Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **9**, No. 6, pp. 1420-1437

DOI: 10.4208/aamm.2016.m1420
December 2017

A New L^2 Projection Method for the Oseen Equations

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Received 21 January 2016; Accepted (in revised version) 9 January 2017

Abstract. In this paper, a new type of stabilized finite element method is discussed for Oseen equations based on the local L^2 projection stabilized technique for the velocity field. Velocity and pressure are approximated by two kinds of mixed finite element spaces, $P_l^2-P_1$, $(l=1,2)$. A main advantage of the proposed method lies in that, all the computations are performed at the same element level, without the need of nested meshes or the projection of the gradient of velocity onto a coarse level. Stability and convergence are proved for two kinds of stabilized schemes. Numerical experiments confirm the theoretical results.

AMS subject classifications: 65N12, 65N30

Key words: Oseen equations, L^2 projection method, pressure projection method.

1 Introduction

As a linearized model of the incompressible Navier-Stokes equations, the Oseen problem has attracted much research interest in the analysis of stabilized finite element methods. Mixed finite element methods for the Oseen equations must handle two numerical difficulties: compatibility of velocity and pressure spaces and advection dominated flows.

Stabilized finite element methods could conquer the lack of LBB stability. There are two approaches to design stabilized finite element methods. The first approach is based on the residual of the momentum equation, such as the multiscale enrichment method [2], the residual-free bubble method [18, 19], the least squares method [10, 11] and so on. Another approach is based on the projection stabilization, such as the pressure gradient projection (PGP) method (see [7, 8, 15]), the local pressure gradient stabilization (LPS) method [6] and the polynomial pressure projection stabilization (LPPS) method [9,16,29]. For PGP and LPS methods, the compressibility constraint is relaxed by subtracting the

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discontinuous pressure gradient from its projection onto a piecewise polynomial space. PGP method is based on the global L^2 projection, while LPS method is based on the local L^2 projection which can reduce the computations. LPS method adds terms of the form

$$\sum_{T\in\mathcal{T}_h}\psi_T h_T^2((I-\pi_{2h})\nabla p_l,(I-\pi_{2h})\nabla q_l)_T,$$

where p_l, q_l denote polynomials of degree less than l $(l\geq 1)$, $\psi_T>0$ the stabilized parameter, I the identity operator and π_{2h} the projection onto a coarse level. PGP and LPS methods are not easy to implement, since a special data structure of two-hierarchy mesh is required. As an alternative, LPPS method was introduced with the following term:

$$G_h(p_l,q_l)=\sum_{T\in\mathcal{T}_h}\theta_T((I-\pi_{l-1})p_l,(I-\pi_{l-1})q_l)_T, \tag{1.1}$$

where $\pi_{l-1}:L^2(\Omega)\to P_{l-1}^{dc}(\mathcal{T}_h)$ denotes the local L^2 projection, θ_T the stabilized parameter. In the method, all the computations are performed at the same element level, which simplify the computations. In particular, when pressure is approximated by piecewise linear polynomials, LPPS method's stabilization term has the following relationship:

$$(I-\pi_0)p_1|_T=(I-\pi_0)(\mathbf{x}\cdot\nabla p_1)|_T. \tag{1.2}$$

At present, the most popular approach to solve convection dominated cases is the variational multiscale (VMS) method (see [3,17,22–24,26–28,30,32] and so on), with the stabilized terms of the following form:

$$\sum_{T\in\mathcal{T}_h}\varpi_T((I-\mathcal{Q}_H)\nabla\mathbf{u}_h^l,(I-\mathcal{Q}_H)\nabla\mathbf{v}_h^l)_T$$

or

$$\sum_{T\in\mathcal{T}_h}\varpi_T(\nabla(I-\mathcal{Q}_H)\mathbf{u}_h^l,\nabla(I-\mathcal{Q}_H)\mathbf{v}_h^l)_T,$$

where ϖ_T is the stabilized parameter, $\mathbf{u}_h^l$, $\mathbf{v}_h^l$ denote polynomials of degree less than l $(l\geq 1)$, $\mathcal{Q}_H(H\geq h)$ is a projection onto a coarse level. Similar to LPS method, a special data structure of two-hierarchy mesh is required by VMS methods. Motivated by (1.2), the residual local projection (RELP) method [1,4] based on an enriching space strategy was proposed. Then, [5] used the additional terms of the RELP method and relaxed consistency to propose a local projection method which adds the following stabilized terms

$$\sum_{T\in\mathcal{T}_h}\frac{\delta_T}{\nu}(\chi_h(\mathbf{x}\cdot(\nabla\mathbf{u}_h^1)\beta),\chi_h(\mathbf{x}\cdot(\nabla\mathbf{v}_h^1)\beta))_T+\frac{\varrho_T}{\nu}(\chi_h(\beta\cdot\mathbf{x}\nabla\cdot\mathbf{u}_h^1),\chi_h(\beta\cdot\mathbf{x}\nabla\cdot\mathbf{v}_h^1))_T \tag{1.3}$$

to solve convection dominated, where δ_T and ϱ_T are the stabilized parameters, $\mathbf{u}_h^1$ is approximated by continuous piecewise linear polynomial, β is a advection field, $\chi_h:=$

$I-\pi_0$. In [12], Feng etc. proposed a new projection-based stabilized method for steady convection-dominated convection-diffusion equations, and discussed the connections between the proposed method and artificial viscosity method, Streamline Upwind Petrov Galerkin method, and VMS methods. The numerical results showed that the proposed method has good numerical performance in the stabilized methods for steady convection-dominated convection-diffusion equations. Can this method be used to incompressible Navier-Stokes equations?

In this paper, we will discuss the local L^2 projection stabilized technique [12] for Oseen equations. Velocity and pressure are approximated by two kinds of mixed finite element spaces, $P_l^2-P_1$ $(l=1,2)$. The stabilized term reads:

$$S_h(\mathbf{u}_h,\mathbf{v}_h)=\sum_{T\in\mathcal{T}_h}\varsigma_T((I-\pi_{l-1})\mathbf{u}_h^l,(I-\pi_{l-1})\mathbf{v}_h^l)_T,$$

where ς_T is the stabilized parameter, and $\mathbf{u}_h^l$, $\mathbf{v}_h^l$ are polynomials of degree less than l $(l\geq 1)$. A main advantage of the proposed methods lies in that, all the computations are performed at the same element level, without the need of nested meshes or the projection of the gradient of velocity onto a coarse level. Stability and convergence are proved for two kinds of stabilized schemes. Numerical experiments confirm the theoretical results, and show that L^2 projection method has better numerical performance than VMS methods.

The rest of the paper is organized as follows. Section 2 introduces the local L^2 projection method for the Oseen equations employing $P_l^2-P_1$, $(l=1,2)$ mixed finite element spaces. Section 3 shows stability and convergence of $P_1^2-P_1$ stabilized method. Section 4 shows stability and convergence of $P_2^2-P_1$ stabilized method. Finally, in section 5, we end our presentation with some numerical experiments.

Throughout the paper, we use notation $a\lesssim b$ (or $a\gtrsim b$) to represent that there exists a constant C, independent of h, ν and α, such that $a\leq Cb$ (or $a\geq Cb$).

2 Notation and scheme

Let $\Omega\subset\mathbb{R}^2$ be a bounded polygonal domain with boundary $\partial\Omega$. We consider the following Oseen problem:

$$-\nu\Delta\mathbf{u}+(\beta\cdot\nabla)\mathbf{u}+\alpha\mathbf{u}+\nabla p=\mathbf{f} \qquad \text{in } \Omega, \tag{2.1a}$$

$$\nabla\cdot\mathbf{u}=0 \qquad \text{in } \Omega, \tag{2.1b}$$

$$\mathbf{u}=0 \qquad \text{on } \partial\Omega, \tag{2.1c}$$

where $\mathbf{u}$ denotes the velocity field, p the pressure, $\mathbf{f}$ the body force, $\nu=1/Re>0$ the fluid viscosity. Here we assume $\beta\in(W^{1,\infty}(\Omega))^2$ with $\nabla\cdot\beta=0$.

We introduce some notations as follows. For an arbitrary open set T, we denote by $H^k(T)$ the usual Sobolev space consisting of functions defined on T with derivatives of order up to k being square-integrable, with norm $\|\cdot\|_{k,T}$ and semi-norm $|\cdot|_{k,T}$. In particular, $H^0(T)=L^2(T)$. When $T=\Omega$, we abbreviate $\|\cdot\|_{k,\Omega}$, $|\cdot|_{k,\Omega}$ to $\|\cdot\|_k$ and $|\cdot|_k$, respectively. We use the same notations of norms and semi-norms as above for corresponding vector or tensor spaces. We use $(\cdot,\cdot)_T$ to denote inner product of $L^2(T)$. When $T=\Omega$, we abbreviate $(\cdot,\cdot)_\Omega$ to $(\cdot,\cdot)$. For any $v\in L^\infty(\Omega)$, we denote

$$\|v\|_{0,\infty}:=\sup_{x\in\Omega}|v(x)|.$$

Define the spaces

$$\mathbf{V}:=(H_0^1(\Omega))^2,\quad Q:=L_0^2(\Omega)=\Big\{q\in L^2(\Omega):\int_\Omega q dx=0\Big\}.$$

Then we have the following weak formulation for the system (2.1): Find $(\mathbf{u},p)\in\mathbf{V}\times Q$ such that

$$A((\mathbf{u},p),(\mathbf{v},q))=(\mathbf{f},\mathbf{v}),\quad \text{for all } (\mathbf{v},q)\in\mathbf{V}\times Q, \tag{2.2}$$

where for any $\mathbf{u},\mathbf{v}\in\mathbf{V}$, $p,q\in Q$,

$$A((\mathbf{u},p),(\mathbf{v},q)):=a(\mathbf{u},\mathbf{v})+b(\mathbf{u},\mathbf{v})+d(\mathbf{u},\mathbf{v})-c(\mathbf{v},p)+c(\mathbf{u},q),$$

and

$$\begin{aligned} &a(\mathbf{u},\mathbf{v}):=\nu(\nabla\mathbf{u},\nabla\mathbf{v}), && b(\mathbf{u},\mathbf{v}):=((\beta\cdot\nabla)\mathbf{u},\mathbf{v}),\\ &d(\mathbf{u},\mathbf{v}):=\alpha(\mathbf{u},\mathbf{v}), && c(\mathbf{v},q):=(q,\nabla\cdot\mathbf{v}). \end{aligned}$$

Let $\{\mathcal{T}_h\}_h$ be shape regular triangulations of $\bar\Omega$ with the mesh size $h:=\max_{T\in\mathcal{T}_h}h_T$, where h_T is the diameter of triangular $T\in\mathcal{T}_h$. Let ε_h be the set of all interior edges, and define

$$\|u\|_{\varepsilon_h}:=\Big(\sum_{e\in\varepsilon_h}\int_e u^2 ds\Big)^{\frac12}.$$

Let $P_{k+1}(T)$ be the set of polynomial on T with degree no more than $k+1$. For any $k\geq 0$, define

$$\begin{aligned} &P_{k+1}(\mathcal{T}_h)=\{v\in H_0^1(\Omega):v|_T\in P_{k+1}(T),\text{ for all } T\in\mathcal{T}_h\},\\ &P_k^{dc}(\mathcal{T}_h)=\{v\in L_0^2(\Omega):v|_T\in P_k(T),\text{ for all } T\in\mathcal{T}_h\}, \end{aligned}$$

and finite element spaces

$$\mathbf{V}_{1,h}:=(P_1(\mathcal{T}_h))^2\cap V,\quad \mathbf{V}_{2,h}:=(P_2(\mathcal{T}_h))^2\cap V,\quad Q_h=:P_1(\mathcal{T}_h)\cap Q.$$

A well-known approximation result [20] is that for all $\mathbf{u}\in(H^2(\Omega))^2$, there exists a function $\mathbf{w}_h\in\mathbf{V}_h^1$, such that

$$\|\mathbf{u}-\mathbf{w}_h\|_0+h^{1/2}\|\mathbf{u}-\mathbf{w}_h\|_{\varepsilon_h}+h|\mathbf{u}-\mathbf{w}_h|_1\lesssim h^2\|\mathbf{u}\|_2. \tag{2.3}$$

When the mesh scales can't resolve the smallest scale in fluid flows, we must add stabilized term into the weak formulation (2.2) to smear out the effect from the unresolve scales. In this paper, we will analyze the L^2 projection method for Oseen equations. Let $\pi_{l-1}:\mathbf{V}\to(P_{l-1}^{dc}(\mathcal{T}_h))^2 (l=1,2)$ be the local L^2 projection with the following properties:

$$
\begin{aligned}
&(\mathbf{u},\mathbf{v}_h)_T=(\pi_{l-1}\mathbf{u},\mathbf{v}_h)_T, && \text{for all } \mathbf{u}\in\mathbf{V},\quad \mathbf{v}_h\in(P_{l-1}^{dc}(\mathcal{T}_h))^2, && \text{(2.4a)}\\
&\|\pi_{l-1}\mathbf{u}\|_{0,T}\le C\|\mathbf{u}\|_{0,T}, && \text{for all } \mathbf{u}\in\mathbf{V}, && \text{(2.4b)}\\
&\|\mathbf{u}-\pi_{l-1}\mathbf{u}\|_{0,T}\le Ch^l\|\mathbf{u}\|_{l,T}, && \text{for all } \mathbf{u}\in\mathbf{V}\cap(H^l(\mathcal{T}_h))^2. && \text{(2.4c)}
\end{aligned}
$$

Let $S_h(\cdot,\cdot)$ denote the stabilized term with the following form:

$$S_h(\mathbf{u}_h,\mathbf{v}_h):=\sum_{T\in\mathcal{T}_h}\varsigma_T((I-\pi_{l-1})\mathbf{u}_h,(I-\pi_{l-1})\mathbf{v}_h)_T,\quad \text{for all } \mathbf{u}_h,\mathbf{v}_h\in\mathbf{V}_{l,h}, \tag{2.5}$$

where ς_T is stabilized parameter.

Since $P_1^2-P_1$ pair does not satisfy the so-called inf-sup condition, we use LPPS method [9] to overcome this defect. Suppose $\vartheta_0:Q\to P_0^{dc}(\mathcal{T}_h)$ be local L^2 projection with the following properties:

$$
\begin{aligned}
&(p,q_h)_T=(\vartheta_0 p,q_h)_T, && \text{for all } p\in Q,\ q_h\in P_0^{dc}(\mathcal{T}_h), && \text{(2.6a)}\\
&\|\vartheta_0 p\|_{0,T}\le C\|p\|_{0,T}, && \text{for all } p\in Q, && \text{(2.6b)}\\
&\|p-\vartheta_0 p\|_{0,T}\le Ch\|p\|_{1,T}, && \text{for all } p\in H^1(\mathcal{T}_h)\cap Q. && \text{(2.6c)}
\end{aligned}
$$

Pressure stabilized term has the following form:

$$G_h(p_h,q_h)=\sum_{T\in\mathcal{T}_h}\theta_T((I-\vartheta_0)p_h,(I-\vartheta_0)q_h)_T, \tag{2.7}$$

where θ_T is the stabilized parameter.

In this paper, we will analyze two kinds of stabilized schemes. Velocity and pressure are approximated by two kinds of mixed finite element spaces, $P_l^2-P_1$ (l=1,2). The $P_1^2-P_1$ stabilized method reads as: Find $(\mathbf{u}_h,p_h)\in\mathbf{V}_{1,h}\times Q_h$ such that

$$B_h^1((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h))=(\mathbf{f},\mathbf{v}_h),\quad \text{for all } (\mathbf{v}_h,q_h)\in\mathbf{V}_{1,h}\times Q_h, \tag{2.8}$$

where

$$B_h^1((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)):=A((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h))+S_h(\mathbf{u}_h,\mathbf{v}_h)+G_h(p_h,q_h).$$

The $P_2^2-P_1$ stabilized method reads as: Find $(\mathbf{u}_h,p_h)\in\mathbf{V}_{2,h}\times Q_h$ such that

$$B_h^2((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h))=(\mathbf{f},\mathbf{v}_h),\quad \text{for all } (\mathbf{v}_h,q_h)\in\mathbf{V}_{2,h}\times Q_h, \tag{2.9}$$

where

$$B_h^2((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)):=A((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h))+S_h(\mathbf{u}_h,\mathbf{v}_h).$$

3 Stability and convergence of $P_1^2-P_1$ stabilized method

In this section, we will discuss the stability and convergence of the scheme (2.8). Before proving the stability of the method (2.8), we first introduce some notations. Let

$$\varsigma_{\max}:=\max_{T\in\mathcal{T}_h}\varsigma_T,\ \ \varsigma_{\min}:=\min_{T\in\mathcal{T}_h}\varsigma_T,\ \ \theta_{\max}:=\max_{T\in\mathcal{T}_h}\theta_T,\ \ \theta_{\min}:=\min_{T\in\mathcal{T}_h}\theta_T,\ \ \text{and}\ \ h_{\min}:=\min_{T\in\mathcal{T}_h}h_K.$$

For any $(\mathbf{v},q)\in\mathbf{V}_{1,h}\times Q_h$, we define

$$\begin{aligned}|||(\mathbf{v},q)|||_h^2:=&\nu|\mathbf{v}|_1^2+\alpha||\mathbf{v}||_0^2+\sum_{T\in\mathcal{T}_h}(\varsigma_T||(I-\pi_0)\mathbf{v}||_{0,T}^2+\theta_T||(I-\vartheta_0)q||_{0,T}^2)\\&+(\nu+\alpha+\varsigma_{\max}h^2)||q||_0^2.\end{aligned}\tag{3.1}$$

It is easy to check that $|||\cdot|||_h$ is a norm on the spaces $\mathbf{V}_{1,h}\times Q_h$.

3.1 Stability

Theorem 3.1. *Assume*

$$\max\Big\{\nu,\alpha,\varsigma_{\max}h^2,\frac{\nu+\alpha+\varsigma_{\max}h^2}{\theta_{\min}},||\beta||_{0,\infty}\Big\}\le C.$$

For any $(\mathbf{u}_h,p_h)\in\mathbf{V}_{1,h}\times Q_h$, it holds:

$$C_s|||(\mathbf{u}_h,p_h)|||_h\le\sup_{(\mathbf{v}_h,q_h)\in\mathbf{V}_{1,h}\times Q_h}\frac{B_h^1((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h))}{|||(\mathbf{v}_h,q_h)|||_h}.\tag{3.2}$$

Here, the constant C_s is independent of h, ν, α.

Proof. For any $(\mathbf{u}_h,p_h)\in\mathbf{V}_{1,h}\times Q_h$, since

$$b(\mathbf{u_h},\mathbf{u_h})=((\beta\cdot\nabla)\mathbf{u}_h,\mathbf{u}_h)=\frac{1}{2}(\beta\cdot\nabla(\mathbf{u}_h\cdot\mathbf{u}_h),1)=-\frac{1}{2}(\nabla\cdot\beta,\mathbf{u}_h\cdot\mathbf{u}_h)=0,$$

we have

$$\begin{aligned}B_h^1((\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h))=&\sum_{T\in\mathcal{T}_h}\big(\varsigma_T||(I-\pi_0)\mathbf{u}_h||_{0,T}^2+\theta_T||(I-\vartheta_0)p_h||_{0,T}^2\big)\\&+\nu|\mathbf{u}_h|_1^2+\alpha||\mathbf{u}_h||_0^2.\end{aligned}\tag{3.3}$$

For any fixed pressure $p_h\in Q_h\subset L_0^2(\Omega)$, there exists $\mathbf{w}\in\mathbf{V}$ such that

$$\int_\Omega p_h\nabla\cdot\mathbf{w}d\mathbf{x}\ge C||p_h||_0||\mathbf{w}||_1.$$

Let $\mathbf{w}_h \in \mathbf{V}_{1,h}$ be the Scott-Zhang [31] or Clément [14] interpolation of $\mathbf{w}$. By (2.3), it holds

$$||\mathbf{w}-\mathbf{w}_h||_0+h^{\frac{1}{2}}||\mathbf{w}-\mathbf{w}_h||_{\varepsilon_h}\lesssim h|\mathbf{w}|_1 \quad \text{and} \quad ||\mathbf{w}_h||_1\lesssim ||\mathbf{w}||_1.$$

Setting $\tilde{\mathbf{w}}_h=\frac{||p_h||_0}{||\mathbf{w}_h||_1}\mathbf{w}_h$, we have

$$||\tilde{\mathbf{w}}_h||_1=||p_h||_0. \tag{3.4}$$

For simplicity of the notation, we still use $\mathbf{w}_h$ to denote $\tilde{\mathbf{w}}_h$. By (2.14) and (2.21) of [9], we have

$$|(p_h,\nabla\cdot\mathbf{w}_h)|\geq (C_1||p_h||_0-C_2h||\nabla p_h||_0)||\mathbf{w}_h||_1$$

and

$$\sum_{T\in\mathcal{T}_h}||(I-\vartheta_0)p_h||_{0,T}\geq C_3h||\nabla p_h||_0.$$

So it holds

$$\int_\Omega p_h\nabla\cdot\mathbf{w}_h d\mathbf{x}\geq C_1||p_h||_0^2-\tilde{C}_2\sum_{T\in\mathcal{T}_h}||(I-\vartheta_0)p_h||_{0,T}||p_h||_{0,T}.$$

Setting $(\mathbf{v}_h,q_h)=(-(\nu+\alpha+\varsigma_{\max}h^2)\mathbf{w}_h,0)$, it holds

$$\begin{aligned}
&B_h^1((\mathbf{u}_h,p_h),(-(\nu+\alpha+\varsigma_{\max}h^2)\mathbf{w}_h,0))\\
\geq&-(\nu+\alpha+\varsigma_{\max}h^2)(a(\mathbf{u}_h,\mathbf{w}_h)+S_h(\mathbf{u}_h,\mathbf{w}_h)+b(\mathbf{u}_h,\mathbf{w}_h)+d(\mathbf{u}_h,\mathbf{w}_h))\\
&\quad+C_1(\nu+\alpha+\varsigma_{\max}h^2)||p_h||_0^2-\tilde{C}_2(\nu+\alpha+\varsigma_{\max}h^2)\sum_{T\in\mathcal{T}_h}||(I-\vartheta_0)p_h||_{0,T}||p_h||_{0,T}\\
\triangleq&-(\nu+\alpha+\varsigma_{\max}h^2)I_1+C_1(\nu+\alpha+\varsigma_{\max}h^2)||p_h||_0^2\\
&\quad-\tilde{C}_2(\nu+\alpha+\varsigma_{\max}h^2)\sum_{T\in\mathcal{T}_h}||(I-\vartheta_0)p_h||_{0,T}||p_h||_{0,T}.
\end{aligned}\tag{3.5}$$

By Young's inequality and

$$\frac{\nu+\alpha+\varsigma_{\max}h^2}{\theta_{\min}}\leq C,$$

we have

$$\begin{aligned}
&\tilde{C}_2(\nu+\alpha+\varsigma_{\max}h^2)\sum_{T\in\mathcal{T}_h}||(I-\vartheta_0)p_h||_{0,T}||p_h||_{0,T}\\
\leq&\frac{C_1}{6}(\nu+\alpha+\varsigma_{\max}h^2)||p_h||_0^2+\frac{C\cdot\tilde{C}_2^2}{C_1}\sum_{T\in\mathcal{T}_h}\theta_T||(I-\vartheta_0)p_h||_{0,T}^2.
\end{aligned}\tag{3.6}$$

By Cauchy-Schwardz inequality and (3.4), we have

$$
\begin{aligned}
\alpha I_1 \le& \alpha\big(\nu|\mathbf{u}_h|_1|\mathbf{w}_h|_1+||\beta||_{0,\infty}|\mathbf{w}_h|_1||\mathbf{u}_h||_0+\alpha||\mathbf{u}_h||_0||\mathbf{w}_h||_0+S_h(\mathbf{u}_h,\mathbf{u}_h)^{\frac{1}{2}}S_h(\mathbf{w}_h,\mathbf{w}_h)^{\frac{1}{2}}\big)\\
\le& C\alpha||p_h||_0(|||(\mathbf{u}_h,0)|||_h+||\mathbf{u}_h||_0).
\end{aligned}
\tag{3.7}
$$

Integration by parts, and using Cauchy-Schwardz inequality and (3.4) again, we have

$$
\begin{aligned}
(\nu+\varsigma_{\max}h^2)I_1 \le& (\nu+\varsigma_{\max}h^2)\big(\nu|\mathbf{u}_h|_1|\mathbf{w}_h|_1+||\beta||_{0,\infty}|\mathbf{u}_h|_1||\mathbf{w}_h||_0+\alpha||\mathbf{u}_h||_0||\mathbf{w}_h||_0\\
&+S_h(\mathbf{u}_h,\mathbf{u}_h)^{\frac{1}{2}}S_h(\mathbf{w}_h,\mathbf{w}_h)^{\frac{1}{2}}\big)\\
\le& C(\nu+\varsigma_{\max}h^2)||p_h||_0(|||(\mathbf{u}_h,0)|||_h+|\mathbf{u}_h|_1).
\end{aligned}
\tag{3.8}
$$

By (3.7) and (3.8), it holds

$$
\begin{aligned}
(\nu+\alpha+\varsigma_{\max}h^2)I_1 \le& C\big((\nu+\varsigma_{\max}h^2+\alpha)|||(\mathbf{u}_h,0)|||_h+\alpha||\mathbf{u}_h||_0\\
&+(\nu+\varsigma_{\max}h^2)|\mathbf{u}_h|_1\big)||p_h||_0\\
\le& \frac{C_1}{3}(\nu+\varsigma_{\max}h^2+\alpha)||p_h||_0^2+\frac{C}{C_1}|||(\mathbf{u}_h,0)|||_h^2.
\end{aligned}
\tag{3.9}
$$

Combining the above inequalities, we get

$$
\begin{aligned}
&B_h^1((\mathbf{u}_h,p_h),(-(\nu+\alpha+\varsigma_{\max}h^2)\mathbf{w}_h,0))\\
\ge& \frac{C_1}{2}(\nu+\alpha+\varsigma_{\max}h^2)||p_h||_0^2-\frac{C\cdot\tilde{C}_2^2}{C_1}\sum_{T\in\mathcal{T}_h}\theta_T||(I-\vartheta_0)p_h||_{0,T}^2-\frac{C}{C_1}|||(\mathbf{u}_h,0)|||_h^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&B_h^1((\mathbf{u}_h,p_h),(\mathbf{u}_h-\lambda(\nu+\alpha+\varsigma_{\max}h^2)\mathbf{w}_h,p_h))\\
\ge& \Big(1-\frac{C}{C_1}\lambda\Big)|||(\mathbf{u}_h,0)|||_h^2+\frac{\lambda C_1}{2}(\nu+\alpha+\varsigma_{\max}h^2)||p_h||_0^2\\
&+\Big(1-\lambda\frac{C\cdot\tilde{C}_2^2}{C_1}\Big)\sum_{T\in\mathcal{T}_h}\theta_T||(I-\vartheta_0)p_h||_{0,T}^2.
\end{aligned}
$$

Choosing

$$
\hat{\lambda}=\min\Big\{\frac{C_1}{2C},\frac{C_1}{2\tilde{C}_2^2C}\Big\},
\tag{3.10}
$$

then we have

$$
1-\frac{C}{C_1}\hat{\lambda}\ge\frac{1}{2},\quad 1-\frac{C\cdot\tilde{C}_2^2}{C_1}\hat{\lambda}\ge\frac{1}{2}.
$$

Now we choose $(\mathbf{v}_h,q_h)=(\mathbf{u}_h-\hat{\lambda}(\nu+\alpha+\varsigma_{\max}h^2)\mathbf{w}_h,p_h)$, and then we get

$$\begin{aligned}
&B_h^1((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h))\\
\geq&\frac{1}{2}\Big(|||(\mathbf{u}_h,0)|||_h^2+\hat{\lambda}C_1(\nu+\alpha+\varsigma_{\max}h^2)||p_h||_0^2+\sum_{T\in\mathcal{T}_h}\theta_T||(I-\vartheta_0)p_h||_{0,T}^2\Big)\\
\geq&C|||(\mathbf{u}_h,p_h)|||_h^2. \qquad (3.11)
\end{aligned}$$

By (3.4), (3.10) and Young's inequality, it holds

$$\begin{aligned}
|||(\mathbf{v}_h,q_h)|||_h\leq&|||(\mathbf{u}_h,p_h)|||_h+\hat{\lambda}(\nu+\alpha+\varsigma_{\max}h^2)|||(\mathbf{w}_h,0)|||_h\\
\leq&|||(\mathbf{u}_h,p_h)|||_h+\hat{\lambda}(\nu+\alpha+\varsigma_{\max}h^2)^{\frac{3}{2}}||p_h||_0\\
\leq&(1+\min\{\frac{C_1}{2\tilde{C}_2^2},\frac{C_1}{2}\})|||(\mathbf{u}_h,p_h)|||_h. \qquad (3.12)
\end{aligned}$$

As a result, a combination of (3.11) and (3.12) yields the desired result. □

3.2 Convergence

Lemma 3.1. *Let $(\mathbf{u},p)\in(\mathbf{V}\cap(H^2(\Omega))^2)\times(Q\cap H^1(\Omega))$ and $(\mathbf{u}_h,p_h)\in\mathbf{V}_{1,h}\times Q_h$ be the solutions of the problems* (2.2) *and* (2.8)*, respectively. Then, the consistent error is*

$$\begin{aligned}
R((\mathbf{u},p),(\mathbf{v}_h,q_h)):=&B_h^1((\mathbf{u}-\mathbf{u}_h,p-p_h),(\mathbf{v}_h,q_h))\\
=&S_h(\mathbf{u},\mathbf{v}_h)+G_h(p,q_h). \qquad (3.13)
\end{aligned}$$

Furthermore, the following estimate holds true

$$|R((\mathbf{u},p),(\mathbf{v}_h,q_h))|\lesssim(\sqrt{\varsigma_{\max}}h||\mathbf{u}||_2+\sqrt{\theta_{\max}}h|p|_1)|||(\mathbf{v}_h,q_h)|||_h. \qquad (3.14)$$

Proof. The consistent error (3.13) follows from the definition of $B_h^1((\cdot,\cdot),(\cdot,\cdot))$, (2.2) and (2.8). The estimate (3.14) follows from properties of the projection π_0 and ϑ_0. □

Theorem 3.2. *Assume*

$$\max\Big\{\nu,\alpha,\varsigma_{\max}h^2,\frac{\nu+\alpha+\varsigma_{\max}h^2}{\theta_{min}},||\beta||_{0,\infty}\Big\}\leq C.$$

Let $(\mathbf{u},p)\in(\mathbf{V}\cap(H^2(\Omega))^2)\times(Q\cap H^1(\Omega))$ and $(\mathbf{u}_h,p_h)\in\mathbf{V}_{1,h}\times Q_h$ be the solutions of the problems (2.2) *and* (2.8)*, respectively. Then,*

$$\begin{aligned}
&|||(\mathbf{u}-\mathbf{u}_h,p-p_h)|||_h\\
\lesssim&\max\left\{\sqrt{\nu},\sqrt{\frac{1}{\nu+\alpha+\varsigma_{\max}h^2}},\sqrt{\varsigma_{\max}}\right\}h||\mathbf{u}||_2\\
&+\max\left\{\sqrt{\nu+\alpha+\varsigma_{\max}h^2+\theta_{\max}},\sqrt{\theta_{\max}},\sqrt{\frac{1}{\nu+\varsigma_{\min}h_{\min}^2}}\right\}h|p|_1.
\end{aligned}$$

Proof. Let $(I_h\mathbf{u},J_hp)$ be an interpolation pair of $(\mathbf{u},p)$ in $\mathbf{V}_{1,h}\times Q_h$, where I_h is the Lagrange interpolation operator and $J_h:L^2(\Omega)\to Q_h$ is the L^2 projection operator [13]. By Theorem 3.1, we have

$$\begin{aligned}
&|||(\mathbf{u}_h-I_h\mathbf{u},p_h-J_hp)|||_h\\
\le& C_s^{-1}\sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_{1,h}\times Q_h}\frac{B_h^1((\mathbf{u}_h-I_h\mathbf{u},p_h-J_hp),(\mathbf{w}_h,r_h))}{|||(\mathbf{w}_h,r_h)|||_h}\\
\le& C_s^{-1}\sup_{(\mathbf{w}_h,r_h)\in\mathbf{v}_{1,h}\times Q_h}\frac{B_h^1((\mathbf{u}_h-\mathbf{u},p_h-p),(\mathbf{w}_h,r_h))}{|||(\mathbf{w}_h,r_h)|||_h}\\
&+C_s^{-1}\sup_{(\mathbf{w}_h,r_h)\in\mathbf{v}_{1,h}\times Q_h}\frac{B_h^1((\mathbf{u}-I_h\mathbf{u},p-J_hp),(\mathbf{w}_h,r_h))}{|||(\mathbf{w}_h,r_h)|||_h}.
\end{aligned}\tag{3.15}$$

By the definition of $|||\cdot|||_h$ and Cauchy-Schwardz inequality, we get

$$\begin{aligned}
B_h^1((\mathbf{u}-I_h\mathbf{u},p-J_hp),(\mathbf{w}_h,r_h))\le&|||(\mathbf{u}-I_h\mathbf{u},p-J_hp)|||_h|||(\mathbf{w}_h,r_h)|||_h\\
&+b(\mathbf{u}-I_h\mathbf{u},\mathbf{w}_h)+c(\mathbf{u}-I_h\mathbf{u},r_h)-c(\mathbf{w}_h,p-J_hp).
\end{aligned}\tag{3.16}$$

Using the stabilities of π_0 and ϑ_0, the approximation properties of I_h and J_h, we obtain

$$\begin{aligned}
|||(\mathbf{u}-I_h\mathbf{u},p-J_hp)|||_h\lesssim&(\sqrt{\nu}h+\sqrt{\alpha}h^2+\sqrt{\varsigma_{\max}}h^2)||\mathbf{u}||_2\\
&+\sqrt{\nu+\alpha+\varsigma_{\max}h^2+\theta_{\max}}h|p|_1.
\end{aligned}\tag{3.17}$$

Integration by parts and using Cauchy-Schwardz inequality, there hold

$$\begin{aligned}
b(\mathbf{u}-I_h\mathbf{u},\mathbf{w}_h)=&-(\beta\cdot\nabla\mathbf{w}_h,\mathbf{u}-I_h\mathbf{u})\lesssim||\beta||_{0,\infty}h^2|\mathbf{u}|_2|\mathbf{w}_h|_1\\
\lesssim&\frac{||\beta||_{0,\infty}}{(\nu+\varsigma_{\min}h_{\min}^2)^{1/2}}h^2|\mathbf{u}|_2|||(\mathbf{w}_h,r_h)|||_h,
\end{aligned}$$

and

$$c(\mathbf{u}-I_h\mathbf{u},r_h)\lesssim h|\mathbf{u}|_2||r_h||_0\lesssim\sqrt{\frac{1}{\nu+\alpha+\varsigma_{\max}h^2}}h|\mathbf{u}|_2||(\mathbf{w}_h,r_h)||_h.$$

Applying the approximation property of J_h yields

$$c(\mathbf{w}_h,p-J_hp)\lesssim|\mathbf{w}_h|_1||p-J_hp||_0\lesssim\sqrt{\frac{1}{\nu+\varsigma_{\min}h_{\min}^2}}h|p|_1|||(\mathbf{w}_h,r_h)|||_h.$$

By the above estimates, it follows

$$\begin{aligned}
&B_h^1((\mathbf{u}-I_h\mathbf{u},p-J_hp),(\mathbf{w}_h,r_h))\\
\lesssim&\left(\max\left\{\sqrt{\nu},\sqrt{\frac{1}{\nu+\alpha+\varsigma_{\max}h^2}}\right\}h||\mathbf{u}||_2\right.\\
&\left.+\max\left\{\sqrt{\nu+\alpha+\varsigma_{\max}h^2+\theta_{\max}},\sqrt{\frac{1}{\nu+\varsigma_{\min}h_{\min}^2}}\right\}h|p|_1\right)|||(\mathbf{w}_h,r_h)|||_h. \qquad (3.18)
\end{aligned}$$

Lemma 3.1, (3.15) and (3.18) lead to

$$\begin{aligned}
&|||(\mathbf{u}_h-I_h\mathbf{u},p_h-J_hp)|||_h\\
\lesssim&\max\left\{\sqrt{\nu},\sqrt{\frac{1}{\nu+\alpha+\varsigma_{\max}h^2}},\sqrt{\varsigma_{\max}}\right\}h||\mathbf{u}||_2\\
&+\max\left\{\sqrt{\nu+\alpha+\varsigma_{\max}h^2+\theta_{\max}},\sqrt{\theta_{\max}},\sqrt{\frac{1}{\nu+\varsigma_{\min}h_{\min}^2}}\right\}h|p|_1. \qquad (3.19)
\end{aligned}$$

By triangular inequality, (3.17) and (3.19), it is easy to obtain the desired result

$$|||(\mathbf{u}-\mathbf{u}_h,p-p_h)|||_h\le|||(\mathbf{u}-I_h\mathbf{u},p-J_hp)|||_h+|||(\mathbf{u}_h-I_h\mathbf{u},p_h-J_hp)|||_h.$$

Thus, we complete the proof. □

By the definition of the norm $|||(\cdot,\cdot)|||_h$ and Theorem 4.2, we have the following results.

Corollary 3.1. *Assume*

$$\max\left\{\nu,\alpha,\varsigma_{\max}h^2,\frac{\nu+\alpha+\varsigma_{\max}h^2}{\theta_{min}},||\beta||_{0,\infty}\right\}\le C.$$

Let $(\mathbf{u},p)\in(\mathbf{V}\cap(H^2(\Omega))^2)\times(Q\cap H^1(\Omega))$ *and* $(\mathbf{u}_h,p_h)\in\mathbf{V}_{1,h}\times Q_h$ *be the solutions of the problems* (2.2) *and* (2.8)*, respectively. Then, we have*

$$\begin{aligned}
|\mathbf{u}-\mathbf{u}_h|_1\lesssim&\frac{1}{\sqrt{\nu}}\max\left\{\sqrt{\nu},\sqrt{\frac{1}{\nu+\alpha+\varsigma_{\max}h^2}},\sqrt{\varsigma_{\max}}\right\}h||\mathbf{u}||_2\\
&+\frac{1}{\sqrt{\nu}}\max\left\{\sqrt{\nu+\alpha+\varsigma_{\max}h^2+\theta_{\max}},\sqrt{\theta_{\max}},\sqrt{\frac{1}{\nu+\varsigma_{\min}h_{\min}^2}}\right\}h|p|_1,
\end{aligned}$$

and

$$\begin{aligned}
||p-p_h||_0\lesssim&\max\left\{\sqrt{\nu},\sqrt{\frac{1}{\nu+\alpha+\varsigma_{\max}h^2}},\sqrt{\varsigma_{\max}}\right\}h||\mathbf{u}||_2\\
&+\max\left\{\sqrt{\nu+\alpha+\varsigma_{\max}h^2+\theta_{\max}},\sqrt{\theta_{\max}},\sqrt{\frac{1}{\nu+\varsigma_{\min}h_{\min}^2}}\right\}h|p|_1.
\end{aligned}$$

4 Stability and convergence of $P_2^2-P_1$ stabilized method

In this section, we give the stability and convergence of the scheme (2.8). For any $(\mathbf{v},q)\in\mathbf{V}_{2,h}\times Q_h$, we define

$$|||(\mathbf{v},q)|||_*^2:=\nu|\mathbf{v}|_1^2+\alpha||\mathbf{v}||_0^2+\sum_{T\in\mathcal{T}_h}\varsigma_T||(I-\pi_1)\mathbf{v}||_{0,T}^2+(\nu+\alpha+\varsigma_{\max}h^2)||q_h||_0. \tag{4.1}$$

It is easy to check that $|||\cdot|||_*$ is a norm on $\mathbf{V}_{2,h}\times Q_h$. We have the following results.

Theorem 4.1. *The new L^2 projection method defined in* (2.9) *satisfies the following stability property. Assume*

$$\max\Big\{\nu,\alpha,\varsigma_{\max}h^2,\frac{\nu+\alpha+\varsigma_{\max}h^2}{\theta_{\min}},||\beta||_{0,\infty}\Big\}\le C.$$

For all $(\mathbf{u}_h,p_h)\in\mathbf{V}_{2,h}\times Q_h$, there holds:

$$C_s|||(\mathbf{u}_h,p_h)|||_*\le\sup_{(\mathbf{v}_h,q_h)\in\mathbf{V}_{2,h}\times Q_h}\frac{B_h^2((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h))}{|||(\mathbf{v}_h,q_h)|||_*}.$$

Here, the constant C_s is independent of h, ν, α.

Proof. The pair $\mathbf{V}_{2,h}\times Q_h$ fulfills the discrete inf$-$sup condition, i.e., there exists a positive constant C_0 such that

$$\inf_{q_h\in Q_h}\sup_{\mathbf{v}_h\in\mathbf{V}_{2,h}}\frac{(q_h,\nabla\cdot\mathbf{v}_h)}{||\mathbf{v}_h||_1||q_h||_0}\ge C_0.$$

By the discrete inf$-$sup condition, for all $p_h\in Q_h$, there exists a $\mathbf{w}_h\in\mathbf{V}_{2,h}$, such that (see [20, page 118, Remark 1.4])

$$(\nabla\cdot\mathbf{w}_h,p_h)=-||p_h||_0^2 \quad\text{and}\quad ||\mathbf{w}_h||_1\le C_3||p_h||_0.$$

Following the same proof line as the proof of Theorem 3.1, we obtain the desired result. Thus, we complete the proof. □

Theorem 4.2. *Assume*

$$\max\Big\{\nu,\alpha,\varsigma_{\max}h^2,\frac{\nu+\alpha+\varsigma_{\max}h^2}{\theta_{min}},||\beta||_{0,\infty}\Big\}\le C.$$

Let $(\mathbf{u},p)\in(\mathbf{V}\cap(H^3(\Omega))^2)\times(Q\cap H^2(\Omega))$ and $(\mathbf{u}_h,p_h)\in\mathbf{V}_{2,h}\times Q_h$ be the solutions of the problems (2.2) *and* (2.9)*, respectively. Then, we have*

$$\begin{aligned}|||(\mathbf{u}-\mathbf{u}_h,p-p_h)|||_*\lesssim&\max\Big\{\sqrt{\nu},\sqrt{\frac{1}{\nu+\alpha+\varsigma_{\max}h^2}},\sqrt{\varsigma_{\max}}\Big\}h^2||\mathbf{u}||_3\\&+\max\Big\{\sqrt{\nu+\alpha+\varsigma_{\max}h^2},\sqrt{\frac{1}{\nu+\varsigma_{\min}h_{\min}^2}}\Big\}h^2|p|_2,\end{aligned}$$

$$|\mathbf{u}-\mathbf{u}_h|_1 \lesssim \frac{1}{\sqrt{\nu}} \max\left\{\sqrt{\nu}, \sqrt{\frac{1}{\nu+\alpha+\varsigma_{\max}h^2}}, \sqrt{\varsigma_{\max}}\right\} h^2 ||\mathbf{u}||_3$$
$$+\frac{1}{\sqrt{\nu}} \max\left\{\sqrt{\nu+\alpha+\varsigma_{\max}h^2}, \sqrt{\frac{1}{\nu+\varsigma_{\min}h_{\min}^2}}\right\} h^2 |p|_2,$$

and

$$||p-p_h||_0 \lesssim \max\left\{\sqrt{\nu}, \sqrt{\frac{1}{\nu+\alpha+\varsigma_{\max}h^2}}, \sqrt{\varsigma_{\max}}\right\} h^2 ||\mathbf{u}||_3$$
$$+\max\left\{\sqrt{\nu+\alpha+\varsigma_{\max}h^2}, \sqrt{\frac{1}{\nu+\varsigma_{\min}h_{\min}^2}}\right\} h^2 |p|_2.$$

5 Numerical experiments

In this section, we present some numerical results to verify our theoretical results. We list the numerical results of $e_p = ||p-p_h||_0/||p||_0$ for the pressure error, $e_{\mathbf{u}} = |\mathbf{u}-\mathbf{u}_h|_1/|\mathbf{u}|_1$ for the velocity error and $e_{\mathbf{u}p} = |||(\mathbf{u}-\mathbf{u}_h, p-p_h)|||_h / \sqrt{|\mathbf{u}|_1^2 + ||p||_0^2}$.

5.1 Convergence validations

Example 5.1. We consider $\Omega = (0,1)\times(0,1)$, $\beta = (\sin(y), \cos(y))^T$, $\alpha = 0$, and set $\mathbf{f}$ and the boundary conditions such that the exact solution is given by

$$\mathbf{u} = (e^x \sin(x), e^x \cos(y))^T, \quad p = -e^x + e - 1.$$

In this example, velocity and pressure are approximated by two kinds of mixed finite element spaces, $P_l^2 - P_1$ $(l = 1,2)$. Numerical results are listed in Tables 1-2.

Table 1: The results of (2.8): Example 5.1, $\varsigma_T = 3$, $\theta_T = 0.5$.

ν	Error	4×4	8×8	16×16	32×32	64×64	Order
10^{-4}	$\bar{e}_{\mathbf{u}}$	0.63637	0.36709	0.15202	0.06116	0.02498	1.17
	$\bar{e}_p$	0.92294	0.20941	0.04931	0.01197	0.00318	2.05
	$\bar{e}_{\mathbf{u}p}$	0.20525	0.04070	0.00857	0.00202	0.00058	2.17
10^{-6}	$\bar{e}_{\mathbf{u}}$	0.62882	0.38095	0.17078	0.08452	0.04222	0.97
	$\bar{e}_p$	0.91663	0.21164	0.05063	0.01268	0.00319	2.04
	$\bar{e}_{\mathbf{u}p}$	0.20333	0.04122	0.00886	0.00215	0.00054	2.14
10^{-8}	$\bar{e}_{\mathbf{u}}$	0.62874	0.38110	0.17100	0.08495	0.04307	0.97
	$\bar{e}_p$	0.91657	0.21167	0.05065	0.01269	0.00321	2.04
	$\bar{e}_{\mathbf{u}p}$	0.20331	0.04122	0.00886	0.00216	0.00054	2.14

Table 2: The results of (2.9): Example 5.1, $\varsigma_T=3$.

ν	Error	4×4	8×8	16×16	32×32	64×64	Order
	$\bar{e}_{\mathbf{u}}$	0.05096	0.02203	0.00805	0.00211	0.00037	1.78
10^{-4}	$\bar{e}_p$	0.00967	0.00234	0.00056	0.00013	0.00003	2.08
	$\bar{e}_{\mathbf{u}p}$	0.00217	0.00046	0.00011	0.00002	0.00000	2.29
	$\bar{e}_{\mathbf{u}}$	0.05325	0.02562	0.01245	0.00601	0.00285	1.06
10^{-6}	$\bar{e}_p$	0.00975	0.00239	0.00060	0.00015	0.00004	1.98
	$\bar{e}_{\mathbf{u}p}$	0.00219	0.00046	0.00011	0.00003	0.00001	2.08
	$\bar{e}_{\mathbf{u}}$	0.05328	0.02567	0.01253	0.00615	0.00313	1.02
10^{-8}	$\bar{e}_p$	0.00975	0.00239	0.00060	0.00015	0.00004	1.98
	$\bar{e}_{\mathbf{u}p}$	0.00219	0.00046	0.00011	0.00003	0.00001	2.08

Example 5.2. We consider $\Omega=(0,1)\times(0,1)$, $\beta=\frac{1}{\sqrt{5}}(2.0,1.0)^T$, $\alpha=0$. Set $\mathbf{f}$ and the boundary conditions such that the exact solution is given by

$$\mathbf{u}=(\sin(x)\cos(y),-\cos(x)\sin(y))^T, \quad p=x^2+y^2-\frac{2}{3}.$$

Numerical results are listed in Tables 3-4. Velocity field and pressure are approximated by $P_2^2-P_1$ pair, we make a compare with L^2 projection method (2.9) and VMS methods, which add terms of the form:

$$S_{VMS1}(\mathbf{u}_h,\mathbf{v}_h)=\sum_{T\in\mathcal{T}_h}\nu_{V1}((I-\mathcal{Q}_0)\nabla\mathbf{u}_h,(I-\mathcal{Q}_0)\nabla\mathbf{v}_h)_T \tag{5.1}$$

and

$$S_{VMS2}(\mathbf{u}_h,\mathbf{v}_h)=\sum_{T\in\mathcal{T}_h}\nu_{V2}(\nabla(I-\mathcal{Q}_1)\mathbf{u}_h,\nabla(I-\mathcal{Q}_1)\mathbf{v}_h)_T. \tag{5.2}$$

Here $\varsigma_T=3.0$, $\nu_{V1}=0.4$, $\nu_{V2}=0.1$, $\mathcal{Q}_0:(L^2(\Omega))^{2\times2}\to(P_0^{dc}(\mathcal{T}_h))^{2\times2}$ and $\mathcal{Q}_1:(L^2(\Omega))^2\to(P_1^{dc}(\mathcal{T}_h))^2$ denote local L^2 projection operators.

From Tables 1-4, we can conclude the following conclusions:

Table 3: The results of (2.8): Example 5.2, $\varsigma_T=7$, $\theta_T=0.5$.

ν	Error	4×4	8×8	16×16	32×32	64×64	Order
	$\bar{e}_{\mathbf{u}}$	0.22526	0.17247	0.08538	0.04171	0.02246	0.83
10^{-4}	$\bar{e}_p$	0.31010	0.08629	0.02207	0.00570	0.00165	1.89
	$\bar{e}_{\mathbf{u}p}$	0.11869	0.02320	0.00496	0.00124	0.00040	2.05
	$\bar{e}_{\mathbf{u}}$	0.22508	0.17420	0.08843	0.04623	0.02376	0.81
10^{-6}	$\bar{e}_p$	0.31013	0.08632	0.02206	0.00565	0.00144	1.94
	$\bar{e}_{\mathbf{u}p}$	0.11866	0.02319	0.00494	0.00121	0.00030	2.16
	$\bar{e}_{\mathbf{u}}$	0.22508	0.17422	0.08847	0.04631	0.02392	0.81
10^{-8}	$\bar{e}_p$	0.31013	0.08632	0.02206	0.00565	0.00144	1.94
	$\bar{e}_{\mathbf{u}p}$	0.11866	0.02319	0.00494	0.00121	0.00030	2.16

Table 4: The comparison between L^2 projection method (2.9) and VMS methods: Example 5.2, $P_2^2-P_1$, $\nu=10^{-6}$.

Method	Error	4×4	8×8	16×16	32×32	64×64	order
(2.9)	$\bar{e}_{\mathbf{u}}$	0.04739	0.02408	0.01177	0.00573	0.00259	1.05
	$\bar{e}_p$	0.01580	0.00394	0.00098	0.00025	0.00006	2.01
VMS1	$\bar{e}_{\mathbf{u}}$	0.11382	0.06262	0.03346	0.01748	0.00902	0.91
	$\bar{e}_p$	0.04533	0.01164	0.00297	0.00075	0.00019	1.97
VMS2	$\bar{e}_{\mathbf{u}}$	0.11411	0.06150	0.03141	0.01592	0.00838	0.94
	$\bar{e}_p$	0.02442	0.00613	0.00154	0.00041	0.00013	1.89

1. $\bar{e}_p$ and $\bar{e}_{\mathbf{u}p}$ are of second order convergence of stabilized method (2.9). These are conformable to the convergence results in Theorem 3.2. The comparison between L^2 projection method and VMS methods, shows that two kinds of methods have the same convergence rate, and L^2 projection method has smaller relative error using the same mesh $\mathcal{T}_h$.

2. An unexpected second order convergence appears for $\bar{e}_p$ and $\bar{e}_{energy}$ of stabilized method (2.8). $\bar{e}_{\mathbf{u}}$ is of one order convergence.

5.2 Boundary layer problem

Example 5.3. We consider $\Omega=(0,1)\times(0,1)$, $\beta=(1,1)$, $\alpha=0$, $\nu=10^{-2}$ and set $\mathbf{f}$ and the boundary conditions such that the exact solution is given by

$$\mathbf{u}=\begin{pmatrix} \dfrac{e^{\frac{y}{\nu}}-e^{\frac{1}{\nu}}}{1-e^{\frac{1}{\nu}}} \\ \dfrac{e^{\frac{x}{\nu}}-e^{\frac{1}{\nu}}}{1-e^{\frac{1}{\nu}}} \end{pmatrix}, \quad p=x-y.$$

In this example, velocity filed and pressure approximated by $P_2^2-P_1$ pair, we use two kinds of methods to compare: L^2 projection method (2.9) and VMS methods. Here $h=\frac{\sqrt{2}}{32}$ (Fig. 1), $\varsigma_T=3.0$, $\nu_{V1}=0.9$, $\nu_{V2}=7.0$. In Figs. 2-4, we give elevations of the second component of velocity field $\mathbf{u}_h$. In Figs. 5-6, we give the numerical solutions of the second component of velocity field $\mathbf{u}_h$ by using two kinds of methods, where we set $y=0.1$ and $y=0.9$, respectively.

From Figs. 2-6, we can can draw the following conclusions:

- Velocity field and pressure are approximated by $P_2^2-P_1$ pair, two kinds of methods can deal with boundary layer problem effectively.

- Compared with VMS methods, numerical solutions of L^2 projection method (2.9) are more close to the exact solutions.

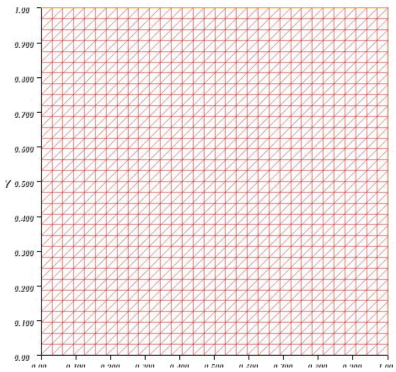

Figure 1: Grid: $\mathcal{T}_h, h=\frac{\sqrt{2}}{32}$.

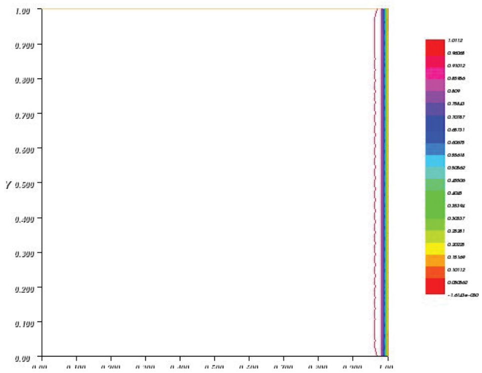

Figure 2: (2.9).

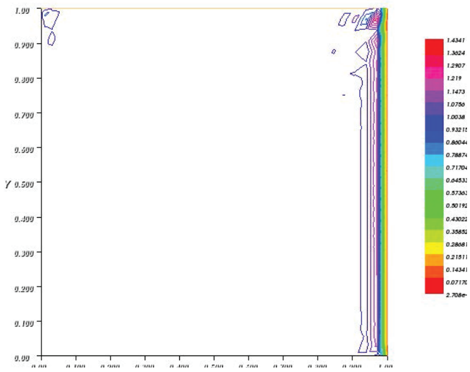

Figure 3: VMS1: $P_2^2-P_1$.

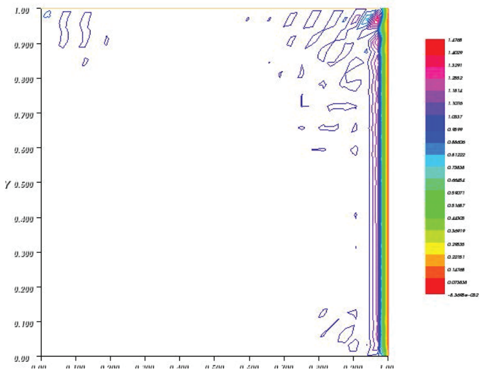

Figure 4: VMS2: $P_2^2-P_1$.

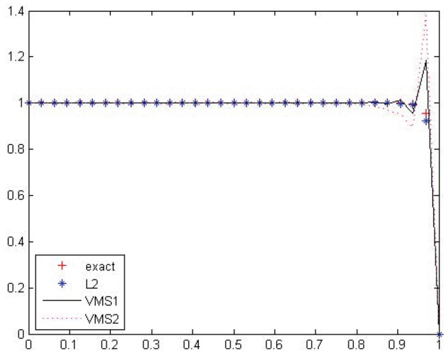

Figure 5: $y=0.1$.

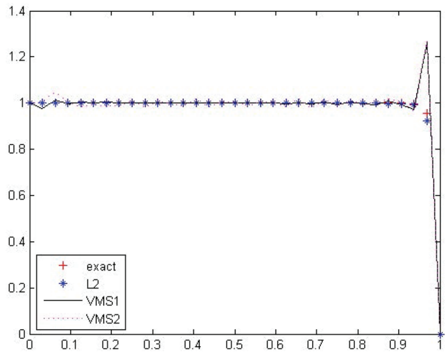

Figure 6: $y=0.9$.

Acknowledgments

This work is supported by NSF of China (Nos. 11071184, 11271273, 11371275, 41674141), NSF of Shanxi Province (No. 2012011015-6), STIP of Higher Education Institutions in

Shanxi (No. 20111121), Young Scholars Development Fund of SWPU (No. 201599010041), Young Science and Technology Innovation Team of SWPU (No. 2015CXTD07), and Key Program of SiChuan Provincial Department of Education (No. 16ZA0066).

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