General Solutions for a Class of Inverse Quadratic Eigenvalue Problems

Xiaoqin Tan and Li Wang

Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China.

Received 10 April 2013; Accepted (in revised version) 2 October 2013
Available online 24 February 2014

Abstract. Based on various matrix decompositions, we compare different techniques for solving the inverse quadratic eigenvalue problem, where \(n \times n\) real symmetric matrices \(M, C\) and \(K\) are constructed so that the quadratic pencil \(Q(\lambda) = \lambda^2M + \lambda C + K\) yields good approximations for the given \(k\) eigenpairs. We discuss the case where \(M\) is positive definite for \(1 \leq k \leq n\), and a general solution to this problem for \(n+1 \leq k \leq 2n\). The efficiency of our methods is illustrated by some numerical experiments.

AMS subject classifications: 65F18

Key words: Quadratic eigenvalue problem, inverse quadratic eigenvalue problem, partially prescribed spectral information.

1. Introduction

For \(n \times n\) complex matrices \(M, C\) and \(K\), the quadratic eigenvalue problem (QEP) involves finding the eigenpairs \((\lambda, x)\) such that \(Q(\lambda)x = 0\), where

\[
Q(\lambda) = Q(\lambda; M, C, K) = \lambda^2M + \lambda C + K
\]

(1.1)
is a so-called quadratic pencil defined by \(M, C\) and \(K\). The scalars \(\lambda\) and the corresponding nonzero vectors \(x\) are the eigenvalues and eigenvectors of the pencil, respectively. It is known that the QEP has \(2n\) finite eigenvalues over the complex field, provided that the leading matrix coefficient \(M\) is nonsingular. The "direct" problem is of course to find the eigenvalues and eigenvectors when the coefficient matrices \(M, C\) and \(K\) are given (cf. [5] and references therein), while the inverse quadratic eigenvalue problem (IQEP) is to determine the matrix coefficients \(M, C\) and \(K\) from a prescribed set of eigenvalues and eigenvectors (cf. [16] and references therein).

*Corresponding author. Email addresses: wlsb@163.com (L. Wang), wangli1@nju.edu.cn (L. Wang)

The IQEP has received much attention because of the wide variety of its applications — including structural design [9], control design for second-order systems [6, 16], finite element model updating for damped or gyroscopic systems [7], system identification [1] and inverse problems for damped vibration systems [12]. Some general reviews and extensive bibliographies in this regard can be found in Refs. [3] and [4].

The formulation of an IQEP depends upon the type of eigen-information available, the conditions imposed upon the matrix coefficients, and the techniques used to decompose the matrix constituted by the given eigenvectors. The IQEP studied by Ram & Elhay [17] is for symmetric tridiagonal coefficients where instead of prescribed eigenpairs, two sets of eigenvalues are given. Based on the spectral theory of matrix polynomials, Lancaster et al. [8, 11, 13] considered the IQEP with: (1) Hermitian matrices \( M, C \) and \( K \), (2) real symmetric matrices \( M, C \) and \( K \), and (3) real symmetric positive definite or semi-definite matrices \( M, C \) and \( K \), so that the quadratic pencil \( Q(\lambda) \) has complete information on the eigenvalues and eigenvectors. We deal with the inverse problem with \( k \) given eigenpairs, where \( M \) is required to be real symmetric positive definite, and \( C \) and \( K \) are \( n \times n \) real symmetric matrices. For \( 1 \leq k \leq n \), Yuan et al. [18] gave a detailed discussion involving QR decomposition, while for \( n + 1 \leq k \leq 2n \) Kuo et al. [10] studied the general solution to this problem with QR decomposition.

Our main concern is as follows: for a given eigen-information pair \( (\Lambda, X) \), find real symmetric matrices \( M, C \) and \( K \) where \( M \) is positive definite such that

\[
MX\Lambda^2 + CX\Lambda + KX = 0 \tag{1.2}
\]

is satisfied. Our motivation is to find a more efficient method to solve this problem, and the techniques we investigate below are the Rank Revealing QR (RRQR), SVD and UTV factorizations where \( U \) and \( V \) are orthogonal matrices, while \( T \) is an upper-two-diagonal matrix.

Since \( M, C \) and \( K \) are in \( \mathbb{R}^{n \times n} \), we can transform the given complex eigenpairs into real eigenpairs. To facilitate the discussion, let the real eigenpairs constitute the pair of matrices \( (\Lambda, X) \in \mathbb{R}^{k \times k} \times \mathbb{R}^{n \times k} \) such that

\[
\Lambda = \text{diag} \left\{ \lambda_1^{[2]}, \lambda_2^{[2]}, \cdots, \lambda_l^{[2]}, \lambda_{2l+1}, \cdots, \lambda_k \right\}, \tag{1.3}
\]

with

\[
\lambda_j^{[2]} = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix} \in \mathbb{R}^{2 \times 2}, \quad \beta_j \neq 0 \quad \text{for} \ j = 1, 2, \cdots, l \tag{1.4}
\]

and

\[
X = \{x_{1R}, x_{1I}, \cdots, x_{lR}, x_{lI}, x_{2l+1}, \cdots, x_k\}, \tag{1.5}
\]

where \( x_{iR} \) and \( x_{iI} \) denote the real and imaginary parts of the corresponding eigenvector, respectively. Then the original eigenpairs can be described by the matrices

\[
\bar{\Lambda} = R^H \Lambda R = \text{diag} \left\{ \lambda_1, \lambda_2, \cdots, \lambda_{2l-1}, \lambda_{2l}, \lambda_{2l+1}, \cdots, \lambda_k \right\}
\]
General Solutions for a Class of Inverse Quadratic Eigenvalue Problems

and

\[ \tilde{X} = XR = \text{diag} \{ x_1, x_2, \cdots, x_{2l-1}, x_{2l}, x_{2l+1}, \cdots, x_k \} \in \mathbb{C}^{n \times k}, \tag{1.6} \]

where

\[
R = \text{diag} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \cdots, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, I_{k-2l} \right\} \text{ with } i^2 = -1,
\]

\[
x_{2j-1} = \frac{1}{\sqrt{2}} x_{jR} + \frac{i}{\sqrt{2}} x_{jI}, \quad x_{2j} = \frac{1}{\sqrt{2}} x_{jR} - \frac{i}{\sqrt{2}} x_{jI},
\]

\[
\lambda_{2j-1} = \alpha_j + i\beta_j, \quad \lambda_{2j} = \alpha_j - i\beta_j, \quad \text{for } j = 1, 2, \cdots, l.
\]

Here \( x_j \) and \( \lambda_j \) are real-valued for \( j = 2l + 1, \cdots, k \). Thus our IQEP involves finding a real-valued quadratic pencil \( Q(\lambda) \) with matrix coefficients possessing a certain specified structure so that \( Q(\lambda_j)x_j = 0 \) for all \( j = 1, 2, \cdots, k \).

For convenience, let us denote the set of diagonal elements of \( \tilde{\Lambda} \) (the spectrum of \( \Lambda \)) by \( \sigma(\Lambda) \), and write \( (\Lambda, X) \) for an eigen-information pair of the quadratic pencil \( Q(\lambda) \). In addition, we make the following assumptions:

(1) the eigenvalue matrix \( \Lambda \) in (1.3) has simple eigenvalues;

(2) the eigenvector matrix \( X \) in (1.5) has full rank, and the matrix \( \begin{bmatrix} X & X\Lambda \end{bmatrix} \) is of full column rank.

In Section 2, we prove that the above ISQEP is always solvable with our techniques, and representations of the solution sets are then produced. In Section 3, we present some numerical results to support our main results and for comparison with existing methods.

### 2. Main Results

#### 2.1. Results for \( 1 \leq k \leq n \)

In this subsection, we solve the ISQEP for a given matrix pair \( (\Lambda, X) \in \mathbb{R}^{k \times k} \times \mathbb{R}^{n \times k} \) \((k \leq n)\) defined by (1.3), (1.4) and (1.5).

**Theorem 2.1.** Given a matrix pair \( (\Lambda, X) \in \mathbb{R}^{k \times k} \times \mathbb{R}^{n \times k} \) \((k \leq n)\) as in (1.3), (1.4) and (1.5), let

\[
X = Q \begin{pmatrix} R & 0 \\ \end{pmatrix} P^T = Q_1 RP^T \tag{2.1}
\]

be the RRQR decomposition of \( X \), where \( Q = ( Q_1 \quad Q_2 ) \in O^{n \times n} \) (a set of orthogonal \( n \times n \) real matrices) with \( Q_1 \in \mathbb{R}^{n \times k} \), \( P \in O^{k \times k} \) (a set of orthogonal \( k \times k \) real matrices) and \( R \) an upper triangular matrix. Let \( S = RP^T \Lambda PR^{-1} \). The general solution to the ISQEP is then

\[
M = Q \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} Q^T, \quad C = Q \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} Q^T, \tag{2.2}
\]

\[
K = Q \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} Q^T.
\]
where:

(i) \( \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathbb{R}^{n \times n} \) is an arbitrary symmetric positive definite matrix;

(ii) \( C_{22} = C_{22}^T \) and \( K_{22} = K_{22}^T \in \mathbb{R}^{(n-k) \times (n-k)} \) are arbitrary symmetric matrices;

(iii) \( C_{21} = C_{12}^T \in \mathbb{R}^{(n-k) \times k} \), where \( C_{21} \) is arbitrary;

(iv) 
\[
C_{11} = C_{11}^T = -(M_{11} S + S^T M_{11} + R^{-T} P^T D P R^{-1}) \in \mathbb{R}^{k \times k};
\]

(v) 
\[
M_{11} S + S^T M_{11} + R^{-T} P^T D P R^{-1} \in \mathbb{R}^{k \times k}, \quad \text{and}
\]

(vi) 
\[
M_{21} S + C_{21} S \in \mathbb{R}^{(n-k) \times k}.
\]

Here
\[
D = \text{diag}\left\{ \begin{pmatrix} \epsilon_1 & \eta_1 \\ \eta_1 & -\epsilon_1 \end{pmatrix}, \ldots, \begin{pmatrix} \epsilon_l & \eta_l \\ \eta_l & -\epsilon_l \end{pmatrix}, \epsilon_{2l+1}, \ldots, \epsilon_k \right\},
\]

where \( \epsilon_i \) and \( \eta_i \) are arbitrary real numbers.

Proof. Substituting (2.1) and (2.2) into (1.2) gives

\[
M_{11} R P^T \Lambda^2 + C_{11} R P^T \Lambda + K_{11} R P^T = 0,
\]

\[
M_{21} R P^T \Lambda^2 + C_{21} R P^T \Lambda + K_{21} R P^T = 0.
\]

Post-multiplying the above two equations by \( P R^{-1} \) yields

\[
M_{11} S^2 + C_{11} S + K_{11} = 0,
\]

\[
M_{21} S^2 + C_{21} S + K_{21} = 0,
\]

where \( S = R P^T \Lambda P R^{-1} \). Thus finding \( M, C \) and \( K \) satisfying (1.2) is equivalent to finding the submatrices \( M_{11}, M_{21}, C_{11}, C_{21}, K_{11} \) and \( K_{21} \) that satisfy (2.6) and (2.7). Clearly, it follows from (2.7) that \( K_{21} \) is determined by (2.4) where \( M_{21} \) and \( C_{21} \) are arbitrary. As \( M \) and \( K \) are required to be symmetric positive definite and symmetric, respectively, in (2.2) \( M_{11} \) is symmetric positive definite and \( K_{11} \) is symmetric. From (2.6) it follows that

\[
K_{11} = -(M_{11} S^2 + C_{11} S).
\]

Let \( M_{11} \) be an arbitrary symmetric positive definite matrix. We need to find a symmetric matrix \( C_{11} \) such that \( K_{11} \) in (2.8) is symmetric — i.e.

\[
(M_{11} S^2 + C_{11} S)^T = M_{11} S^2 + C_{11} S.
\]
After rearrangement, (2.9) becomes
\[ C_{11} S - S^T C_{11} = -M_{11} S^2 + (S^2)^T M_{11} , \] (2.10)
which has a particular solution
\[ C_{11}^0 = -(M_{11} S + S^T M_{11}) . \] (2.11)

Next we consider the homogeneous equation
\[ C_{11} S - S^T C_{11} = 0 . \] (2.12)
Substituting \( S = R P^T A R^{-1} \) into (2.12) yields
\[ (R P^T)^T C_{11} R P^T \Lambda - \Lambda^T (R P^T)^T C_{11} R P^T = 0 . \] (2.13)
Corresponding to the structure, we have \( s = k - l \) and partition \((R P^T)^T C_{11} R P^T\) as
\[ (R P^T)^T C_{11} R P^T = \begin{pmatrix} \Gamma_{11} & \cdots & \Gamma_{1l} \\ \vdots & \ddots & \vdots \\ \Gamma_{1l} & \cdots & \Gamma_{ll} \end{pmatrix} , \] (2.14)
where \( \Gamma_{jj} \) is a \( 2 \times 2 \) matrix for \( 1 \leq j \leq l \) and \( \Gamma_{jj} \) is a \( 1 \times 1 \) matrix for \( l + 1 \leq j \leq s \).
Substituting (2.14) into (2.13), and using assumption (2) and the same technique as in Ref. [19], we obtain that \( \Gamma_{ij} = 0 \) for \( j \neq i \),
\[ \Gamma_{jj} \lambda_j^{[2]} - (\lambda_j^{[2]})^T \Gamma_{jj} = 0 , \quad j = 1, 2, \ldots, l \] (2.15)
and
\[ \Gamma_{i+j,l+j} \lambda_{2l+j} - \lambda_{2l+j} \Gamma_{i+j,l+j} = 0 , \quad j = 1, 2, \ldots, s-l \] (2.16)
Since \( \lambda_j^{[2]} \) has the form in (1.4) with \( \beta_j \neq 0 \), it is easy to see that the general solution of (2.5) has the form
\[ \Gamma_{jj} = \begin{pmatrix} \epsilon_j & \eta_j \\ \eta_j & -\epsilon_j \end{pmatrix} , \quad j = 1, 2, \ldots, l \] (2.17)
where \( \epsilon_j, \eta_j \) are arbitrary real numbers and (2.16) holds for any real numbers \( \Gamma_{i+j,l+j} = \epsilon_{l+j} \). Thus the general solution of the homogeneous equation (2.12) has the form
\[ C_{11} = (R P^T)^{-T} D (R P^T)^{-1} , \]
where \( D \) is defined in (2.5). Together with (2.11), this produces the general solution of (2.10):
\[ C_{11} = -(R P^T)^{-T} D (R P^T)^{-1} - M_{11} S - S^T M_{11} . \] (2.18)
Substituting (2.18) into (2.8) yields (2.3). From (2.17) and related discussion, the matrix \( D \) is symmetric. This completes the proof.

Theorem 2.1 shows the solution to the ISQEP is underdetermined, and using this theorem we can construct a solution to the ISQEP.
2.2. Results for $n + 1 \leq k \leq 2n$

To solve the ISQEP ($n + 1 \leq k \leq 2n$), we cite the following lemma [2], and then obtain the general solution of the ISQEP in a parameterized form.

**Lemma 2.1.** (cf. Ref. [2]) There exist real symmetric matrices $M$, $C$ and $K$ satisfying the equation (1.2) if and only if

$$X^T CX = -(\Lambda^T X^T MX + X^T MX \Lambda) + D,$$

(2.19)

$$X^T KX = \Lambda^T X^T MX \Lambda - \Lambda^T D$$

(2.20)

for some $D \in D_A$, where

$$D_A = \left\{ D \in \mathbb{R}^{k \times k} | D = D^T, D \Lambda = \Lambda^T D \right\}.$$

Assume that the singular value decomposition of $X^T$ is

$$X^T = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} Q^T = U_1 \Sigma Q_1^T$$

(2.21)

where $U = (U_1, U_2) \in O(\mathbb{R}^{k \times k})$ with $U_1 \in \mathbb{R}^{k \times n}$, $Q \in O(\mathbb{R}^{n \times n})$ and

$$\Sigma = \text{diag} \{ \sigma_1, \sigma_2, \ldots, \sigma_n \} > 0.$$

Then it follows from (2.21) that $X = Q \Sigma U_1^T = (U_1 \Sigma Q_1^T)^T$ and $XU_2 = 0$, and denoting

$$M_r = (\Sigma Q_1^T)M(\Sigma Q_1^T)^T, \quad C_r = (\Sigma Q_1^T)C(\Sigma Q_1^T)^T, \quad K_r = (\Sigma Q_1^T)K(\Sigma Q_1^T)^T$$

(2.22)

we have the following result.

**Lemma 2.2.** Let $M_r$, $C_r$ and $K_r$ be defined as in (2.22). Then there are real symmetric matrices $M$, $C$ and $K$ satisfying (1.2) if and only if

$$C_r = -[(U_1^T \Lambda^T U_1)M_r + M_r(U_1^T \Lambda U_1)] + U_1^T DU_1,$$

(2.23)

$$K_r = (U_1^T \Lambda^T U_1)M_r(U_1^T \Lambda U_1) - U_1^T \Lambda^T DU_1,$$

(2.24)

$$M_r(U_1^T \Lambda U_2) = U_1^T DU_2$$

(2.25)

for some $D \in D(\Lambda, X)$, where

$$D(\Lambda, X) = \left\{ D \in D_A | U_1^T DU_2 = 0 \right\}.$$  

(2.26)

**Proof.** (Necessity) Suppose that the real symmetric matrices $M$, $C$ and $K$ satisfy (1.2). From Lemma 2.1, it follows that (2.19) and (2.20) hold for some matrix $D \in D(\Lambda, X)$. Then from (2.19) we have

$$\begin{pmatrix} U_1^T DU_1 & U_1^T DU_2 \\ U_2^T DU_1 & U_2^T DU_2 \end{pmatrix} = U^T DU$$

$$= U^T (X^T CX + \Lambda^T X^T MX + X^T MX \Lambda) U$$

$$= \begin{pmatrix} C_r + (U_1^T \Lambda^T U_1)M_r + M_r(U_1^T \Lambda U_1) & M_r(U_1^T \Lambda U_2) \\ (U_2^T \Lambda^T U_1)M_r & 0 \end{pmatrix},$$
from which we get
\[ C_r = \left[ (U_1^T \Lambda U_1)M_r + M_r(U_1^T \Lambda U_1) \right] + U_1^T D U_1, \tag{2.27} \]
\[ M_r(U_1^T \Lambda U_2) = U_1^T D U_2, \tag{2.28} \]
\[ U_2^T D U_2 = 0. \tag{2.29} \]

Similarly, from (2.20) we get
\[ K_r = (U_1^T \Lambda U_1)M_r(U_1^T \Lambda U_1) - U_1^T \Lambda D U_1, \tag{2.30} \]
\[ U_1^T \Lambda D U_2 = (U_1^T \Lambda U_1)M_r(U_1^T \Lambda U_2), \tag{2.31} \]
\[ U_2^T \Lambda D U_2 = (U_2^T \Lambda U_1)M_r(U_1^T \Lambda U_2). \tag{2.32} \]

This shows that (2.23), (2.24) and (2.25) hold for some \( D \in D(\Lambda, X) \).

(Sufficiency) Suppose that the real symmetric matrices \( M, C \) and \( K \) satisfy (2.23), (2.24) and (2.25) for some \( D \in D(\Lambda, X) \). Then
\[
\begin{pmatrix}
U_1^T D U_1 & U_1^T D U_2 \\
U_2^T D U_1 & U_2^T D U_2
\end{pmatrix}
= \begin{pmatrix}
C_r + (U_1^T \Lambda U_1)M_r + M_r(U_1^T \Lambda U_1) & M_r(U_1^T \Lambda U_2) \\
(U_2^T \Lambda U_1)M_r & 0
\end{pmatrix}.
\]
The equality (2.19) is then established. From (2.25) and (2.26) it is easy to derive \( (D - U_1 M, U_1^T \Lambda) U_2 = 0 \), which produces (2.31) and (2.32) immediately. Together with (2.24), we therefore have (2.30), (2.31) and (2.32), whence (2.20). Thus from Lemma 2.1, we obtain (1.2).

Next we consider the solvability of the matrix equation (2.25). First we note the following result concerning its coefficient matrix \( U_1^T \Lambda U_2 \):

**Lemma 2.3.** (cf. Ref. [2]) The matrix \( U_1^T \Lambda U_2 \) in (2.25) is of full column rank.

The following result then gives the general solution of the matrix equation (2.25).

**Lemma 2.4.** (cf. Refs. [14, 15]) Let \( B = U_1^T \Lambda U_2 \). Then for any \( D \in D(\Lambda, X) \), the matrix equation \( M_r B = U_1^T D U_2 \) for \( M_r \) is solvable, and moreover, \( M_r \) is given by
\[
M_r = V \begin{pmatrix}
B^T U_1^T D U_2 \\
Z^T U_1^T D U_2 \\
U_2^T D U_2
\end{pmatrix} V^T \tag{2.33}
\]
where \( W^T = W \in \mathbb{R}^{(2n-k) \times (2n-k)} \) is arbitrary, and
\[
V = \begin{pmatrix}
B(B^T B)^{-1} & Z
\end{pmatrix}
\]
with \( Z \in \mathbb{R}^{n \times (2n-k)} \) satisfying \( B^T Z = 0 \) and \( Z^T Z = I_{2n-k} \).

With Lemmas 2.2 and 2.4, we have the main result that completely characterises the ISQEP \((n + 1 \leq k \leq 2n)\) in the following theorem.
Theorem 2.2. Let \( R = U_1(\Sigma Q^T)^{-T} = U_1 \Sigma^{-T} Q^T \) and \( V \) be defined as in Lemma 2.4. Then the general solution of the ISQEP can be represented in terms of \( W \) and \( D \) in the following parameterized form:

\[
M = (\Sigma Q^T)^{-1} V \begin{pmatrix} B^T U_1^T D U_2 & \alpha \end{pmatrix} V^T (\Sigma Q^T)^{-T},
\]

\[
C = R^T D \Lambda^T X^T M - M X \Lambda R,
\]

\[
K = R^T \Lambda^T X^T M X \Lambda R - R^T \Lambda^T D R,
\]

where \( W^T = W \in \mathbb{R}^{(2n-k) \times (2n-k)} \) and \( D \in D(\Lambda, X) \) are arbitrary.

Let the \( UTV \) decomposition of \( X^T \) be

\[
X^T = U \begin{pmatrix} T & 0 \end{pmatrix} q^T = U_1 T Q^T,
\]

where \( U = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \in O(\mathbb{R}^{k \times k}) \), with \( U_1 \in \mathbb{R}^{k \times n} \), \( Q \in O(\mathbb{R}^{n \times n}) \) and

\[
T = \begin{pmatrix} * & * & 0 & \cdots & 0 & 0 \\
0 & * & * & \cdots & 0 & 0 \\
0 & 0 & * & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & * & * \\
0 & 0 & 0 & \cdots & 0 & * \\
\end{pmatrix}.
\]

Then from (2.34) it follows that \( X = Q T U_1^T = (U_1 T Q^T)^T \) and \( X U_2 = 0 \). Denoting

\[
M_r = (T Q^T) M (T Q^T)^T, \quad C_r = (T Q^T) C (T Q^T)^T, \quad K_r = (T Q^T) K (T Q^T)^T,
\]

from Lemma 2.2 and Lemma 2.4 we similarly get the solution of ISQEP \((n + 1 \leq k \leq 2n)\) as follows.

Theorem 2.3. Let \( R = U_1(T Q^T)^{-T} = U_1 T^{-T} Q^T \) and \( V \) be defined as in Lemma 2.4. Then the general solution of the ISQEP can be represented in terms of \( W \) and \( D \) in the following parameterized form:

\[
M' = (T Q^T)^{-1} V \begin{pmatrix} B^T U_1^T D U_2 & \alpha \end{pmatrix} V^T (\Sigma Q^T)^{-T},
\]

\[
C' = R^T D \Lambda^T X^T M - M X \Lambda R,
\]

\[
K' = R^T \Lambda^T X^T M X \Lambda R - R^T \Lambda^T D R,
\]

where \( W^T = W \in \mathbb{R}^{(2n-k) \times (2n-k)} \) and \( D \in D(\Lambda, X) \) are arbitrary.
Let the RRQR decomposition of $X^T$ be

$$X^T = Q \begin{pmatrix} T \\ 0 \end{pmatrix} P^T = Q_1 T P^T$$

(2.36)

where $Q = (Q_1 \quad Q_2) \in O_{R}^{k \times k}$, with $Q_1 \in R^{k \times n}$, $P \in O_{R}^{n \times n}$ and $T$ an $n \times n$ upper triangular matrix. From (2.36), it follows that $X = P^T Q_1^T = (Q_1 T P^T)^T$ and $X Q_2 = 0$.

Finally, denoting

$$M_r = (T P^T) M (T P^T)^T, \quad C_r = (T P^T) C (T P^T)^T, \quad K_r = (T P^T) K (T P^T)^T,$$

(2.37)

from Lemma 2.2 and Lemma 2.4 we similarly get the solution of ISQEP ($n + 1 \leq k \leq 2n$) as follows.

**Theorem 2.4.** Let $R = Q_1 (T P^T)^{-1} = Q_1 T^{-1} T P^T$ and $V$ be defined as those in Lemma 2.4. Then the general solution of the ISQEP can be represented in the following parameterised forms in terms of $W$ and $D$:

$$M'' = (T P^T)^{-1} V \begin{pmatrix} B^T U_1^T D U_2 \\ Z^T U_1^T D U_2 \\ W \end{pmatrix} V^T (T P^T)^{-T},$$

$$C'' = R^T D R - R^T \Lambda^T X^T M - MX \Lambda R,$$

$$K'' = R^T \Lambda^T X^T M X \Lambda R - R^T \Lambda^T D R,$$

(2.38)

(2.39)

(2.40)

where $W^T = W \in R^{(2n-k) \times (2n-k)}$ and $D \in D(\Lambda, X)$ are arbitrary.

### 3. Numerical Experiments

In this section, we present some numerical examples to illustrate the solutions constructed in Sections 2. We report all the numerical results in five significant digits using MATLAB with full precision on a PC, where $\hat{\lambda}_i$ are the computed eigenvalues of $Q(\lambda)$.

**Example 3.1.** Consider the ISQEP where the partial eigen-structure $(\Lambda, X) \in C^{5 \times 5} \times C^{5 \times 5}$ is as in (1.3) and (1.5), with $\lambda_1 = -0.31828 - 0.86754 i = \hat{\lambda}_2$, $\lambda_3 = -0.95669 + 0.17379 i = \hat{\lambda}_4$, $\lambda_5 = -4.4955$, and the corresponding eigenvectors

$$x_1 = \hat{x}_2 = \begin{pmatrix} 15.159 - 11.123 i \\ -77.470 - 14.809 i \\ 2.8190 - 10.275 i \\ 0.3821 + 16.329 i \\ 57.042 + 18.419 i \end{pmatrix}, \quad x_3 = \hat{x}_4 = \begin{pmatrix} 65.621 + 34.379 i \\ 22.625 - 24.189 i \\ -37.062 - 15.825 i \\ -9.6496 - 14.401 i \\ -0.61893 - 25.609 i \end{pmatrix}, \quad x_5 = \begin{pmatrix} 2.2245 \\ 1.6586 \end{pmatrix}.$$

It is easy to check that the matrix pair $(\Lambda, X) \in C^{5 \times 5} \times C^{5 \times 5}$ satisfies the assumptions (1) and (2).

According to Theorem 2.1, we get the solution with $1 \leq k \leq n$, with the accuracy of the approximated eigenvalues shown in Table 1.
Table 1: Absolute errors for ISQEP \((1 \leq k \leq n)\) with decomposition of \(X\).

| Eigenvalues | \(|\lambda_i - \lambda_j|\) (RRQR) | \(|\lambda_i - \lambda_j|\) (QR) |
|-------------|-------------------------------|-------------------------------|
| \(\lambda_1 = \lambda_2\) | 9.6273e−11 | 2.1615e−10 |
| \(\lambda_3 = \lambda_4\) | 7.3066e−11 | 1.6542e−10 |
| \(\lambda_5\) | 9.5035e−14 | 1.6858e−12 |

**Example 3.2.** Consider the ISQEP with the partial eigen-information \((\Lambda, X) \in \mathbb{C}^{6 \times 6} \times \mathbb{C}^{5 \times 6}\) as in (1.3) and (1.5), \(\lambda_1 = -0.31828 - 0.86754i = \tilde{\lambda}_2\), \(\lambda_3 = -0.95669 + 0.17379i = \tilde{\lambda}_4\), \(\lambda_5 = -4.4955\), \(\lambda_6 = 1.5135\), and the corresponding eigenvectors

\[
\begin{align*}
x_1 &= \tilde{x}_2 = \begin{pmatrix} 15.159 - 11.123i \\ -77.470 - 14.809i \\ 2.1930 - 10.275i \\ 0.3821 + 16.329i \\ 57.042 + 18.419i \\ 0 \end{pmatrix}, & x_3 &= \tilde{x}_4 = \begin{pmatrix} 65.621 + 34.379i \\ 22.625 - 24.189i \\ -37.062 - 15.825i \\ -9.6496 - 14.401i \\ -0.61893 - 25.609i \\ 0 \end{pmatrix}, & x_5 &= \tilde{x}_6 = \begin{pmatrix} 2.2245 \\ -0.86754 \\ 0.17379 \\ 0 \end{pmatrix}.
\end{align*}
\]

It is easy to check that the matrix pair \((\Lambda, X) \in \mathbb{C}^{6 \times 6} \times \mathbb{C}^{5 \times 6}\) satisfies the assumptions (1) and (2).

From Theorems 2.2, 2.3 and 2.4, we get the solutions for \(n + 1 \leq k \leq 2n\), with the accuracy of the approximated eigenvalues as shown in Table 2.

**Table 2: Absolute errors for ISQEP \((n + 1 \leq k \leq 2n)\) with decomposition of \(X\).**

| Eigenvalues | \(|\lambda_i - \lambda_j|\) (RRQR) | \(|\lambda_i - \lambda_j|\) (QR) |
|-------------|-------------------------------|-------------------------------|
| \(\lambda_1 = \lambda_2\) | 8.3090e−3 | 5.3768e−2 |
| \(\lambda_3 = \lambda_4\) | 5.0347e−3 | 6.9076e−3 |
| \(\lambda_5\) | 3.1807e−1 | 1.8448 |
| \(\lambda_6\) | 4.4399e−3 | 9.1065e−2 |

| Eigenvalues | \(|\lambda_i - \lambda_j|\) (SVD) | \(|\lambda_i - \lambda_j|\) (UTV) |
|-------------|-------------------------------|-------------------------------|
| \(\lambda_1 = \lambda_2\) | 3.6599e−4 | 5.0891e−3 |
| \(\lambda_3 = \lambda_4\) | 7.7763e−5 | 1.1036e−3 |
| \(\lambda_5\) | 3.7208e−2 | 3.5517e−1 |
| \(\lambda_6\) | 1.2133e−4 | 1.7605e−1 |

From the above numerical results, in terms of the eigenvalues we observe that the RRQR and SVD methods are superior to the QR method for both \(1 \leq k \leq n\) and for \(n + 1 \leq k \leq 2n\), as is the UTV method for most of the eigenvalues.

**Example 3.3.** Consider a \(20 \times 20\) triplet \((M_0, C_0, K_0)\) with \(M_0\) the identity matrix, \(C_0\) and \(K_0\) five-diagonal matrices with \(C_0(i, i) = 5\), \(C_0(i, j) = 2\) if \(|i - j| = 1\), \(C_0(i, j) = -1\) if \(|i - j| = 2\), and \(K_0(i, i) = 3\), \(K_0(i, j) = 1\) if \(|i - j| = 1\), \(K_0(i, j) = -2\) if \(|i - j| = 2\). We first compute all 40 eigenpairs of \(Q_0(\lambda) = \lambda^2 I + \lambda C_0 + K_0\) and \((\Lambda, X) \in \mathbb{R}^{12 \times 12} \times \mathbb{R}^{20 \times 12}\), chosen from those 40 computed eigenpairs of \(Q_0(\lambda)\), where the selected eigenvalues are \(\lambda_1 = -0.9505 + 0.4397i = \overline{\lambda}_2\), \(\lambda_3 = -1.4268 + 0.6214i = \overline{\lambda}_4\), \(\lambda_5 = -5.9454\), \(\lambda_6 = -6.4673\), \(\lambda_7 = -6.8169\), \(\lambda_8 = -7.1928\), \(\lambda_9 = -7.1919\), \(\lambda_{10} = -7.0824\), \(\lambda_{11} = -7.0993\), \(\lambda_{12} = -6.8732\), and the corresponding eigenvectors are as follows:
General Solutions for a Class of Inverse Quadratic Eigenvalue Problems

\[
\begin{align*}
V_1 &= \begin{pmatrix} -0.5819 - 0.0011i \\ 0.7117 + 0.2883i \\ -0.3770 - 0.2476i \\ -0.0252 + 0.0584i \\ 0.4376 + 0.0956i \\ -0.6585 - 0.2201i \\ 0.5302 + 0.2225i \\ -0.1440 - 0.0860i \\ -0.3065 - 0.0927i \\ 0.6102 + 0.2148i \\ 0.3065 + 0.0927i \\ 0.1440 + 0.0860i \\ -0.5302 - 0.2225i \\ 0.6585 + 0.2201i \\ -0.4376 - 0.0956i \\ 0.0252 - 0.0584i \\ 0.3770 + 0.2476i \\ -0.7117 - 0.2883i \\ 0.5819 + 0.0011i \end{pmatrix}, & V_2 &= \begin{pmatrix} -0.5819 + 0.0011i \\ 0.7117 - 0.2883i \\ -0.3770 + 0.2476i \\ -0.0252 - 0.0584i \\ 0.4376 - 0.0956i \\ -0.6585 + 0.2201i \\ 0.5302 - 0.2225i \\ -0.1440 + 0.0860i \\ -0.3065 + 0.0927i \\ 0.6102 - 0.2148i \\ 0.3065 - 0.0927i \\ 0.1440 - 0.0860i \\ -0.5302 + 0.2225i \\ 0.6585 - 0.2201i \\ -0.4376 + 0.0956i \\ 0.0252 + 0.0584i \\ 0.3770 - 0.2476i \\ -0.7117 + 0.2883i \\ 0.5819 - 0.0011i \end{pmatrix}, & V_3 &= \begin{pmatrix} 0.4900 + 0.0810i \\ -0.5539 + 0.2377i \\ -0.1750 + 0.1668i \\ 0.3123 - 0.0431i \\ -0.5691 + 0.2056i \\ 0.4157 - 0.1985i \\ 0.0373 + 0.0327i \\ -0.4621 + 0.1560i \\ 0.5517 - 0.2153i \\ -0.2423 + 0.0985i \\ 0.5517 - 0.2153i \\ -0.4621 + 0.1560i \\ 0.0373 + 0.0327i \\ 0.4157 - 0.1985i \\ 0.3123 - 0.0431i \\ 0.1750 - 0.1668i \\ -0.5539 + 0.2377i \\ 0.4900 - 0.0810i \end{pmatrix}\\
V_4 &= \begin{pmatrix} 0.4900 - 0.0810i \\ -0.5539 - 0.2377i \\ -0.1750 - 0.1668i \\ 0.3123 + 0.0431i \\ -0.5691 - 0.2056i \\ 0.4157 + 0.1985i \\ -0.4621 - 0.1560i \\ 0.0373 - 0.0327i \\ -0.2423 - 0.0985i \\ 0.5517 + 0.2153i \\ -0.4621 - 0.1560i \\ 0.0373 - 0.0327i \\ 0.4157 + 0.1985i \\ 0.3123 + 0.0431i \\ 0.1750 + 0.1668i \\ -0.5539 - 0.2377i \\ 0.4900 + 0.0810i \end{pmatrix}, & V_5 &= \begin{pmatrix} -0.1682 \\ -0.0864 \\ 0.1546 \\ 0.1405 \\ 0.1044 \\ 0.1044 \\ 0.1044 \\ 0.1044 \\ 0.1044 \\ 0.1044 \\ 0.1044 \\ 0.1044 \\ 0.1044 \\ 0.1044 \\ 0.1044 \end{pmatrix}, & V_6 &= \begin{pmatrix} 0.1546 \\ 0.1405 \\ 0.1044 \\ 0.1044 \\ 0.1044 \\ 0.1044 \end{pmatrix}, & V_7 &= \begin{pmatrix} 0.1044 \\ 0.1044 \\ 0.1044 \\ 0.1044 \\ 0.1044 \\ 0.1044 \end{pmatrix}\\
V_8 &= \begin{pmatrix} -0.0294 \\ -0.0478 \\ 0.0532 \\ 0.1048 \\ 0.0818 \\ -0.0139 \\ -0.1146 \\ -0.1390 \\ -0.0618 \\ 0.0618 \\ 0.1390 \\ 0.1390 \\ 0.1146 \\ 0.0139 \\ -0.0818 \\ -0.1048 \\ -0.0532 \\ 0.0170 \\ 0.0478 \\ 0.0294 \end{pmatrix}, & V_9 &= \begin{pmatrix} 0.0119 \\ 0.0457 \\ 0.0583 \\ 0.0459 \\ 0.1390 \\ 0.0459 \\ -0.0717 \\ 0.0459 \\ 0.1390 \\ 0.0748 \\ 0.0119 \end{pmatrix}, & V_{10} &= \begin{pmatrix} 0.0454 \\ 0.0585 \\ 0.0555 \\ 0.0574 \\ 0.0585 \\ 0.0454 \end{pmatrix}, & V_{11} &= \begin{pmatrix} 0.0367 \\ 0.1043 \\ 0.0936 \\ -0.0191 \\ -0.0015 \\ -0.0367 \end{pmatrix}, & V_{12} &= \begin{pmatrix} 0.0204 \\ 0.0584 \\ 0.0887 \\ 0.1214 \\ 0.1214 \\ 0.1214 \end{pmatrix}.
\end{align*}
\]
Here $(\Lambda, X)$ satisfies the assumptions (1) and (2) in Section 1. Then according to Theorem 2.1, by choosing $M = M_0, D = \text{diag}((1, 1,\cdots, 1))$ and then randomly generating $C_{22} = C_{22}^T, K_{22} = K_{22}^T$, and $C_{12} = C_{12}^T$, we get $M_i, C_i, K_i, i = 1, 2$ for QR and RRQR decompositions of matrix $X$, respectively. The residuals are estimated by

$$
\|M_1X\Lambda^2 + C_1X\Lambda + K_1X\|_2 = 108.5162,$$

$$
\|M_2X\Lambda^2 + C_2X\Lambda + K_2X\|_2 = 2.6976 \times 10^{-14}.
$$

These results again show that the RRQR method can be much superior to the QR method.

4. Acknowledgments

The authors would like to thank referees for their many helpful comments and suggestions which improve our manuscript greatly. The work is supported by the National Natural Science Foundation of China under grant 10971102, and the Natural Science Foundation of Jiangsu Province of China under grant BK2009398.

References


