not exceeding $x$, and similarly, $\lfloor x \rfloor$ denotes the smallest integer exceeding $x$.

**Remark 3.1.** Noting that $q t_{n,m} - q t_{n,1} = q(c_m - c_1) < h$, then for each fixed $n$, $\{q t_{n,i}\}_{i=1}^m$ at most belongs to two subintervals of $I_h$.

Denoting

$$q^I = \left[ \frac{q}{1 - q c_1} \right], \quad q^{II} = \left[ \frac{q}{1 - q c_m} \right], \tag{3.5}$$

the following lemma characterize the “overlap” of the images $q t$ of the collocation points.

**Lemma 3.1.** (cf. [6]) Let $q \in (0, 1)$ and $0 < c_1 < \cdots < c_m \leq 1$ be given, and assume that $I_h$ is a uniform mesh with mesh diameter $h = T / N$. Then

(i) For $0 \leq n < q^I$, we have $\{q t_{n,i}\}_{i=1}^m \subset (t_n, t_{n+1})$;

(ii) For $q^I \leq n < q^{II}$, there exists $v_n \in \{1, \cdots, m - 1\}$ so that

$$q t_{n,i} \in \begin{cases} (t_{n-1}, t_n], & 1 \leq i \leq v_n, \\ (t_n, t_{n+1}], & v_n < i \leq m; \end{cases}$$

(iii) For $q^{II} \leq n \leq N - 1$, then $q t_{n,i} \leq t_n$ ($i = 1, \cdots, m$), therefore we have two cases:

(iii) \{q t_{n,i}\}_{i=1}^m \text{ belong to one interval, which means } q_{n,i} = q n \text{ and } \{q t_{n,i}\}_{i=1}^m \subset (t q_n, t q_{n+1}).
(iiiib) \( \{ q_{tn_i} \}_{i=1}^m \) belong to two intervals, then there exists \( v_n \in \{ 1, \ldots, m - 1 \} \), such that
\[
q_{tn_i} \in \begin{cases} 
(t_{qn_i}, t_{qn_i+1}), & 1 \leq i \leq v_n, \\
(t_{qn_i+1}, t_{qn_i+2}), & v_n < i \leq m.
\end{cases}
\]

For \( q_I \leq n \leq N - 1 \), we can consider the case (iiia) as a special state of case (iiib) with \( v_n = m \).

Next, we shall study the images \( q_k(t) \) \( (k = 1, 2, \cdots, p) \) of the collocation points of the multiple proportional vanishing delays (i.e., \( \{ q_{tn_i} \}_{i=1}^m \)). Without loss of generality, we assume \( q_1 \geq q_2 \cdots \geq q_p \), otherwise, one can rearrange \( \{ q_k \}_{k=1}^p \) and rename them such that the inequalities still hold. Similar to the single proportional vanishing delays, we introduce following notations
\[
q_I^k = \left\lceil \frac{q_k - c_1}{1 - q_k} \right\rceil, \quad q_{II}^k = \left\lceil \frac{q_k - c_m}{1 - q_k} \right\rceil, \quad q_n^{(k)} = \left\lfloor q_k(n + c_i) \right\rfloor, \quad q_n^{(k)} = \min_{1 \leq i \leq m} \{ q_n^{(k)} \}, \quad \gamma_{ni}^{(k)} = q_k(n + c_i) - q_n^{(k)}, \quad k = 1, 2, \cdots, p. \tag{3.6a}
\]

From the definitions (3.6) and \( c_1 \leq c_m \), it is obvious that
\[
q_k^I \geq q_k^I, \quad k = 1, 2, \cdots, p; \quad q_1^{II} \geq q_2^{II} \cdots \geq q_p^{II}, \quad q_1^I \geq q_2^I \cdots \geq q_p^I. \tag{3.7}
\]

In order to give the numerical schemes, it is necessary to study the images \( q_k(t) \) \( (k = 1, 2, \cdots, p) \) of the collocation points, which is equivalent to enumerate all possible arrangements of \( \{ q_k^I, q_k^{II} \}_{k=1}^p \) under constrains (3.7). Before solve this problem, we first introduce some concepts and a lemma about Catalan number.

**Definition 3.1.** A Dyck word is a string consisting of \( p \) X’s and \( p \) Y’s such that no initial segment of the string has more Y’s than X’s.

**Definition 3.2.** Given a \( p \times p \) square cells, a monotonic path is one which starts in the lower-left corner, finishes in the upper-right corner, and consists entirely of edges pointing rightwards or upwards.

**Lemma 3.2.** (see \([15]\)) The number of monotonic paths which do not pass above the diagonal (as showed in Fig. 1) is \( C_p \) (Catalan number), which is give by
\[
C_p = C_{2p}^p - C_{2p}^{p-1} = \frac{1}{p+1}C_{2p}^p = \frac{1}{p}C_{2p}^{p-1}, \quad p \geq 1,
\]
where \( C_n^k = n!/k!(n-k)! \) is the combination number.

The following theorem gives the number of rangeements of \( \{ q_k^I, q_k^{II} \}_{k=1}^p \) with constrains (3.7).

**Theorem 3.1.** The following three problems are equivalent.
Next, we use the monotonic paths which do not pass above the diagonal increasing as \( p \), where \( C \) is the Catalan number. Furthermore, the number problem (P1) is

\[
I = C_p = C_{2p} - C_{2p-1},
\]

where \( C_p \) is the Catalan number.

**Proof.** For a given choose \( p \) from \( 2p \) position, it can denotes ordered series \( q^2_k \geq q^1_k \cdots \geq q^1_p \) or \( p \) X’s without order; the rest \( p \) positions denote ordered series \( q^1_k \geq q^1_k \cdots \geq q^1_p \) or \( p \) Y’s without order. Setting \( p \) X’s represent \( q^2_k \) and \( p \) Y’s represent \( q^1_k \), the constrains \( q^2_k \geq q^1_k \), \( k = 1, 2, \ldots, p \) equivalent to no initial segment of the string has more Y’s than X’s. Hence, problem (P1) is equivalent to problem (P2).

Let X stand for “move right” and Y stand for “move up”, then no initial segment of the string has more Y’s than X’s equivalent to paths which do not pass above the diagonal. Therefore, problem (P2) is equivalent to problem (P3). Using these equivalences and Lemma 3.2, we have the conclusion. \( \square \)

Theorem 3.1 tells us the number of rangements of \( \{q^2_k, q^1_k\}_{k=1}^p \) with constrains (3.7) is equal to the number of monotonic paths which do not pass above the diagonal, and shows the number is Catalan number \( C_p \). But we still need an algorithm to enumerate all \( C_p \) cases, and following algorithm or the proof of Lemma 3.2 solve this problem.

**Algorithm 3.1.** For \( l = p, p - 1, \ldots, 1 \), enumerate the cases that the number of elements of \( \{q^2_k \geq q^1_k, k = 1, 2 \cdots, p\} \) is \( l \).

Using above algorithm and mathematical induction, we can get the formulation for Catalan number in Lemma 3.2. It is easy to see that the number under unit squares the monotonic paths which do not pass above the diagonal increasing as \( l \) decreasing. Next, we use \( p = 2 \) and \( p = 3 \) as illustrations for this algorithm.
Remark 3.2. When \( p = 2 \), \( C_2 = C_{2 \times 2}^2 - C_{2 \times 2}^1 = 2 \), the two monotonic paths are shown in Fig. 2.

Then corresponding rangements are

Case 2.1: \( q_1^{II} \geq q_2^{II} \geq q_1^I \geq q_2^I \)

Case 2.2: \( q_1^{II} \geq q_1^I > q_2^{II} \geq q_2^I \)

When \( p = 3 \), \( C_3 = C_{2 \times 3}^3 - C_{2 \times 3}^2 = 5 \), the five monotonic paths are shown in Fig. 3. Then corresponding rangements are

Case 3.1: \( q_1^{III} \geq q_2^{III} \geq q_3^{III} \geq q_4^I \geq q_5^I \)

Case 3.2: \( q_1^{III} \geq q_2^{III} \geq q_1^I \geq q_3^{III} \geq q_4^I \geq q_5^I \)

Case 3.3: \( q_1^{III} \geq q_1^I > q_2^{III} \geq q_3^{III} \geq q_4^I \geq q_5^I \)

Case 3.4: \( q_1^{III} \geq q_2^{III} \geq q_1^I \geq q_2^I > q_3^{III} \geq q_4^I \geq q_5^I \)

Case 3.5: \( q_1^{III} \geq q_1^I > q_2^{III} \geq q_2^I > q_3^{III} \geq q_5^I \)

Although Theorem 3.1 shows that the number of all possible rangements of \( \{q_k^I, q_k^{II}\}_{k=1}^p \) under constrains (3.7) is the Catalan number \( C_n = C_{2n}^n - C_{2n}^{n-1} \), the collocation schemes for each rangement are totally different. Once the \( q_k^I \)'s in (1.2) are given, we should compute \( q_k^{II} \) and \( q_k^{III} \), and specify the corresponding case by using Theorem 3.1 and Algorithm 3.1, and then adopt concrete collocation scheme for this case finally.

![Diagram of monotonic paths which do not pass above the diagonal when \( p = 2 \).](image1)

![Diagram of monotonic paths which do not pass above the diagonal when \( p = 3 \).](image2)
3.2 Collocation schemes for VIFEMPDs with Cases 2.1 and 3.1

In this subsection, we shall give the numerical schemes for VIFEMPDs with \( p = 2 \) (Case 2.1) and \( p = 3 \) (Case 3.1) corresponding to Remark 3.2 as illustrations.

Using notations (3.6), for each collocation point \( t_{n,j} \in X_h \), the collocation equation (3.2) becomes

\[
u_h(t_{n,j}) = \sum_{k=1}^{p} \left[ a_k(t_{n,j}) \sum_{j=1}^{m} L_j(q_{n,ij}^{(k)})\right] + f(t_{n,j}) + (\mathcal{V} u_h)(t_{n,j}) + \sum_{k=1}^{p} (\mathcal{V}_{q_k} u_h)(t_{n,j}), \tag{3.8}
\]

where

\[
(\mathcal{V} u_h)(t_{n,j}) = h \sum_{l=0}^{n-1} \sum_{j=1}^{m} \left( \int_{0}^{1} K_0(t_{n,j}, t_l + sh)L_j(s)ds \right) U_{l,j}
+ h \sum_{j=1}^{m} \left( \int_{0}^{c_i} K_0(t_{n,j}, t_l + sh)L_j(s)ds \right) U_{n,j},
\]

and for \( k = 1, \cdots, p, \)

\[
(\mathcal{V}_{q_k} u_h)(t_{n,j}) = h \sum_{l=0}^{q_{n,ij}^{(k)}} \sum_{j=1}^{m} \left( \int_{0}^{1} K_k(t_{n,j}, t_l + sh)L_j(s)ds \right) U_{l,j}
+ h \sum_{j=1}^{m} \left( \int_{0}^{\tau_{n,j}^{(k)}} K_k(t_{n,j}, t_{\tau_{n,j}^{(k)}} + sh)L_j(s)ds \right) U_{n,j}^{(k)}.
\]

For simplification, we introduce some notations \((k = 1, 2, 3; n, l = 0, 1, \cdots, N-1; i, j = 1, \cdots, m)\)

\[
\begin{align*}
B_{n}^{(k)} &= (a_k(t_{n,j})L_j(q_{n,ij}^{(k)})_{ij}), & F_n &= (f(t_n + c_1 h), \cdots, f(t_n + c_m h))^T, \\
M_n &= \left( \int_{0}^{c_i} K_0(t_{n,j}, t_l + sh)L_j(s)ds \right)_{ij}, & M_{n,j} &= \left( \int_{0}^{1} K_0(t_{n,j}, t_l + sh)L_j(s)ds \right)_{ij}, \\
N_{n,j}^{(k)} &= \left( \int_{0}^{1} K_k(t_{n,j}, t_l + sh)L_j(s)ds \right)_{ij}, & R_{n,j}^{(k)} &= \left( \int_{0}^{\tau_{n,j}^{(k)}} K_k(t_{n,j}, t_{\tau_{n,j}^{(k)}} + sh)L_j(s)ds \right)_{ij}, \\
U_{n} &= (U_{n,1}, \cdots, U_{n,m})^T, & T_{n,j}^{(k)} &= \text{diag}(1, \cdots, 1, 0, \cdots, 0).
\end{align*}
\]

3.2.1 Collocation schemes for Case 2.1: \( q_1^{1} \geq q_2^{1} \geq q_1^{2} \geq q_2^{1} \)

We have five phases with respect to interval \((t_n, t_{n+1}]\) for this case.

**Phase I** \((q_1 \text{ and } q_2 \text{ complete overlap})\)

\[
0 \leq n < q_2^{1} = \left[ \frac{q_2 - q_1}{1 - q_2} \right].
\]
We know that \(\{q_k t_{n,j}\}_{i=1}^m \subset (t_{n-1}, t_{n+1})\), \(k = 1, 2\) only belong to one interval with \(q^{(k)}_n = n\). Then the corresponding matrix form for Eq. (3.8) is given by

\[
\begin{bmatrix}
I_n - \sum_{k=1}^{2} (B^{(k)}_n + hR^{(k)}_{n,n}) - hM_n
\end{bmatrix} U_n = F_n + h \sum_{l=0}^{n-1} \left( M_{n,l} + \sum_{k=1}^{2} N^{(k)}_{n,l} \right) U_l.
\tag{3.9}
\]

**Phase II** \((q_1\) complete overlap and \(q_2\) partial overlap\)

\[
q_2^I \leq n < q_1^I = \left[ \frac{q_1}{1 - q_1} c_1 \right].
\]

Here, \(q^{(1)}_{n,j} = n\) for all values of \(i = 1, \ldots, m\) and for given integer \(n\), let \(v^{(2)}_n\) with \(1 \leq v^{(2)}_n < m\) such that

\[
\begin{align*}
& t_{n-1} < q_2 t_{n,j} \leq t_n, \quad i = 1, \ldots, v^{(2)}_n, \\
& t_n < q_2 t_{n,j} < t_{n+1}, \quad i = v^{(2)}_n + 1, \ldots, m.
\end{align*}
\]

Then the collocation equation (3.8) can be led to the linear algebraic system

\[
\begin{align*}
& \left[ I_n - B^{(1)}_n - hR^{(1)}_{n,n} - (I_n - T_{v^{(2)}_n}) (B^{(2)}_n + hR^{(2)}_{n,n}) - hM_n \right] U_n \\
& = T_{v^{(2)}_n} (B^{(2)}_n + hR^{(2)}_{n,n-1}) U_{n-1} + h \sum_{l=0}^{n-1} (M_{n,l} + N^{(1)}_{n,l}) U_l + F_n \\
& + h \sum_{l=0}^{n-2} N^{(2)}_{n,l} U_l + (I_n - T_{v^{(2)}_n}) hN^{(2)}_{n,n-1} U_{n-1}.
\end{align*}
\tag{3.10}
\]

**Phase III** \((q_1\) and \(q_2\) partial overlap\)

\[
q_1^I \leq n < q_2^{II} = \left[ \frac{q_2}{1 - q_2} c_m \right].
\]

For given \(n\) and \(k = 1, 2\), we know that

\[
\{q_k t_{n,j}\}_{i=1}^m \subset (t_{n-1}, t_{n+1}),
\]

and there exist \(v^{(k)}_n \in \{1, \ldots, m - 1\}\), such that

\[
\begin{align*}
& t_{n-1} < q_1 t_{n,j} \leq t_n, \quad i = 1, \ldots, v^{(1)}_n, \\
& t_n < q_1 t_{n,j} < t_{n+1}, \quad i = v^{(1)}_n + 1, \ldots, m.
\end{align*}
\]

The corresponding linear algebraic system yields

\[
\begin{align*}
& \left[ I_n - \sum_{k=1}^{2} (I_n - T_{v^{(k)}_n}) (B^{(k)}_n + hR^{(k)}_{n,n}) - hM_n \right] U_n \\
& = \sum_{k=1}^{2} T_{v^{(k)}_n} (B^{(k)}_n + hR^{(k)}_{n,n-1}) U_{n-1} + F_n + h \sum_{l=0}^{n-1} M_{n,l} U_l \\
& + h \sum_{l=0}^{n-2} \left( \sum_{k=1}^{2} N^{(k)}_{n,l} \right) U_l + \sum_{k=1}^{2} (I_n - T_{v^{(k)}_n}) hN^{(k)}_{n,n-1} U_{n-1}.
\end{align*}
\tag{3.11}
\]
Phase IV \((q_1\text{ partial overlap and }q_2\text{ non-overlap})\)

\[ q_2^{II} \leq n < q_1^{II} = \left\lfloor \frac{q_1}{1 - q_1} \right\rfloor. \]

According to Lemma 3.1, we know that \(\{q_1t_{n,i}\}_{i=1}^{m} \subset \left(t_{n-1}, t_{n+1}\right)\) and \(q_2t_{n,i} \leq t_n \ (i = 1, \ldots, m)\), which means, for given \(n\), there exist two integers \(v_n^{(1)} \in \{1, \ldots, m - 1\}\) and \(v_n^{(2)} \in \{1, \ldots, m\}\) such that

\[
\begin{align*}
q_n^{(1)} &= n - 1(i = 1, \ldots, v_n^{(1)}), & q_n^{(1)} &= n(i = v_n^{(1)} + 1, \ldots, m), \\
q_n^{(2)} &= q_n^{(1)}(i = 1, \ldots, v_n^{(2)}), & q_n^{(2)} &= q_n^{(2)} + 1(i = v_n^{(2)} + 1, \ldots, m).
\end{align*}
\]

The linear algebraic system for this phase satisfies

\[
\begin{align*}
[I_n - (I_n - T_{v_n^{(1)}})(B_n^{(1)} + hR_n^{(1)}) - hM_n]U_n &= T_{v_n^{(1)}}(B_n^{(1)} + hR_{n,hn}^{(1)})U_{n-1} + T_{v_n^{(2)}}(B_n^{(2)} + hR_{n,hdn}^{(2)})U_{q_n^{(2)} + 1} + F_n \\
&+ h \sum_{l=0}^{n-1} M_{n,l}U_l + h \sum_{l=0}^{n-2} N_{n,l}^{(1)}U_l + h \sum_{l=0}^{q_n^{(2)} - 1} N_{n,l}^{(2)}U_l + (I_n - T_{v_n^{(1)}})hN_{n,hn-1}^{(1)}U_{n-1} \\
&+ (I_n - T_{v_n^{(2)}})(B_n^{(2)}U_{q_n^{(2)} + 1} + hN_{n,hdn}^{(2)}U_{q_n^{(2)} + 1} + hR_{n,hdn}^{(2)}U_{q_n^{(2)} + 1}).
\end{align*}
\]

(3.12)

Phase V \((q_1\text{ and }q_2\text{ non-overlap})\)

\[ q_1^{II} \leq n \leq N - 1. \]

Given \(n\), there exists integers \(v_n^{(k)} \ (k = 1, 2) \in \{1, \ldots, m\}\), such that

\[
\begin{align*}
\begin{cases}
i_{q_n^{(1)}} < q_1t_{n,i} \leq t_{q_n^{(1)} + 1}, & i = 1, \ldots, v_n^{(1)}, \\
i_{q_n^{(1)} + 1} < q_1t_{n,i} < t_{q_n^{(1)} + 2}, & i = v_n^{(1)} + 1, \ldots, m, \\
i_{q_n^{(2)}} < q_2t_{n,i} \leq t_{q_n^{(2)} + 1}, & i = 1, \ldots, v_n^{(2)}, \\
i_{q_n^{(2)} + 1} < q_2t_{n,i} < t_{q_n^{(2)} + 2}, & i = v_n^{(2)} + 1, \ldots, m.
\end{cases}
\end{align*}
\]

The system of linear equations describing the last phase is given by

\[
(I_n - hM_n)U_n = \sum_{k=1}^{2} T_{v_n^{(k)}}(B_n^{(k)} + hR_{n,hdn}^{(k)})U_{q_n^{(k)} + 1} + F_n + h \sum_{l=0}^{n-1} M_{n,l}U_l + \sum_{k=1}^{2} h \sum_{l=0}^{q_n^{(k)} - 1} N_{n,l}^{(k)}U_l \\
+ \sum_{k=1}^{2} (I_n - T_{v_n^{(k)}})(B_n^{(k)}U_{q_n^{(k)} + 1} + hN_{n,hdn}^{(k)}U_{q_n^{(k)} + 1} + hR_{n,hdn}^{(k)}U_{q_n^{(k)} + 1}).
\]

(3.13)
3.2.2 Collocation schemes for Case 3.1: $q_1^{II} \geq q_2^{II} \geq q_3^{II} \geq q_1^1 \geq q_2^1 \geq q_3^1$

We have seven phases with respective to interval $(t_n, t_{n+1})$ for this case.

**Phase I** ($q_1$, $q_2$ and $q_3$ complete overlap)

$$0 \leq n < q_3^1 = \left\lceil \frac{q_3}{1 - q_3 c_1} \right\rceil.$$\[3.13\]

We know that $\{q_k t_{n,i}\}_{i=1}^m \subset (t_n, t_{n+1}), k = 1, 2, 3$ only belong to one interval with $q_{n,i} = n$. Then the corresponding matrix form for Eq. (3.8) is given by

$$\left[ I_m - \sum_{k=1}^3 (B_{n}^{(k)} + h R_{n,i}) - h M_n \right] U_n = F_n + h \sum_{l=0}^{n-1} \left( M_{n,l} + \sum_{k=1}^3 N_{n,l}^{(k)} \right) U_l + F_{n-1}.$$\[3.14\]

**Phase II** ($q_1$ and $q_2$ complete overlap, $q_3$ partial overlap)

$$q_1^1 \leq n < q_2^1 = \left\lceil \frac{q_2}{1 - q_2 c_1} \right\rceil.$$\[3.15\]

Here, $q_{n,i} = n, k = 1, 2$ for all values of $i = 1, \ldots, m$. For given integer $n$, let $v_n^{(3)}$ with $1 \leq v_n^{(3)} < m$ be such that

$$t_{n-1} < q_3 t_{n,i} \leq t_n, \quad i = 1, \ldots, v_n^{(3)},$$

$$t_n < q_3 t_{n,i} < t_{n+1}, \quad i = v_n^{(3)} + 1, \ldots, m.$$\[3.16\]

Then the collocation equation (3.8) leads to

$$\left[ I_m - \sum_{k=1}^3 (B_{n}^{(k)} + h R_{n,i}) - (I_m - T_{v_n^{(3)}}) (B_{n}^{(3)} + h R_{n,i}) - h M_n \right] U_n$$

$$= T_{v_n^{(3)}} (B_n^{(3)} + h R_{n,n-1}) U_{n-1} + h \sum_{l=0}^{n-2} \left( M_{n,l} + \sum_{k=1}^2 N_{n,l}^{(k)} \right) U_l + F_{n-1} + F_n.$$\[3.17\]

**Phase III** ($q_1$ complete overlap, $q_2$ and $q_3$ partial overlap)

$$q_2^1 \leq n < q_1^1 = \left\lceil \frac{q_1}{1 - q_1 c_1} \right\rceil.$$\[3.18\]

According to Lemma 3.1, we know that $q_{n,i}^{(1)} = n$ for all values of $i = 1, \ldots, m$ and $\{q_k t_{n,i}\}_{i=1}^m \subset (t_{n-1}, t_{n+1}), k = 2, 3$, which means for given $n$, there exist two integers $v_n^{(2)} \in \{1, \ldots, m - 1\}$ and $v_n^{(3)} \in \{1, \ldots, m - 1\}$ such that

$$q_{n,i}^{(2)} = n - 1(i = 1, \ldots, v_n^{(2)}), \quad q_{n,i}^{(2)} = n(i = v_n^{(2)} + 1, \ldots, m),$$

$$q_{n,i}^{(3)} = n - 1(i = 1, \ldots, v_n^{(3)}), \quad q_{n,i}^{(3)} = n(i = v_n^{(3)} + 1, \ldots, m).$$\[3.19\]
The linear algebraic system corresponding to this phase can be written as

\[
\left[ I_m - (B_n^{(1)} + hR_{n,n}) - \sum_{k=2}^{3} (I_m - T_{v_n^{(k)}})(B_n^{(k)} + hR_{n,n}^{(k)}) - hM_n \right] U_n
\]

\[
= \sum_{k=2}^{3} T_{v_n^{(k)}}(B_n^{(k)} + hR_{n,n-1}^{(k)})U_{n-1} + h \sum_{l=0}^{n-1} (M_{n,l} + N_{n,l}^{(1)}) U_l
\]

\[
+ F_n + h \sum_{k=2}^{3} \left[ \sum_{l=0}^{n-2} N_{n,l}^{(k)} U_l + (I_m - T_{v_n^{(k)}})N_{n,n-1}^{(k)} U_{n-1} \right].
\]  

(3.16)

**Phase IV** \((q_1, q_2 \text{ and } q_3 \text{ partial overlap})\)

\[
q_1 \leq n < q_3^{II} = \left\lfloor \frac{q_3}{1 - q_3^c} \right\rfloor.
\]

Using Lemma 3.1, we have \(\{q_k t_{n,j}\}_{j=1}^{m} \subset (t_{n-1}, t_{n+1})\), \(k = 1, 2, 3\), which means for given \(n\), there exist three integers \(v_n^{(k)} \in \{1, \cdots, m-1\}\), \(k = 1, 2, 3\) such that

\[
q_{n,j}^{(k)} = n - 1(i = 1, \cdots, v_n^{(k)}), \quad q_{n,j}^{(k)} = n(i = v_n^{(k)} + 1, \cdots, m), \quad k = 1, 2, 3.
\]

The equation is given by

\[
\left[ I_m - \sum_{k=1}^{3} (I_m - T_{v_n^{(k)}})(B_n^{(k)} + hR_{n,n}^{(k)}) - hM_n \right] U_n
\]

\[
= \sum_{k=1}^{3} T_{v_n^{(k)}}(B_n^{(k)} + hR_{n,n-1}^{(k)})U_{n-1} + F_n + h \sum_{l=0}^{n-1} M_{n,l} U_l
\]

\[
+ h \sum_{k=1}^{3} \left[ \sum_{l=0}^{n-2} N_{n,l}^{(k)} U_l + (I_m - T_{v_n^{(k)}})N_{n,n-1}^{(k)} U_{n-1} \right].
\]  

(3.17)

**Phase V** \((q_1 \text{ and } q_2 \text{ partial overlap, } q_3 \text{ non-overlap})\)

\[
q_3^{II} \leq n < q_2^{II} = \left\lfloor \frac{q_2}{1 - q_2^c} \right\rfloor.
\]

By Lemma 3.1, we obtain \(\{q_k t_{n,j}\}_{j=1}^{m} \subset (t_{n-1}, t_{n+1})\), \(k = 1, 2\) and \(q_3 t_{n,i} \leq t_n\) for \(i = 1, \cdots, m\), which means for given \(n\), there exist three integers \(v_n^{(k)} \in \{1, \cdots, m-1\}\), \(k = 1, 2\), and \(v_n^{(3)} \in \{1, \cdots, m\}\) so that

\[
q_{n,j}^{(k)} = n - 1(i = 1, \cdots, v_n^{(k)}), \quad q_{n,j}^{(k)} = n(i = v_n^{(k)} + 1, \cdots, m), \quad k = 1, 2,
\]

\[
q_{n,j}^{(3)} = q_{n}^{(3)}(i = 1, \cdots, v_n^{(3)}), \quad q_{n,j}^{(3)} = q_{n}^{(3)} + 1(i = v_n^{(3)} + 1, \cdots, m).
\]
The collocation scheme for this phase can be written as

\[
\left[ I_m - \sum_{k=1}^{2} (I_m - T^{(k)}_{v_n}) (B^{(k)}_n + hR^{(k)}_{n,n}) - hM_n \right] U_n
\]

\[
= \sum_{k=1}^{2} T^{(k)}_{v_n} (B^{(k)}_n + hR^{(k)}_{n,n-1}) U_{n-1} + h \sum_{k=1}^{2} \left( \sum_{l=0}^{n-2} N^{(k)}_{n,l} U_l + (I_m - T^{(k)}_{v_n}) N^{(k)}_{n,n-1} U_{n-1} \right)
\]

\[
+ T^{(3)}_{v_n} (B^{(3)}_n + hR^{(3)}_{n,a_{(3)}}) U_{q_n(3)} + F_n + h \sum_{l=0}^{n-1} M_{n,l} U_l + h \sum_{l=0}^{q_n^{(3)} - 1} N^{(3)}_{n,l} U_l
\]

\[
+ (I_m - T^{(3)}_{v_n}) (B^{(3)}_n U_{q_n(3)} + hN^{(3)}_{n,a_{(3)}} U_{q_n(3)} + hR^{(3)}_{n,a_{(3)}} U_{q_n(3)} + 1). \quad (3.19)
\]

**Phase VI** (\(q_1\) partial overlap, \(q_2\) and \(q_3\) non-overlap)

\[
q^{II}_{1} \leq n < q^{II}_{1} = \left\lfloor \frac{n}{1 - q_1} \cdot c_n \right\rfloor.
\]

From Lemma 3.1, we see \(\{q_1 t_{n,i}\}_{i=1}^{m} \subset (t_{n-1}, t_{n+1})\) and \(q_k t_{n,i} \leq t_{n}, k = 2, 3\) for \(i = 1, \cdots, m\), which means for given \(n\), there exist three integers \(v_n^{(1)} \in \{1, \cdots, m - 1\}\) and \(v_n^{(k)} \in \{1, \cdots, m\}, k = 2, 3\) such that

\[
q^{(1)}_{n,i} = n - 1(i = 1, \cdots, v_n^{(1)}), \quad q^{(1)}_{n,i} = n(i = v_n^{(1)} + 1, \cdots, m),
\]

\[
q^{(k)}_{n,i} = q^{(k)}_n(i = 1, \cdots, v_n^{(k)}), \quad q^{(k)}_{n,i} = q^{(k)}_n + 1(i = v_n^{(k)} + 1, \cdots, m), \quad k = 2, 3.
\]

The linear algebraic system corresponding to this phase given by

\[
\left[ I_m - (I_m - T^{(k)}_{v_n}) (B^{(k)}_n + hR^{(k)}_{n,n}) - hM_n \right] U_n
\]

\[
= T^{(k)}_{v_n} (B^{(k)}_n + hR^{(k)}_{n,n-1}) U_{n-1} + h \sum_{l=0}^{n-2} N^{(k)}_{n,l} U_l + (I_m - T^{(k)}_{v_n}) hN^{(k)}_{n,n-1} U_{n-1}
\]

\[
+ \sum_{k=2}^{3} T^{(k)}_{v_n} (B^{(k)}_n + hR^{(k)}_{n,a_{(k)}}) U_{q_n(3)} + F_n + h \sum_{l=0}^{n-1} M_{n,l} U_l + \sum_{k=2}^{3} q^{(k)}_{n,i} - 1 \sum_{l=0}^{q_n^{(k)} - 1} N^{(k)}_{n,l} U_l
\]

\[
+ \sum_{k=2}^{3} (I_m - T^{(k)}_{v_n}) (B^{(k)}_n U_{q_n(3)} + hN^{(k)}_{n,a_{(k)}} U_{q_n(3)} + hR^{(k)}_{n,a_{(k)}} U_{q_n(3)} + 1 U_{q_n(3)} + 1). \quad (3.20)
\]

**Phase VII** (\(q_1\), \(q_2\) and \(q_3\) non-overlap)

\[
q^{II}_{1} \leq n \leq N - 1.
\]

Here, there is no longer any overlap of the images \(q_k t_{n,i}, k = 1, 2, 3\) with interval \((t_n, t_{n+1})\). For each value of \(n\), there exist integers \(v_n^{(k)} \in \{1, \cdots, m\}, k = 1, 2, 3\), such that

\[
q^{(k)}_{n,i} = q^{(k)}_n(i = 1, \cdots, v_n^{(k)}), \quad q^{(k)}_{n,i} = q^{(k)}_n + 1(i = v_n^{(k)} + 1, \cdots, m), \quad k = 1, 2, 3.
\]
The collocation scheme for the last phase is given by:

\[
(I_m - h M_n) U_n = \sum_{k=1}^{3} T_{n, (i)} \left( B_{n, (i)}^{(k)} + h R_{n, (i)}^{(k)} \right) U_{n, (i)} + F_n + h \sum_{l=0}^{n-1} M_{n, l} U_l + h \sum_{k=1}^{3} \sum_{l=0}^{q_{n, (i)}^{(k)} - 1} N_{n, l}^{(k)} U_l + \sum_{k=1}^{3} (I_m - T_{n, (i)}) \left( B_{n}^{(k)} U_{n, (i)} + h N_{n, q_{n, (i)}^{(k)}} U_{n, (i)} + h R_{n, q_{n, (i)}^{(k)} + 1} U_{n, (i)} + 1 \right). 
\]  

(3.21)

4 Theoretic results for the collocation solution on uniform mesh \( I_h \)

In this section, we shall present the existence and uniqueness of the collocation solution. Using traditional technique (see [6]), we give a convergence result about collocation method for VIFEMPDs (1.2) with \( p = 2 \), and then using projection operators, we propose a theorem about convergence of VIFEMPDs for general \( p \). The last part of this section is devoted to some comments on superconvergence of collocation method for VIFEMPDs.

4.1 The existence and uniqueness of the collocation solution

In order to study existence and uniqueness, we firstly introduce a lemma as follows

**Lemma 4.1.** (see [10]) Consider multiple delays functional equation (2.4), if \( a_k, f \in C^v (I) \) for some integer \( v \geq 1 \) and

\[
\sum_{k=1}^{p} \| a_k \|_\infty < 1,
\]

then there exists an \( \tilde{h} > 0 \) (depend only on \( q_k \)), for any uniform mesh \( I_h \) with \( h < \tilde{h} \), the Eq. (3.8) with \( V = V_{q_k} = 0 \) defines a unique collocation solution \( u_h \in S_{m-1}^{(-1)} \) for all \( q_k \in (0, 1) \).

Using above lemma, the existence of a unique solution for (3.8) is given by following theorem.

**Theorem 4.1.** Assume that the functions \( a_k, f \) and \( K_k \) in (1.2) satisfy

(i) \( a_k, f \in C(I), K_0 \in C(D) \) and \( K_k \in C(D_{q_k}), k = 1, 2, \cdots, p \);

(ii) \( \sum_{k=1}^{p} \| a_k \|_\infty < 1 \).

Then there exists a constant \( \tilde{h} > 0 \) (depend only on \( q_k \)), for any uniform mesh \( I_h \) with \( h < \tilde{h} \), the Eq. (3.8) defines a unique collocation solution \( u_h \in S_{m-1}^{(-1)} \) for all \( q_k \in (0, 1) \).
Proof. We only discuss when \( p = 2 \) with Case 2.1 here, the proof for the other case is similar. For the Phase I, the collocation solution of (3.8) can be rewritten as (3.9), and according to Lemma 4.1, we know that there exists a constant \( \bar{h}_1 > 0 \), such that for any \( h < \bar{h}_1 \), the matrix \( I_m - \sum_{k=1}^p B_n^{(k)} \) is nonsingular. Then we obtain
\[
\| I_m - \sum_{k=1}^p B_n^{(k)} - h \left( M_n + \sum_{k=1}^p R_n^{(k)} \right) \| \leq h \| M_n + \sum_{k=1}^p R_n^{(k)} \|
\]
which means \( I_m - \sum_{k=1}^p B_n^{(k)} - h (M_n + \sum_{k=1}^p R_n^{(k)}) \) is nonsingular for \( h \) small enough. Therefore the linear algebraic systems (3.9) has a unique solution. The statement that the linear algebraic system (3.10) in Phase II, (3.11) in Phase III, (3.12) in Phase IV and (3.13) in Phase V has a unique solution, can be carried out in a similar way. \( \square \)

4.2 The convergence results for collocation solution

We now analyze the attainable global convergence order of the collocation solution \( u_h \in S_{m-1}^{(1)} (I_h) \) for the VIFEMPDs (1.2). First, we give the analysis of convergence by the traditional method for (1.2) with \( p = 2 \).

**Theorem 4.2.** Let \( u_h \in S_{m-1}^{(1)} (I_h) \) be the collocation solution with \( p = 2 \) defined in Section 3.2. Assume that the functions \( a_k, f \) and \( K_k \) in (1.2) satisfy

(i) \( a_k, f \in C^m (I), K_0 \in C^m (D) \) and \( K_k \in C^m (D_{q_k}), k = 1, 2; \)

(ii) \( \sum_{k=1}^2 \| a_k \|_\infty < 1. \)

Then there exists a constant \( \bar{h} > 0 \), for any uniform mesh \( I_h \) with \( h < \bar{h} \), the following estimate holds
\[
\| u - u_h \|_\infty \leq C h^m.
\]
Here, the constant \( C \) is independent on \( h \).

**Proof.** By Theorem 2.2, we have \( u \in C^m (I) \). Using Peano’s Theorem for interpolation to \( y \) on \( e_n \), we obtain
\[
u(t_n + sh) = \sum_{j=1}^m L_j(s) Y_{n,j} + h^m R_{m,n} (s), \quad s \in (0, 1], \tag{4.1}
\]
where \( L_j \) \((j = 1, 2, \ldots, m)\) are the Lagrange interpolation basis defined in (3.4) and \( Y_{n,j} = u(t_{n,j}) \). Here, the Peano remainder term and Peano kernel are given by
\[
R_{m,n} (s) = \int_0^1 \tilde{K}_m (s, z) u^{(m)} (t_n + zh) dz, \\
\tilde{K}_m (s, z) = \frac{1}{(m-1)!} \left\{ (s - z)^{m-1} - \sum_{k=1}^m L_k (s) (c_k - z)^{m-1} \right\}.
\]
Let $E_{n,i} = Y_{n,i} - U_{n,i}$, using formulations (3.3) and (4.1), the collocation error $e_h = y - u_h$ can be written as

$$e_h(t_n + sh) = \sum_{j=1}^{m} L_j(s) E_{n,j} + h^m R_{m,n}(s), \quad s \in (0,1].$$

(4.2)

Subtracting (3.8) from (1.2) with $p = 2$, we obtain the error at collocation points satisfy following equation

$$e_h(t_{n,i}) = \sum_{k=1}^{2} a_k(t_{n,i}) e_h(q_k t_{n,i}) + \int_{0}^{t} K_0(t,s)e_h(s)ds + \sum_{k=1}^{2} \int_{0}^{q_k t} K_k(t,s)e_h(s)ds,$$

(4.3)

with $e_h(t_{n,i}) = E_{n,i}$. Using (4.2) and (4.3), we get

$$E_{n,i} = \sum_{k=1}^{2} t_1^{(k)} + r_2 + \sum_{k=1}^{2} t_3^{(k)},$$

(4.4)

where

$$r_1^{(k)} = a_k(t_{n,i}) \left( \sum_{j=1}^{m} L_j \gamma_{n,j}^{(k)} E_{n,j} + h^m R_{m,n,j}^{(k)}(\gamma_{n,j}^{(k)}) \right), \quad k = 1, 2,$$

$$r_2 = h \sum_{l=0}^{n-1} \int_{0}^{t_{l+1}} K_0(t_{n,i}, t_l + sh) \left( \sum_{j=1}^{m} L_j(s) E_{l,j} + h^m R_{m,l,j}(s) \right) ds$$

$$+ h \int_{0}^{t_{l+1}} K_0(t_{n,i}, t_l + sh) \left( \sum_{j=1}^{m} L_j(s) E_{n,j} + h^m R_{m,n,j}(s) \right) ds,$$

$$r_3^{(k)} = h \sum_{l=0}^{q_k t_{n,i}^{(k)}-1} \int_{0}^{t_{l+1}} K_k(t_{n,i}, t_l + sh) \left( \sum_{j=1}^{m} L_j(s) E_{l,j} + h^m R_{m,l,j}(s) \right) ds$$

$$+ h \int_{0}^{q_k t_{n,i}^{(k)}} K_k(t_{n,i}, t_l + sh) \left( \sum_{j=1}^{m} L_j(s) E_{n,j} + h^m R_{m,n,j}(s) \right) ds, \quad k = 1, 2.$$

Next, without loss of generality, we shall discuss the convergence of the Case 2.1. For the simplifications, we introduce following notations

$$a_k = \|a_k\|_{\infty}, \quad M_m = \|u^{(m)}\|_{\infty}, \quad \Phi^{(k)}_{n,j} = (a_k(t_{n,j}) R_{m,j}(\gamma_{n,j}^{(k)}))_{i},$$

$$\rho_n = \left( \int_{0}^{t_{l+1}} K_0(t_{n,i}, t_l + sh) R_{m,n}(s) ds \right)_{i}, \quad \rho_{n,l} = \left( \int_{0}^{t_{l+1}} K_0(t_{n,i}, t_l + sh) R_{m,l}(s) ds \right)_{i},$$

$$\rho_{n,j}^{(k)} = \left( \int_{0}^{t_{l+1}} K_k(t_{n,i}, t_l + sh) R_{m,n}(s) ds \right)_{i}, \quad \rho_{n,l}^{(k)} = \left( \int_{0}^{t_{l+1}} K_k(t_{n,i}, t_l + sh) R_{m,l}(s) ds \right)_{i},$$

$$L = \max_j \|L_j\|_{\infty}, \quad k_m = \max_{s \in [0,1]} \left| R_m(s,z) \right| dz, \quad R = \max_{k=0,1,2} \max_{l \in [0,1]} \int_{0}^{l} |K_k(t,s)ds|.$$
For Phase I, we know \( q_{h,i}^{(k)} = n \) \((k = 1, 2)\) for \(i = 1, \cdots, m\), using above notations, the corresponding matrix form for Eq. (4.4) is given by

\[
(I_m - B_n^l)E_n = h^m \sum_{k=1}^{2} \Phi_{n,n}^{(k)} + h \sum_{l=0}^{n-1} \left( M_{n,l} + \sum_{k=1}^{2} N_{n,l}^{(k)} \right) E_l + h^{m+1} \left[ \sum_{l=0}^{n-1} \left( \rho_{n,l} + \sum_{k=1}^{2} \rho_{n,l}^{(k)} \right) + \rho_n + \sum_{k=1}^{2} \rho_{n,n}^{(k)} \right],
\]

with

\[
B_n^l = B_n^{(1)} + B_n^{(2)} + h(M_n + R_n^{(1)} + R_n^{(2)}) \quad \text{and} \quad E_n = (E_{n,1}, \cdots, E_{n,m})^T.
\]

According to Theorem 4.1, the above linear algebraic systems possesses a unique solution for uniform meshes \( I_h \) with \( h \in (0, \bar{h}) \). Thus, there exists a constant \( D_0 \) such that

\[
\| (I_m - B_n^l)^{-1} \|_1 \leq D_0, \quad n = 0, \cdots, q_1 - 1.
\]

Using (4.5), we have

\[
\| E_n \|_1 \leq D_0 \left[ mh^m(-a_1 + a_2)k_m M_m + 3mhK\bar{L} \sum_{l=0}^{n-1} \| E_l \|_1 + 3mk_m M_mNh^m \sum_{l=0}^{n-1} h + 3mk_m M_mNh^{m+1} \right],
\]

which leads to,

\[
\| E_n \|_1 \leq \gamma_{0,1} \sum_{l=0}^{n-1} h \| E_l \|_1 + \gamma_{1,1} M_m h^m, \quad 0 \leq n < q_2,
\]

where

\[
\gamma_{0,1} = 3mK\bar{L}D_0, \quad \gamma_{1,1} = \left( m(-a_1 + a_2)k_m + 3mk_m \bar{K}T + 3mk_m \bar{K}h \right) D_0.
\]

Applying discrete Gronwall inequality to (4.6) yields

\[
\| E_n \|_1 \leq \gamma_{1,1} M_m h^m \exp(\gamma_{0,1} T) := C_1 M_m h^m, \quad 0 \leq n < q_1.
\]

Similar arguments can be done for Phase II to Phase V. Combining these five phases, we know there exists a constant \( C < \infty \) such that

\[
\| E_n \|_1 \leq CM_m h^m, \quad \text{for all} \quad 0 \leq n \leq N - 1.
\]

Substituting the above estimate (4.7) into (4.2) leads to

\[
| e_h(t_n + sh) | \leq (C\bar{L} + k_m) M_m h^m, \quad 0 \leq n \leq N - 1.
\]
This completes the proof.

From the proof presented above on two proportional delays, it seems that the traditional technique is not suitable for analysis about multiple proportional delays, since the situations for multiple proportional delays are too complicated. By using projection operators, we can present another proof of the convergence results on multiple proportional delays which is also hold for proportional delays (see [10]).

**Theorem 4.3.** Let \( u_h \in S_{m-1}^{-1}(I_h) \) is the collocation solution defined in (3.8). Assume that the functions \( a_k, f \) and \( K_k \) in (1.2) satisfy

(i) \( a_k, f \in C^m(I), K_0 \in C^m(D), K_k \in C^m(D_{q_k}), k = 1, 2, \ldots, p; \)

(ii) \( \sum_{k=1}^{p} \| a_k \|_\infty < 1. \)

Then for all sufficiently small \( h > 0 \), we have

\[
\| u - u_h \|_\infty \leq C (\| (I - P_h)f \|_\infty + \| (I - P_h)Ku \|_\infty),
\]

where the operator \( K \) is given by (2.1), \( P_h \) is the Lagrange interpolate projection operator corresponding to the collocation parameters \( \{c_i\} \), and the constant \( C \) is independent on \( h \).

Furthermore, if the exact solution \( u \in W^{m,\infty}(I) \), we obtain

\[
\| u - u_h \|_\infty \leq Ch^m \| u \|_{m,\infty},
\]

where

\[
\| v \|_{m,\infty} := \max_{0 \leq j \leq m} \left( \sup_{t \in I} \left| \frac{d^j v(t)}{dt^j} \right| \right).
\]

**Proof.** The operator formulations for VIFEMPDS (1.2) and its collocation equation (3.8) are given by

\[
\begin{cases}
  u = f + Ku, \\
  u_h = P_h f + P_h K u_h.
\end{cases}
\]

Based on the solvability of the VIFEMPDS and its collocation equation, we obtain

\[
\begin{cases}
  u = (I - K)^{-1} f, \\
  u_h = (I - P_h K)^{-1} P_h f.
\end{cases}
\]

The error between \( u \) and \( u_h \) can be expressed in the form

\[
\begin{align*}
  u - u_h &= (I - K)^{-1} f - (I - P_h K)^{-1} P_h f \\
  &= (I - K)^{-1} (f - P_h f) + (I - K)^{-1} P_h f - (I - P_h K)^{-1} P_h f \\
  &= (I - K)^{-1} (f - P_h f) + (I - P_h K)^{-1} (K - P_h K)(I - K)^{-1} P_h f \\
  &= (I - K)^{-1} (I - P_h) f + (I - P_h K)^{-1} (K - P_h K)(I - K)^{-1} (P_h f - f) \\
  &\quad + (I - P_h K)^{-1} (K - P_h K)(I - K)^{-1} f \\
  &= (I - K)^{-1} (I - P_h) f + (I - P_h K)^{-1} (K - P_h K)(I - K)^{-1} (P_h f - f) \\
  &\quad + (I - P_h K)^{-1} (I - P_h) K u,
\end{align*}
\]
which implies
\[ \| u - u_h \|_\infty \leq C \left( \| (I - P_h) f \|_\infty + \| (I - P_h) K u \|_\infty \right). \] (4.13)

If \( u \in W^{m,\infty} \), from the error estimates of the interpolation operator \( P_h \), we know that
\[ \| (I - P_h) f \|_\infty \leq C h^m \| f \|_{m,\infty}, \] (4.14a)
\[ \| (I - P_h) K u \|_\infty \leq C h^m \| K u \|_{m,\infty} \leq C h^m \| u \|_{m,\infty}, \] (4.14b)
which leads to (4.9) Thus, the proof is complete.

4.3 Comments on superconvergence

In the rest of this section, we discuss the superconvergence of collocation method for VIFEMPDs. Define the iterated collocation solution \( u_{i+1}^h \) associated with \( u_h \) by
\[ u_{i+1}^h(t) = \sum_{k=1}^{p} a_k(t) u_h(q_k t) + f(t) + (V u_h)(t) + \sum_{k=1}^{p} (V q_k u_h)(t) ds. \] (4.15)

Then the iterated error \( e_{i+1}^h = u - u_{i+1}^h \) is given by
\[ e_{i+1}^h(t) = e_i(t) - \delta_i(t) = \sum_{k=1}^{p} a_k(t) e_h(q_k t) + (V e_h)(t) + \sum_{k=1}^{p} (V q_k e_h)(t), \]
where
\[ \delta_i(t) := -u_h(t) + \sum_{k=1}^{p} a_k(t) u_h(q_k t) + f(t) + (V u_h)(t) + \sum_{k=1}^{p} (V q_k u_h)(t). \]

We present a superconvergence result for a special case of (4.15) with \( a_k = 0 \) (\( k = 1, \ldots, p \)).

**Theorem 4.4.** Assume that the functions \( a_k, f \) and \( K_k \) in (1.2) satisfy

(i) \( a_k = 0, f \in C^{m+1}(I), K_0 \in C^{m+1}(D), K_k \in C^{m+1}(D_{q_k}), k = 1, \ldots, p. \)

Let \( u_h \in S_{m-1}(I_h) \) is the collocation solution defined in (3.8) with collocation parameters \( \{c_i\} \)
satisfying the orthogonality condition
\[ J_0 := \int_0^1 \prod_{i=1}^{m} (s - c_i) ds = 0. \]

Then the iterated collocation solution defined by (4.15) is globally superconvergent on I with
\[ \| u - u_{i+1}^h \|_\infty \leq C h^{m+1}, \]
where the constant \( C \) is independent on \( h \).
The proof is similar to collocation method for VIFEPDs (see [6]).

Using Theorem 4.2, it is easy to see that $\delta_h(t) = O(h^m)$, and at collocation points, $\delta_h(t) = 0$. Hence, we obtain $e_h^\ell(t) = e_h(t), t \in X_h$. As for classical VIES, $e_h^\ell$ can exhibit a higher order of convergence for properly chosen collocation points $\{c_i\}$ like Theorem 4.4. But for VIFEMPDs of form (1.2), we point out that $e_h^\ell$ can not achieve a higher convergence order since the existence of the $q$-difference term $\{u(q_k t)\}_{k=1}^p$ and we shall show this numerically in next section.

5 Numerical examples

In this section, we apply the collocation methods described in Section 3 to several VIFEMPDs examples. The first three examples are about the Eq. (1.2) with $p = 2$, and the last example is concerned with $p = 3$. Results of the numerical simulations verify our convergence analysis in Section 4. Simultaneously, we also address, by means of numerical tests in Example 5.2, the iterative collocation method $e_h^\ell$ can not achieve a higher convergence order.

Example 5.1. Consider the VIFEMPDs (1.2) with $p = 2$ and

\[
\begin{align*}
a_1(t) &= \frac{1}{4} \sin t, \quad a_2(t) = \frac{1}{2} t e^{-t}, \\
K_0(t, s) &= e^{-(t+s)}, \quad K_1(t, s) = e^{-(t-s)}, \quad K_2(t, s) = e^{t-s}, \\
f(t) &= t e^{-t} - \frac{1}{4} q_1 t \sin t e^{-q_1 t} - \frac{1}{2} q_2 t^2 e^{-q_2 t} - \frac{1}{2} q_1^2 t^2 e^{-t} - \frac{1}{4} e^{-t} \\
&\quad + \frac{1}{2} t e^{-3t} + \frac{1}{4} e^{-3t} - \frac{1}{4} e^t + \frac{1}{4} e^{(t-2q_2 t)} + \frac{1}{2} q_2 t e^{(t-2q_2 t)}.
\end{align*}
\]

Then the exact solution is $u(t) = te^{-t}$, for $t \in [0, 1]$.

Firstly, for a special case with $q_1 = q_2$, we use the piecewise quadratic space $S_2^{(-1)}(I_h)$ with the collocation parameters

\[
C = \left(\frac{5 - \sqrt{15}}{10}, \frac{5 + \sqrt{15}}{10}\right), \quad q = 0.1 \quad \text{and} \quad C = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right), \quad q = 0.9,
\]

respectively. The results are presented in Fig. 4. It is easy to see that the method is order of three. The result of this experiment, obtained by collocation schemes with $p = 2$, is consistent with the result of single proportional delay (see [6]).

Next, we consider the VIFEMPDs (1.2) with $q_1 \neq q_2$. In our numerical implementation, we use the space $S_2^{(-1)}(I_h)$ with the collocation points

\[
C = \left((5 - \sqrt{15})/10, 1/2, (5 + \sqrt{15})/10\right),
\]

and parameters $(q_1, q_2) = (0.5, 0.1), (q_1, q_2) = (0.99, 0.8)$ respectively. The results are shown in Fig. 5. It is easy to see that the convergence order is three. At the collocation points where

\[
e_h^\ell(t) = e_h(t),
\]
We consider Example 5.1 again, but with consideration parameters $e$.

Comparing the results for Example 5.1 and Example 5.2, we can see that the convergence order can only reach three.

**Example 5.2.** We consider Example 5.1 again, but with $a_1(t) = a_2(t) = 0$, and

$$f(t) = te^{-t} - \frac{1}{2}q_1t^2e^{-t} - \frac{1}{4}e^{-t} + \frac{1}{2}te^{-3t} + \frac{1}{4}e^{-9t} - \frac{1}{4}e^t + \frac{1}{4}e^{t-2q_2t} + \frac{1}{2}q_2te^{(t-2q_2t)}.$$  

Then the exact solution is still $u(t) = te^{-t}$.

In our simulations, we use the piecewise quadratic space $S_2^{(-1)}(I_h)$ with the collocation parameters $C_1 = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right)$, $(q_1, q_2) = (0.5, 0.1)$, and $C_2 = \left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$, $(q_1, q_2) = (0.99, 0.8)$, respectively. Here, the collocation parameters $C_1$ and $C_2$ both satisfy the orthogonality condition in Theorem 4.4. The results presented in Fig. 6 clearly exhibit the theoretical superconvergence, order of four. Comparing the results for Example 5.1 and Example 5.2, we can see that $e^{[h]}$ cannot achieve a higher order convergence due to the existence of the $q$-difference term $\{u(q_kt)\}_{k=1}^{p}$.

**Example 5.3.** Consider the first kind Volterra integral functional equation

$$\int_{q_1t}^{t} \tilde{K}_1(t, s)u(s)ds + \int_{q_2t}^{t} \tilde{K}_2(t, s)u(s)ds = g(t), \quad t \in [0, 1], \quad g(0) = 0, \quad (5.1)$$
and the choices of the delay parameters \((q_1, q_2) = (0.9, 0.2)\) and \((q_1, q_2) = (0.75, 0.5)\). The result is shown in Fig. 7, and the numerical result is consistent with theoretical order, i.e., order of \(m = 3\).
Consider the VIFEMPDs \(1.2\) with versions and valuable suggestions. The authors would also like to thank the referees for many inspiring discussions and valuable suggestions. The authors wish to thank Professor Hermann Brunner for many inspiring discussions.

**Figure 7:** Example 5.3: the errors for \(S_2^{(-1)}(I_h)\), left is by choice \((q_1, q_2) = (0.9, 0.2)\) and right is by \((q_1, q_2) = (0.75, 0.5)\).

**Example 5.4.** Consider the VIFEMPDs \(1.2\) with \(p = 3\), i.e., with three proportional delays. Let

\[
a_1(t) = \frac{1}{8} \sin t, \quad a_2(t) = \frac{1}{4} te^{-t}, \quad a_3(t) = \frac{1}{2} e^{-t},
\]

\[
f(t) = e^t - \frac{1}{8} \sin t e^{ht} - \frac{1}{2} t e^{ht} - \frac{1}{2} e^{ht} - \frac{1}{2} e^{2ht},
\]

\[
+ \frac{1}{2} e^{-t} - q_2 t e^t - \frac{1}{2} e^{t} + \frac{1}{11} e^{10t} + \frac{1}{11} e^{-10t},
\]

\[
K_0(t,s) = e^{-(t+s)}, \quad K_1(t,s) = e^{-(t-s)}, \quad K_2(t,s) = e^{-s}, \quad K_3(t,s) = e^{-10(t-s)},
\]

then the exact solution is \(u(t) = e^t (t \in [0,1])\). In this experiment, we use space \(S_2^{(-1)}(I_h)\) with the collocation parameters \(C=((5 - \sqrt{15})/10, 1/2, (5 + \sqrt{15})/10)\), delay parameters \((q_1, q_2, q_3) = (0.99, 0.9, 0.5)\) and \((q_1, q_2, q_3) = (0.9, 0.75, 0.5)\). The result presented in Fig. 8 implies an order of three.

**Figure 8:** Example 5.4: the errors for \(S_2^{(-1)}(I_h)\), left is by choice \((q_1, q_2, q_3) = (0.99, 0.9, 0.5)\) and right is by \((q_1, q_2, q_3) = (0.9, 0.75, 0.5)\).

**Acknowledgments**

The authors wish to thank Professor Hermann Brunner for many inspiring discussions and valuable suggestions. The authors would also like to thank the referees for...
their comments and suggestions that help improve the manuscript. The first author is partially supported by forefront of science and interdisciplinary innovation projects of Jilin University and NNSF (No. 11071102 of China). The authors would also like to thank high performance computing center of Jilin University of China.

References