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ON THE CONVERGENCE OF THE BRENT METHOD*[1)]

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Abstract

In this paper, we establish the semi-local convergence theorem of the Brent method with regional estimation. By an in-depth investigation in to the algorithm structure of the method, we convert the Brent method into an approximate Newton method with a special error term. Bsaed on such equivalent variation, under a similar condition of the Newton-Kantorovich theorem of the Newton method, we establish a semi-local convergence theorem of the Brent method. This theorem provides a sufficient theoretical basis for initial choices of the Brent method.

1. Introduction

It is well known that the Brent method for solving systems of nonlinear equations

$$F(x)=0, \qquad F: D\subset R^n \longrightarrow R^n \tag{1.1}$$

is to solve the following system:

$$\begin{aligned}
&J(x^{(k)},h^{(k)})(x^{(k+1)}-x^{(k)})+F(x^{(k)})=0,\\
&J(x^{(k)},h^{(k)})e_i=[F(x^{(k)}+h_i^{(k)}e_i)-F(x^{(k)})]/h_i^{(k)}, \quad h_i^{(k)}\neq 0,\\
&h^{(k)T}=[h_1^{(k)},\cdots,h_n^{(k)}], \quad e_i^T=[0,\cdots,0,1,0,\cdots,0],\\
&\qquad\qquad i=1,2,\cdots,n; \quad k=0,1,2,\cdots,
\end{aligned} \tag{1.2}$$

by making use of the orthogonal triangular factorization. Suppose that we have an approximation $x^{(k)}$ to x^*, a solution of (1.1). Then the k-th iterative procedure can be described as follows [1]:

Step 1. Let $y_1^{(k)}=x^{(k)}, Q_1^{(k)}=Q_{n+1}^{(k-1)}$(or $Q_1^{(k)}=I$).

Step 2. Compute the vector

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$$\bar{a}_j^{(k)} = \frac{1}{h_k} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f_j(y_j^{(k)} + h_k Q_j^{(k)} e_j) - f_j(y_j^{(k)}) \\ \vdots \\ f_j(y_j^{(k)} + h_k Q_j^{(k)} e_n) - f_j(y_j^{(k)}) \end{bmatrix}, \tag{1.3}$$

where $h_k \neq 0$ is the difference step corresponding to the index k (we will discuss the choices of h_k in Section 4). Construct an orthogonal matrix (usually by the Household transformation)

$$U_j^{(k)} = \begin{bmatrix} I_{j-1} & 0 \\ 0 & \bar{U}_j^{(k)} \end{bmatrix} \tag{1.4}$$

such that

$$(\bar{a}_j^{(k)})^T U_j^{(k)} = \sigma_j^{(k)} e_j^T, \quad \sigma_j^{(k)} = \pm \|\bar{a}_j^{(k)}\|. \tag{1.5}$$

Let

$$Q_{j+1}^{(k)} = Q_j^{(k)} U_j^{(k)}. \tag{1.6}$$

Step 3. Compute

$$y_{j+1}^{(k)} = y_j^{(k)} - \frac{1}{\sigma_j^{(k)}} f_j(y_j^{(k)}) Q_{j+1}^{(k)} e_j. \tag{1.7}$$

Step 4. If $j < n$, let $j := j + 1$, go to Step 2; otherwise, let $x^{(k+1)} = y_{n+1}^{(k)}$.

Brent[1] applied the Shamanskii technique at each iteration to improve the efficiency of the algorithm. However, it is not essential to the discussion of this paper.

Brent[1] proved that the above algorithm converges locally to x^* with a quadratic convergence order. Since for each iteration the number of function evaluations of the algorithm is $\frac{n(n+3)}{2}$, which is nearly half of what is needed by the Newton method, and since it is also of satisfactory numerical stability, the Brent method has long been regarded as a most effective numerical method for solving nonlinear systems.

The main purpose of this paper is to establish the semi-local convergence theorem of the Brent method with regional estimations. Because of the complexity of the algorithm structure, the classical Kantorovich method can not be applied to this method. By investigation into the algorithm structure of the method, we convert the Brent method into an approximate Newton method with a special error term. Based on such equivalent variation, under a similar condition of the Newton-Kantorovich theorem of the Newton method, for the Brent method we establish an existence-convergence theorem (semilocal convergence theorem), which provides a sufficient theortical basis for initial choices of the Brent method but has not been proved for nearly twenty years.

2. Equivalent Variation of the Brent Method

In this paper, we take the Frobenius norm as matrix norm and the Euclidean norm as vector norm.

Denote

$$\mathcal{F}(y_1^{(k)},\cdots,y_n^{(k)})=\begin{pmatrix} f_1(y_1^{(k)})\\ \vdots\\ f_n(y_n^{(k)})\end{pmatrix},$$

$$a_j^{(k)}=\frac{1}{h_k}\begin{pmatrix} f_j(y_j^{(k)}+h_kQ_j^{(k)}e_1)-f_j(y_j^{(k)})\\ \vdots\\ f_j(y_j^{(k)}+h_kQ_j^{(k)}e_n)-f_j(y_j^{(k)})\end{pmatrix},\quad j=1,2,\cdots,n,$$

$$B_k=\Big[\frac{1}{\sigma_1^{(k)}}Q_2^{(k)}e_1,\cdots,\frac{1}{\sigma_n^{(k)}}Q_{n+1}^{(k)}e_n\Big]\in L(R^n),$$

$$A_k{}^T=\Big[Q_1^{(k)}a_1^{(k)},\cdots,Q_n^{(k)}a_n^{(k)}\Big]\in L(R^n).$$

Using the above notation, we can equivalently express Steps 1–4 of the Brent method in **1** as

$$\begin{cases} x^{(k+1)}=x^{(k)}-B_k\mathcal{F}(y_1^{(k)},\cdots,y_n^{(k)}),\quad y_1^{(k)}=x^{(k)},\\ k=0,1,2,\cdots.\end{cases}\tag{2.1}$$

Suppose that $F'(x^{(k)})^{-1}$ and A_k^{-1} exist for all k. Then, we can further express (2.1) in the following equivalent form:

$$\begin{cases} x^{(k+1)}=x^{(k)}-F'(x^{(k)})^{-1}F(x^{(k)})+R_k,\\ k=0,1,2,\cdots,\end{cases}\tag{2.2}$$

where

$$R_k=W_k+A_k^{-1}V_k,\tag{2.3}$$

$$W_k=[F'(x^{(k)})^{-1}-A_k^{-1}]F(x^{(k)}),\tag{2.4}$$

$$V_k=F(x^{(k)})-A_kB_k\mathcal{F}(y_1^{(k)},\cdots,y_n^{(k)}).\tag{2.5}$$

From (2.2), we know that the Brent method is equivalent to an approximate Newton method with an error term.

Now, we need to estimate the error term R_k.

First of all, by the structure of the orthogonal matrix $U_j^{(k)}$ and the definition of matrices A_k and B_k, we can prove the following lemmas.

Lemma 2.1. *The multiplication of A_k and B_k is a lower triangular matrix, that is,*

$$A_k B_k = \begin{bmatrix} 1 & & & \\ & 1 & & \\ l_{ij}^{(k)} & & \ddots & \\ & & & 1 \end{bmatrix},$$

where

$$l_{ij}^{(k)} = \frac{f_i(y_i^{(k)} + h_k Q_i^{(k)} e_j) - f_i(y_i^{(k)})}{\sigma_j^{(k)} h_k}, \qquad i > j.$$

Proof. It is easy to see from the definition of $U_j^{(k)}$ that

$$U_i^{(k)T} e_j = e_j, \quad i > j. \tag{2.6}$$

For $i < j$, we have

$$l_{ij}^{(k)} = a_i^{(k)T} Q_i^{(k)T} \frac{1}{\sigma_j^{(k)}} Q_{j+1}^{(k)} e_j = \frac{1}{\sigma_j^{(k)}} a_i^{(k)T} Q_i^{(k)T} Q_i^{(k)} U_i^{(k)} \cdots U_j^{(k)} e_j \ (Q_i^{(k)T} Q_i^{(k)} = I)$$

$$= \frac{1}{\sigma_j^{(k)}} a_i^{(k)T} U_i^{(k)} U_{i+1}^{(k)} \cdots U_j^{(k)} e_j. \tag{2.7}$$

From (1.5) and (2.6), we have

$$a_i^{(k)T} U_i^{(k)} U_{i+1}^{(k)} \cdots U_j^{(k)} = \sigma_i^{(k)} e_i{}^T.$$

Thus, from (2.7) we obtain

$$l_{ij}^{(k)} = \frac{1}{\sigma_j^{(k)}} \sigma_i^{(k)} e_i{}^T e_j = 0, \qquad i < j.$$

For $i = j$, we have

$$l_{ii}^{(k)} = a_i^{(k)T} Q_i^{(k)T} \frac{1}{c} \sigma_i^{(k)} Q_{i+1}^{(k)} e_i = \frac{1}{\sigma_i^{(k)}} a_i^{(k)T} Q_i^{(k)T} Q_i^{(k)} U_i^{(k)} e_i = \frac{1}{\sigma_i^{(k)}} \sigma_i^{(k)} e_i{}^T e_i = 1.$$

For $i > j$, we have

$$l_{ij}^{(k)} = a_i^{(k)T} Q_i^{(k)T} \frac{1}{\sigma_j^{(k)}} Q_{j+1}^{(k)} e_j = \frac{1}{\sigma_j^{(k)}} a_i^{(k)T} U_{i-1}^{(k)\,T} \cdots U_{j+1}^{(k)\,T} Q_{j+1}^{(k)\,T} Q_{j+1}^{(k)} e_j (Q_{j+1}^{(k)\,T} Q_{j+1}^{(k)} = I)$$

$$= \frac{1}{\sigma_j^{(k)}} a_i^{(k)T} U_{i-1}^{(k)\,T} \cdots U_{j+1}^{(k)\,T} e_j = \frac{1}{\sigma_j^{(k)}} a_i^{(k)T} e_j$$

$$= \frac{1}{\sigma_j^{(k)}} \cdot \frac{f_i(y_i^{(k)} + h_k Q_i^{(k)} e_j) - f_i(y^{(k)})}{h_k}, \qquad i > j, \quad Q_{j+1}^{(k)\,T} Q_{j+1}^{(k)} = I.$$

Therefore, the lemma is proved.

In order to estimate the error term R_k, we make the following assumption on function $F(x)$: (A) (1) $F : D \subset R^n \longrightarrow R^n$ is continuously differentiable in an open convex set $D_0 \subset D$. There exists $x^{(0)} \in D_0$ such that $F'(x^{(0)})^{-1}$ exists and $\|F'(x^{(0)})^{-1}\| = \beta_0$.

(2) There exists a constant $L > 0$ such that

$$\|F'(x) - F'(y)\| \le L\|x - y\|, \quad \forall x, y \in D_0. \tag{2.8}$$

Lemma 2.2. *Suppose that* (A) *holds. Then there exists a closed neighborhood* $\bar{S}_0(x^{(0)}, r_0) = \{x \in R^n \mid \|x - x^{(0)}\| \le r_0\} \subset D_0$ *and a constant* $\alpha > 0$, *such that for any* $z_1, \cdots, z_n \in S_0(x^{(0)}, r_0)$, *the matrix*

$$\mathcal{F}(z_1, \cdots, z_n) = \begin{bmatrix} \nabla f_1(z_1)^T \\ \vdots \\ \nabla f_n(z_n)^T \end{bmatrix}$$

is invertible and satisfies

$$\|\mathcal{F}'(z_1, \cdots, z_n)^{-1}\| \le \alpha, \tag{2.9}$$

where $\alpha = \beta_0/(1 - \sqrt{n}\beta_0 L r_0)$.

Proof. From (2.8), we have

$$\|\mathcal{F}(z_1, \cdots, z_n) - F'(x^{(0)})\| \le \sqrt{n}L\|z_j - x^{(0)}\| \le \sqrt{n}Lr_0.$$

By the Banach perturbation lemma, if we take $r_0 < 1/\sqrt{n}L\beta_0$, then the matrix $\mathcal{F}(z_1, \cdots, z_n)^{-1}$ exists and satisfies

$$\|\mathcal{F}'(z_1, \cdots, z_n)\| \le \beta_0/(1 - \sqrt{n}L\beta_0 r_0) = \alpha.$$

Therefore, the lemma is true.

Lemma 2.3. *Suppose that* (A) *holds. Then for any* $x \in S_0(x^{(0)}, r_0), F'(x)^{-1}$ *exists and satisfies*

$$\|F'(x)^{-1}\| \le \alpha. \tag{2.10}$$

Letting $z_1 = \cdots = z_n = x$ *in Lemma 2.2, we obtain Lemma 2.3.*

Lemma 2.4. *Suppose that* $F : D \subset R^n \longrightarrow R^n$ *is continuously differentiable in* $D_0 \subset D$, *and that* $F'(x)$ *satisfies* (2.8). *If* $x^{(k)}, y_j^{(k)}, y_j^{(k)} + h_k Q_j^{(k)} e_s \in D_0, j = 1, 2, \cdots, n; s = 1, 2, \cdots, n$, *then the following inequality holds:*

$$\|a_j^{(k)} - Q_j^{(k)T} \nabla f_j(x^{(k)})\| \le \sqrt{n}L(\|y_j^{(k)} - x^{(k)}\| + |h_k|). \tag{2.11}$$

Proof. Denote

$$Q_j^{(k)} = [(q_j^{(k)})_1, \cdots, (q_j^{(k)})_n], \quad (q_j^{(k)})_i \in R^n, \quad i = 1, 2, \cdots, n.$$

Then

$$
\begin{aligned}
&\|a_j^{(k)} - Q_j^{(k)^T}\nabla f_j(x^{(k)})\|^2 \\
&\quad = \sum_{s=1}^{n}\{\frac{1}{h_k}[f_j(y_j^{(k)} + h_k Q_j^{(k)} e_s) - f_j(y^{(k)})] - (q_j^{(k)})_s^T \nabla f_j(x^{(k)})\}^2 \\
&\quad = \sum_{s=1}^{n}[\nabla f_j(y_j^{(k)} + \theta_s h_k Q_j^{(k)} e_s)^T Q_j^{(k)} e_s - \nabla f_j(x^{(k)})^T Q_j^{(k)} e_s]^2 \\
&\quad \le \sum_{s=1}^{n}\|\nabla f_j(y_j^{(k)} + \theta_s h_k Q_j^{(k)} e_s) - \nabla f_j(x^{(k)})\|^2 \ \|Q_j^{(k)} e_s\|^2 \\
&\quad \le \sum_{s=1}^{n} L^2(\|y_j^{(k)} - x^{(k)}\| + |h_k|)^2 = nL^2(\|y_j^{(k)} - x^{(k)}\| + |h_k|)^2,
\end{aligned}
$$

where $\theta_s \in (0,1), \|Q_j^{(k)} e_s\| = \|(q_j^{(k)})_s\| = 1$. Therefore, (2.11) is direct from the above inequality.

Lemma 2.5. *Let $V = (v_{ij}) \in L(R^n)$ be a nonsingular and lower triangular matrix. Then for any i,*

$$|v_{ii}| \ge 1/\|V^{-1}\|, \quad i = 1, 2, \cdots, n.$$

The proof is very simple. We refer the readers to [1].

Lemma 2.6. *Suppose that (A) holds, and that there exist constants $\varepsilon > 0, \delta > 0$ satisfying*

$$0 < |h_k| < \delta, \qquad n^2 L\beta_0(n^{\frac{1}{2}(n-1)}\delta + \varepsilon) < 1.$$

Then there exists a constant $C_1 > 0$ such that for any j, whenever $y_i^{(k)} \in S_\varepsilon(x^{(0)}, \varepsilon) = \{x \in R^n \mid \ \|x - x^{(0)}\| \le \varepsilon\}, \quad i = 1, 2, \cdots, j$, the following estimate holds:

$$|\sigma_s^{(k)}| \ge C_1 > 0, \quad 1 \le s \le j.$$

Proof. Let

$$
M_j^{(k)} = \begin{bmatrix}
\sigma_1^{(k)} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & 0 & \vdots \\
\vdots & \ddots & \sigma^{(k)} & \cdots & \cdots \\
\vdots & & \cdots & \cdots & \cdots \\
\vdots & t_{pq}^{(k)} & & & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}, \qquad
E_p = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-p+1} \end{bmatrix},
$$

where I_{n-p+1} is an $(n-p+1) \times (n-p+1)$ unit matrix, and

$$t_{pq}^{(k)} = e_p^T F'(x^{(0)}) Q_{j+1}^{(k)} e_q.$$

Since for any i,

$$\begin{aligned}
&\left|\frac{1}{h_k}[f_p(y_p^{(k)} + h_k Q_p^{(k)} e_i) - f_p(y_p^{(k)})] - \nabla f_p(y_p^{(k)})^T Q_p^{(k)} e_i\right| \\
&= \frac{1}{|h_k|}|f_p(y_p^{(k)} + h_k Q_p^{(k)} e_i) - f_p(y_p^{(k)}) - \nabla f_p(y_p^{(k)})^T h_k Q_p^{(k)} e_i|, \\
&\le \frac{1}{|h_k|}\|\nabla f_p(y_p^{(k)} + \theta_i^{(k)} h_k Q_p^{(k)} e_i) - \nabla f_p(y_p^{(k)})\|\|h_k Q_p^{(k)} e_i\| \\
&\le \frac{1}{|h_k|} L|\theta_i^{(k)} h_k Q_p^{(k)} e_i\|\|h_K Q_p^{(k)} e_i\| \le \frac{L}{|h_k|}\|h_k Q_p^{(k)} e_i\|^2 = L|h_k|, \qquad (2.12)
\end{aligned}$$

where $\theta_i^{(k)} \in (0,1)$, from the definition of E_p, we have

$$x^T E_p = [0, \cdots, 0, x_p, \cdots, x_n]$$

for any $x = [x_1, \cdots, x_n] \in R^n$. Then we obtain

$$\|[\bar{a}_p^{(k)} - Q_p^{(k)^T} F'(y_p^{(k)}) e_p]^T E_p\|^2 \le n(L|h_k|)^2. \qquad (2.13)$$

Since

$$U_p^{(k)^T} \bar{a}_p^{(k)} = \sigma_p^{(k)} e_p^T, \quad U_i^{(k)} e_p = e_p, \qquad \forall i > p,$$

for any $j > p$,

$$U_j^{(k)^T} U_{j-1}^{(k)\,T} \cdots U_p^{(k)^T} \bar{a}_p^{(k)} = \sigma_p^{(k)} e_p.$$

It follows from the definition of $M_j^{(k)}$ that

$$(U_j^{(k)^T} U_{j-1}^{(k)\,T} \cdots U_p^{(k)^T} \bar{a}_p^{(k)})^T E_p = \sigma_p^{(k)} e_p^T E_p = e_p{}^T M_j^{(k)} E_p. \qquad (2.14)$$

By the definition of U_p and E_p, it is easy to see that

$$U_q E_p = E_p U_q, \quad \forall q \ge p. \qquad (2.15)$$

Thus, from (2.14) and (2.15), we have

$$\begin{aligned}
&\|e_p^T[M_j^{(k)} - F'(y_p^{(k)}) Q_{j+1}^{(k)}] E_p\|^2 = \|e_p^T M_j^{(k)} E_p - e_p^T F'(y_p^{(k)}) Q_{j+1}^{(k)} E_p\|^2 \\
&= \|\bar{a}_p^{(k)^T} U_p^{(k)} U_{p+1}^{(k)} \cdots U_j^{(k)} E_p - e_p^T F'(y_p^{(k)}) Q_p^{(k)} U_p^{(k)} U_{p+1}^{(k)} \cdots U_j^{(k)} E_p\|^2 \\
&= \|\bar{a}_p^{(k)^T} E_p U_p^{(k)} U_{p+1}^{(k)} \cdots U_j^{(k)} - e_p^T F'(y_p^{(k)}) Q_p^{(k)} E_p U_p^{(k)} U_{p+1}^{(k)} \cdots U_j^{(k)}\|^2 \\
&= \|[\bar{a}_p^{(k)^T} - e_p^T F'(y_p^{(k)}) Q_p^{(k)}] E_p U_p^{(k)} U_{p+1}^{(k)} \cdots U_j^{(k)}\|^2 \\
&\le \|[\bar{a}_p^{(k)^T} - e_p^T F'(y_p^{(k)}) Q_p^{(k)}] E_p\| \|U_p^{(k)} U_{p+1}^{(k)} \cdots U_j^{(k)}\|^2 \\
&\le n(L|h_k|)^2 n^n = n^{n+1} L^2 |h_k|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|e_p^T[M_j^{(k)} - F'(x^{(0)})Q_{j+1}^{(k)}]E_p\| \\
&\quad \le \|e_p^T[M_j^{(k)} - F'(y_p^{(k)})Q_{j+1}^{(k)}]E_p\| + \|e_p^T[F'(y_p^{(k)})Q_{j+1}^{(k)} - F'(x^{(0)})Q_{j+1}^{(K)}]E_p\| \\
&\quad \le n^{\frac{n+1}{2}}L|h_k| + \|F'(y_p^{(k)}) - F'(x^{(0)})\|\|Q_{j+1}^{(k)}\|\|E_p\| \\
&\quad \le n^{\frac{n+1}{2}}L|h_k| + nL\|y_p^{(k)} - x^{(0)}\| \le n^{\frac{n+1}{2}}L\delta + nL\varepsilon = nL(n^{\frac{n-1}{2}}\delta + \varepsilon). \qquad (2.16)
\end{aligned}$$

Indeed, the above inequality is an estimation of the last $n-p+1$ components of the p-th row of matrix $(M_j^{(k)} - F'(x^{(0)})Q_{j+1}^{(k)})$. By the definition of $M_j^{(k)}$, the first $p-1$ components of the p-th row of $(M_j^{(k)} - F'(x^{(0)})Q_{j+1}^{(k)})$ are zeroes. Thus, by (2.16), we have

$$\left\|e_p^T[M_j^{(k)} - F'(x^{(0)})Q_{j+1}^{(k)}]\right\| = \|e_p^T[M_j^{(k)} - F'(x^{(0)})Q_{j+1}^{(k)}]E_p\| \le nL(n^{\frac{n-1}{2}}\delta + \varepsilon) \quad (2.17)$$

for any $p \le j$. Since the last $n-j$ rows of $M_j^{(k)}$ and $F'(x^{(0)})Q_{j+1}^{(k)}$ are identical, by (2.17) we have

$$\|M_j^{(k)} - F'(x^{(0)})Q_{j+1}^{(k)}\| \le \sqrt{j}nL(n^{\frac{n-1}{2}}\delta + \varepsilon) \le n^{\frac{3}{2}}L(n^{\frac{n-1}{2}}\delta + \varepsilon).$$

It follows that

$$\begin{aligned}
&\|[F'(x^{(0)})Q_{j+1}^{(k)}]^{-1}\|\|M_j^{(k)} - F'(x^{(0)})Q_{j+1}^{(k)}\| \\
&\quad \le \|{Q_{j+1}^{(k)}}^{-1}\|\|F'(x^{(0)})^{-1}\|\|M_j^{(k)} - F'(x^{(0)})Q_{j+1}^{(k)}\| \\
&\quad \le \sqrt{n}\beta_0 n^{\frac{3}{2}}L(n^{\frac{n-1}{2}}\delta + \varepsilon) = n^2\beta_0 L(n^{\frac{n-1}{2}}\delta + \varepsilon).
\end{aligned}$$

By the Banach perturbation lemma, $(M_j^{(k)})^{-1}$ exists, and

$$\|{M_j^{(k)}}^{-1}\| \le \frac{\sqrt{n}\beta_0}{1 - n^2L\beta_0(n^{\frac{n-1}{2}}\delta + \varepsilon)}.$$

From the special structure of $M_j^{(k)}$, we can find an orthogonal matrix H with the form

$$H = \begin{bmatrix} I_j & 0 \\ 0 & \bar{H} \end{bmatrix}$$

such that $M_j^{(k)}H$ is a lower triangular matrix. Then by Lemma 2.5, for any $1 \le s \le j$, we have

$$\begin{aligned}
|{\sigma_s^{(k)}}^{-1}| &> \frac{1}{\|(M_j^{(k)}H)^{-1}\|} \ge \frac{1}{(\|H^{-1}\|\|{M_j^{(k)}}^{-1}\|)} \ge \frac{1}{\sqrt{n}} \cdot \frac{1 - n^2L\beta_0(n^{\frac{n-1}{2}}\delta + \varepsilon)}{\sqrt{n}\beta_0} \\
&= \frac{1 - n^2L\beta_0(n^{\frac{n-1}{2}}\delta + \varepsilon)}{n\beta_0} = C_1 > 0.
\end{aligned}$$

Therefore, the lemma is proved.

From (2.8), it is easy to see that there exists a constant C_2 such that

$$\|F'(x)| \leq C_2, \quad x \in S_0(x^{(0)}, r_0). \tag{2.18}$$

Lemma 2.7. *Suppose that* (A) *holds, and that there exist constants* $0 < r_1 < \min\{r_0, \varepsilon\}, \eta > 0$, *and* $\delta > 0$ *such that*

$$\|F(x^{(k)})| = \eta_k \leq \eta, \quad 0 < |h_k| < \delta,$$

$$C_2^{-1}[(1 + C_2C_1^{-1})^n - 1]\eta + r_1 < \varepsilon, \quad n^2 L\beta_0(n^{\frac{n-1}{2}}\delta + \varepsilon) < 1.$$

If $x^{(k)} \in S_1(x^{(0)}, r_1) = \{x \in R^n \mid \|x - x^{(0)}\| \leq r_1\} \subset D_0$, *then for any* j, (i) $|f_j(y_j^{(k)})| \leq (1 + C_2C_1^{-1})^{j-1}\|F(x^{(k)}\|$, (ii) $\|y_{j+1}^{(k)} - y_1^{(k)}\| \leq C_2^{-1}[(1 + C_2C_1^{-1})^j - 1]\|F(x^{(k)})\|$, (iii) $y_{j+1}^{(k)} \in S_\varepsilon(x^{(0)}, \varepsilon)$.

Proof. By induction. For $j = 1$, since

$$\|y_1^{(k)} - x^{(0)}\| = \|x^{(k)} - x^{(0)}\| \leq r_1 < \varepsilon,$$

by Lemma 2.6, we have

$$|\sigma_1^{(k)}| \geq C_1 > 0.$$

(i) obviously holds. Since

$$\|y_2^{(k)} - y_1^{(k)}\| = \left\| - \frac{1}{\sigma_1^{(k)}} f_1(y_1^{(k)}) Q_1^{(k)} e_1 \right\| \leq \frac{1}{|\sigma_1^{(k)}|} |f_1(y_1^{(k)})| \|Q_1^{(k)} e_1\| \leq C_1^{-1}|F(x^{(k)})\|,$$

(ii) also holds. For (iii),

$$\|y_2^{(k)} - x^{(0)}\| \leq \|y_2^{(k)} - y_1^{(k)}\| + \|y_1^{(k)} - x^{(0)}\| \leq C_1^{-1}\|F(x^{(k)})\| + r_1 \leq C_1^{-1}\eta + r_1 < \varepsilon.$$

It implies that $y_2^{(k)} \in S_\varepsilon(x^{(0)}, \varepsilon)$.

Suppose that Lemma 2.7 holds for any $j \leq m$. By Lemma 2.6, we have

$$|\sigma_s^{(k)}| \leq C_1, \quad 1 \leq s \leq m + 1.$$

Then for $j = m + 1$, making use of (2.18), we have

$$\begin{aligned} |f_{m+1}(y_{m+1}^{(k)})| &\leq |f_{m+1}(y_{m+1}^{(k)}) - f_{m+1}(y_1^{(k)})| + |f_{m+1}(y_1^{(k)})| \leq C_2\|y_{m+1}^{(k)} - y_1^{(k)}\| \\ &+ \|F(y_1^{(k)})\| \leq C_2C_2^{-1}[(1 + C_2C_1^{-1})^m - 1]\|F(x^{(k)})\| + \|F(x^{(k)})\| \\ &\leq (1 + C_2C_1^{-1})^m\|F(x^{(k)}\|. \end{aligned}$$

For (ii), we have

$$\begin{aligned}\|y_{m+2}^{(k)} - y_1^{(k)}\| &\le \|y_{m+2}^{(k)} - y_{m+1}^{(k)}\| + \|y_{m+1}^{(k)} - y_1^{(k)}\| \\ &= \Big\| - \frac{1}{\sigma_{m+1}^{(k)}} f_{m+1}(y_{m+1}^{(k)}) Q_{m+1}^{(k)} e_{m+1}\Big\| + \|y_{m+1}^{(k)} - y_1^{(k)}\| \\ &= \Big|\frac{1}{\sigma_{m+1}^{(k)}}\Big| |f_{m+1}(y_{m+1}^{(k)})| \|Q_{m+1}^{(k)} e_{m+1}\| + \|y_{m+1}^{(k)} - y_1^{(k)}\| \\ &\le C_2^{-1}(1 + C_2C_1^{-1})^m \|F(x^{(k)})\| + C_2^{-1}[(1 + C_2C_1^{-1})^m - 1]\|F(x^{(k)})\| \\ &= C_2^{-1}[(1 + C_2C_1^{-1})^{m+1} - 1]\|F(x^{(k)})\|.\end{aligned}$$

For (iii),

$$\begin{aligned}\|y_{m+1}^{(k)} - x^0\| &\le \|y_{m+2}^{(k)} - y_1^{(k)}\| + \|y_1^{(k)} - x^0\| \le C_2^{-1}[(1 + c_2c_1^{-1})^{m+1} - 1]\|F(x^k)\| \\ &\quad + \|x^{(k)} - x^{(0)}\| \le c_2^{-1}[(1 + c_2c_1^{-1})^{m+1} - 1]\eta + r_1 < \varepsilon.\end{aligned}$$

It implies that $y_{m+2}^{(k)} \in S_\varepsilon(x^{(0)}, \varepsilon)$. Thus, (i)–(iii) also hold for $j = m+1$. Therefore, the lemma is proved.

Lemma 2.8. *Suppose that the assumptions of Lemma 2.7 hold, and that*

$$\gamma = n^{3/2}\alpha L C_2^{-1}\{[(1 + C_2C_1^{-1})^{n-1} - 1]\eta + C_2\delta\} < 1. \tag{2.19}$$

If $x^{(k)} \in S_1(x^{(0)}, r_1)$, *then*

(i) $\|A_k - F'(x^{(k)})\| \le n^{3/2} L C_2^{-1}\{[(1 + C_2C_1^{-1})^{n-1} - 1]\|F(x^{(k)}\| + C_2|h_k|\}$,

(ii) A_k *is invertible, and satisfies*

$$\|A_k^{-1}\| \le \alpha/(1 - \gamma).$$

Proof. (i) By Lemma 2.4, for any j, we have

$$\begin{aligned}\|a_j^{(k)} - Q_j^{(k)T}\nabla f_j(x^{(k)})\| &\le \sqrt{n}L(\|y_j^{(k)} - x^{(k)}\| + |h_k|) \\ &\le \sqrt{n}L\{C_2^{-1}[(1 + C_2C_1^{-1})^{j-1} - 1]\|F(x^{(k)})\| + |h_k|\}.\end{aligned}$$

It follows that

$$\begin{aligned}\|Q_j^{(k)}a_j^{(k)} - \nabla f_j(x^{(k)})\| &\le \|Q_j^{(k)}\|\|a_j^{(k)} - Q_j^{(k)T}\nabla f_j(x^{(k)})\| \\ &\le \sqrt{n}\cdot\sqrt{n}LC_2^{-1}\{[(1 + C_2C_1^{-1})^{n-1} - 1]\|F(x^{(k)})\| + C_2|h_k|\} \\ &= nLC_2^{-1}\{[(1 + C_2C_1^{-1})^{n-1} - 1]\|F(x^{(k)})\| + C_2|h_k|\}.\end{aligned}$$

Thus,

$$\|A_k - F'(x^{(k)})\| \le n^{3/2}LC_2^{-1}\{[(1 + C_2C_1^{-1})^{n-1} - 1]\|F(x^{(k)})\| + C_2|h_k|\}.$$

(ii) By (i) and (2.19), we have

$$\|F'(x^{(k)})^{-1}\|\|A_k - F'(x^{(k)})\| \le \alpha n^{3/2} L C_2^{-1}\{[(1+C_2C_1^{-1})^{n-1} - 1]\eta + C_2\delta\} = \gamma < 1.$$

Then by the Banach perturbation lemma, A_k^{-1} exists, and

$$\|A_k^{-1}\| \le \|F'(x^{(k)})\|/(1-\gamma) \le \alpha/(1-\gamma).$$

Therefore the lemma is true.

Theorem 2.1. *Suppose that the assumptions of Lemma 2.8 hold. If $x^{(k)} \in S_1(x^{(0)}, r_1)$, then there exists a constant $C_3 > 0$ such that*

$$\|R_k\| \le C_3\{[(1+C_2C_1^{-1}) - 1]\|F(x^{(k)})\| + C_2|h_k|\}\|F(x^{(k)})\|. \tag{2.20}$$

Proof. By (2.4), we have

$$\begin{aligned}\|W_k\| &\le \|A_k^{-1}\|\|A_k - F'(x^{(k)})\|\|F'(x^{(k)})^{-1}\|\|F(x^{(k)})\| \\ &\le \frac{\alpha^2}{1-\gamma}\cdot n^{3/2}LC_2^{-1}\{[(1+C_2C_1^{-1})^{n-1} - 1]\|F(x^{(k)})\| + C_2|h_k|\}\|F(x^{(k)})\| \\ &\le \frac{1}{1-\gamma}\cdot \alpha^2 n^{3/2}LC_2^{-1}\{[(1+C_2C_1^{-1})^{n-1} - 1]\|F(x^{(k)})\| + C_2|h_k|\}\|F(x^{(k)})\|.\end{aligned} \tag{2.21}$$

Denote $[V_k]_j$ as the j-th component of V_k. Then by (2.5) and Lemma 2.1, we have

$$\begin{aligned}[V_k]_j &= f_j(x^{(k)}) - [\sum_{t=1}^{j-1} l_{jt}^{(k)} f_t(y_t^{(k)}) + f_j(y_j^{(k)})] \\ &= f_j(y_1^{(k)}) - f_j(y_j^{(k)}) - \sum_{t=1}^{j-1}\frac{1}{\sigma_t^{(k)}}\cdot\frac{f_j(y_i^{(k)} + h_kQ_i^{(k)}e_t) - f_j(y_i^{(k)})}{h_k}\cdot f_t(y_t^{(k)}) \\ &= \nabla f_j(y_j^{(k)} + \theta_j^{(k)}(y_1^{(k)} - y_j^{(k)}))^T(y_1^{(k)} - y_j^{(k)}) \\ &\quad - \sum_{t=1}^{j-1}\frac{1}{\sigma_t^{(k)}}\nabla f_j(y_j^{(k)} + \xi_{jt}^{(k)}h_kQ_j^{(k)}e_t)^TQ_j^{(k)}e_tf_t(y_t^{(k)}) \\ &= \nabla f_j(y_j^{(k)} + \theta_j^{(k)}(y_1^{(k)} - y_j^{(k)}))^T\sum_{t=1}^{j-1}\frac{1}{\sigma_t^{(k)}}f_t(y_t^{(k)})Q_{t+1}^{(k)}e_t \\ &\quad - \sum_{t=1}^{j-1}\frac{1}{\sigma_t^{(k)}}\nabla f_j(y_j^{(k)} + \xi_{jt}^{(k)}h_kQ_j^{(k)}e_t)^TQ_j^{(k)}e_tf_t(y_t^{(k)}) \\ &= \sum_{t=1}^{j-1}\{\nabla f_j(y_j^{(k)} + \theta_j^{(k)}(y_1^{(k)} - y_j^{(k)}))^TQ_{t+1}^{(k)}e_t \\ &\quad - \nabla f_j(y_j^{(k)} + \xi_{jt}^{(k)}h_kQ_j^{(k)}e_t)^TQ_j^{(k)}e_t\}\frac{1}{\sigma_t^{(k)}}f_t(y_t^{(k)}),\end{aligned} \tag{2.22}$$

where $\theta_j^{(k)}, \xi_{jt}^{(k)} \in (0,1)$. Since for $t < j$,

$$Q_j^{(k)} e_t = Q_{t+1}^{(k)} U_{t+1}^{(k)} U_{t+2}^{(k)} \cdots U_{j-1}^{(k)} e_t = Q_{t+1}^{(k)} e_t,$$

then

$$\begin{aligned}|[V_k]_j| &\le \sum_{t=1}^{j-1} \|\nabla f_j(y_j^{(k)} + \theta_j^{(k)}(y_1^{(k)} - y_j^{(k)})) - \nabla f_j(y_j^{(k)} + \xi_{jt}^{(k)} h_k Q_j^{(k)} e_t)\| \\ &\quad \cdot \|Q_{t+1}^{(k)} e_t\| \Big| \frac{1}{\sigma_t^{(k)}} f_t(y_t^{(k)}) \Big| \le L \sum_{t=1}^{j-1} (\|y_1^{(k)} - y_j^{(k)}\| + |h_k| \|Q_j^{(k)} e_t\|) C_1^{-1} [(1 \\ &\quad + C_2 C_1^{-1})^{t-1} - 1] \|F(x^{(k)})\| \le L C_1^{-1} \{ C_2^{-1} [(1 + C_2 C_1^{-1})^{j-1} - 1] \|F(x^{(k)})\| \\ &\quad + |h_k| \} \sum_{t=1}^{j-1} [(1 + C_2 C_1^{-1})^{t-1} - 1] \|F(x^{(k)})\| \le L C_2^{-2} \{ [(1 + C_2 C_1^{-1})^{n-1} \\ &\quad - 1] \|F(x^{(k)})\| + C_2 |h_k| \} [(1 + C_2 C_1^{-1})^{n-1} - 1] \|F(x^{(k)})\|.\end{aligned}$$

It follows that

$$\|V_k\| \le \sqrt{n} L C_2^{-2} \{ [(1+C_2 C_1^{-1})^{n-1} - 1] \|F(x^{(k)})\| + C_2 |h_k| \} [(1+C_2 C_1^{-1})^{n-1} - 1] \|F(x^{(k)})\|. \tag{2.23}$$

Combining (2.3), (2.21) and (2.23), we obtain

$$\begin{aligned}\|R_k\| &\le \|W_k\| + \|A_k^{-1}\| \|V_k\| \le \{ \frac{\alpha^2}{1-\gamma} n^{3/2} L C_2^{-1} + n^{1/2} L C_2^{-2} [(1 + C_2 C_1^{-1})^{n-1} - 1] \} \\ &\quad \cdot \{ [(1 + C_2 C_1^{-1})^{n-1} - 1] \|F(x^{(k)})\| + C_2 |h_k| \} \|F(x^{(k)})\|.\end{aligned}$$

Let

$$C_3 = \frac{\alpha^2}{1-\gamma} n^{3/2} L C_2^{-1} + n^{1/2} L C_2^{-2} [(1 + C_2 C_1^{-1}) - 1].$$

We can see that (2.20) holds, which completes the proof.

The above discussion implies that the Brent method is an inexact Newton method with an error term satisfying (2.20).

3. The Semi-Local Convergence Theorem

Since the Brent method was presented, the semi-local convergence theorem of the method has not been proved. Here, by making use of the equivalent variation of the Brent method derived in the last section, we establish the semi-local convergence theorem of the method, which is a complement of the convergence theory of the Brent method and provides a theoretical basis for initial choices of the method.

First of all, we introduce the following notations:

$$C_4 = \{C_5 n^{3/2}\alpha L C_2^{-1} + n^{1/2} L C_2^{-2}[(1+C_2C_1^{-1})^{n-1} - 1]\},$$
$$C_5 = \max\{\frac{\alpha}{1-\gamma}, 1\}. \quad C_6 = \max\{C_2, 1\}.$$

It is easy to see that $C_3 \le C_4$.

Lemma 3.1. *Suppose that the assumptions of Lemma* 2.8 *hold, and that*

$$C_6C_4\{[(1+C_2C_1^{-1})^{n-1} - 1]\eta + C_2\delta\} < 1. \tag{3.1}$$

If $x^{(k)} \in S_1(x^{(0)}, r_1) \subset D$, *then*

$$\|F(x^{(k)})\| \le \frac{C_2}{1 - C_2C_3\{[(1+C_2C_1^{-1})^{n-1} - 1]\eta + C_2\delta\}}\|x^{(k+1)} - x^{(k)}\|. \tag{3.2}$$

Proof. By the definition of C_4, C_5 and C_6 together with (3.1), it is easy to verify that (2.19) holds. Thus, the assumptions of Theorem 2.1 hold. From (2.2), we have

$$F(x^{(k)}) = F'(x^{(k)})(x^{(k)} - x^{(k+1)}) + F'(x^{(k)})R_k.$$

It follows that

$$\|F(x^{(k)})\| \le \|F'(x^{(k)})\|\|x^{(k)} - x^{(k+1)}\| + \|F'(x^{(k)})\|\|R_k\| \le C_2\|x^{(k+1)} - x^{(k)}\|$$
$$+ C_2C_3\{[(1+C_2C_1)^{n-1} - 1]\eta + C_2\delta\}\|F(x^{(k)})\|.$$

Also by the definition of C_4, C_5 and C_3 together with (3.1), we obtain

$$C_2C_3\{[(1+C_2C_1^{-1})^{n-1} - 1]\eta + C_2\delta\} \le C_6C_4\{[(1+C_2C_1^{-1})^{n-1} - 1]\eta + C_2\delta\} < 1.$$

Therefore, we have

$$\|F(x^{(k)})\| \le \frac{C_2}{1 - C_2C_3\{[(1+C_2C_1^{-1})^{n-1} - 1]\eta + C_2\delta\}}\|x^{(k+1)} - x^{(k)}\|.$$

The lemma is true.

We can easily prove the following lemma by simple calculation.

Lemma 3.2. *Let* $g(t) = \frac{\tau}{2(1-\tau)} + t + \frac{2-\tau}{1-\tau}\cdot\frac{t}{1-t}$, $0 < t < 1, 0 < \tau < 1$. *If* $0 < 1 < t < 1/4(1-\tau)$, *then* $g(t) < 1 - \tau$.

Consider $x^{(k)} \in S_1(x^{(0)}, r_1)$.*If* $F'(x^{(k)})^{-1}$ *exists, we assume*

$$\|F'(x^{(k)})^{-1}\| \le \beta_k.$$

Let

$$\Delta_k = x^{(k+1)} - x^{(k)}, \quad \zeta_k = \beta_k L\|\Delta_k\|, \quad C_7 = \max\{4C_2, 1\}.$$

Theorem 3.1 (Semi-local convergence theorem). *Suppose that* (A) *holds, and that*

$$0 < |h_k| < \delta, \qquad k = 0, 1, 2, \cdots, \tag{3.3}$$

$$\zeta_0 = \beta_0 L \|\Delta_0\| \le \tau < 1/2, \tag{3.4}$$

$$n^2 L \beta_0 (n^{\frac{n-1}{2}} \delta + \varepsilon) < 1, \tag{3.5}$$

$$\eta_0 \le \min\{\frac{(\varepsilon - r_1) C_2}{[(1 + C_2 C_1^{-1})^{n-1} - 1]}, \frac{\tau r_1}{\beta_0 + 1 - 2\tau}, \frac{4\bar{\gamma} - 1}{2L(\alpha + 1)^2}\}, \quad 1/4 < \bar{\gamma} < 1, \tag{3.6}$$

$$C_7 C_4 \{[(1 + C_2 C_1^{-1})^{n-1} - 1]\eta_0 + C_2 \delta\} < 1 - 2\tau. \tag{3.7}$$

Then, starting from $x^{(0)}$,the sequence $\{x^{(k)}\}$ generated by the Brent method remains in $S_1(x^{(0)}, r_1)$, and converges to the unique solution x^* of (1.1) *in $\bar{S}_1(x^{(0)}, r_1)$.*

In order to prove Theorem 3.1, we establish the following lemmas.

Denote

$$\Gamma = [(1 + C_2 C_1^{-1})^{n-1} - 1]\eta_0 + C_2 \delta.$$

Lemma 3.3. *Suppose that the assumptions of Theorem 3.1 hold. Then for any k, if $x^{(1)}, x^{(2)}, \cdots, x^{(k)} \in S_1(x^{(0)}, r_1)$, the following inequality holds:*

$$\eta_s \le \bar{\gamma} \eta_{s-1}, \qquad s = 1, 2, \cdots, k.$$

Proof. By induction. For $k = 0$, it is easy to verify that, under the assumptions of Theorem 3.1, the assumptions of Theorem 2.1 hold. However, at this moment, η is replaced by η_0. Thus, from Theorem 2.1, we have

$$F(x^{(1)}) = F(x^{(1)}) - F(x^{(0)}) - F'(x^{(0)})(x^{(1)} - x^{(0)}) + F'(x^{(0)}) R_0.$$

It follows that

$$\begin{aligned} \eta_1 = \|F(x^{(1)})\| &\le (1/2) L \|x^{(1)} - x^{(0)}\|^2 + C_2 \|R_0\| \le (1/2) L \| - F'(x^{(0)})^{-1} F(x^{(0)}) \\ &+ R_0\|^2 + C_2 \|R_0\| = (1/2) L (\|F'(x^{(0)})^{-1}\| \|F(x^{(0)})\| + \|R_0\|)^2 + C_2 \|R_0\| \\ &= (1/2) L (\alpha \|F(x^{(0)})\| + C_3 \Gamma \|F(x^{(0)})\|)^2 + C_2 C_3 \Gamma \|F(x^{(0)})\|. \end{aligned} \tag{3.8}$$

By (3.7), we have

$$C_3 \Gamma \le C_4 \Gamma \le C_7 C_4 \Gamma < 1 - 2\tau < 1, \quad C_2 C_3 \Gamma \le \frac{1}{4} C_7 C_4 \Gamma < \frac{1}{4}(1 - 2\tau) < \frac{1}{4}.$$

Hence, from (3.8) and (3.7), we obtain

$$\eta_1 \le \Big[\frac{1}{2} L (\alpha + 1)^2 \eta_0 + \frac{1}{4}\Big] \eta_0 \le \Big[\frac{1}{2} L (\alpha + 1)^2 \cdot \frac{4\bar{\gamma} - 1}{2L(\alpha + 1)^2} + \frac{1}{4}\Big] \eta_0 = \bar{\gamma} \eta_0.$$

Suppose that Lemma 3.3 also holds for any $k \le m$. We can easily prove that the assumptions of Theorem 2.1 hold when η is replaced by $\eta_1, \eta_2, \cdots, \eta_m$. Then, by the same argument as in the case $k = 0$, we can prove

$$\eta_{m+1} \le \bar{\gamma} \eta_m.$$

Therefore, the lemma is proved by induction.

Lemma 3.4. *Suppose that the assumptions of Theorem 3.1 hold. Then, for any k, if $x^{(1)}, x^{(2)}, \cdots, x^{(k)}, x^{(k+1)} \in S_1(x^{(0)}, r_1)$ and $\theta_k \in \tau$, the following inequality holds:*

$$\|\Delta_{k+1}\| < (1-\tau)\|\Delta_k\|. \tag{3.9}$$

Proof. Since $x_k, x_{k+1} \in S_1(x^{(0)}, r_1)$, then $F'(x^{(k)})^{-1}, F'(x^{(k+1)})^{-1}$ exist. Let

$$G_k = F'(x^{(k)})^{-1}[F'(x^{(k)}) - F(x^{(k+1)})].$$

Since

$$\|G_k\| \le \|F'(x^{(k)})^{-1}\|\|F'(x^{(k)}) - F'(x^{(k+1)})\| \le \beta_k L\|x^{(k)} - x^{(k+1)}\| = \beta_k L\|\Delta\| = \zeta_k,$$

by the Neumann lemma, we obtain

$$\|(I - G_k)^{-1}\| \le 1/(1-\zeta_k).$$

Let $\beta_{k+1} = \beta_k/(1-\zeta_k)$. From the identity

$$F'(x^{(k+1)})^{-1} = (I - G_k)^{-1}F'(x^{(k)})^{-1},$$

we have

$$\|F'(x^{(k+1)})^{-1}\| \le \|(I-G_k)^{-1}\|\|F'(x^{(k)})^{-1}\| \le \beta_k/(1-\zeta_k) = \beta_{k+1}.$$

By Lemma 3.3, if $x^{(1)}, x^{(2)}, \cdots, x^{(k)}, x^{(k+1)} \in S_1(x^{(0)}, r_1)$, then

$$\eta_s \le \eta_0, \quad s = 1, 2, \cdots, k+1.$$

Thus, the assumption of Theorem 2.1 and Lemma 3.1 hold for $s = 1, 2, \cdots, k, k+1$. Then, we have

$$F(x^{(k+1)}) = F(x^{(k+1)}) - F(x^{(k)}) - F'(x^{(k)})(x^{(k+1)} - x^{(k)}) + F'(x^{(k)})R_k.$$

It follows that

$$\begin{aligned}\Delta_{k+1} &= -F'(x^{(k+1)})^{-1}F(x^{(k+1)}) + R_{k+1} = -(I-G_k)^{-1}F'(x^{(k)})^{-1}[F(x^{(k+1)}) \\ &\quad - F(x^{(k)}) - F'(x^{(k)})(x^{(k+1)} - x^{(k)}) + F'(x^{(k)})R_k] + R_{k+1} \\ &= -(I-G_k)^{-1}\{F'(x^{(k)})^{-1}[F(x^{(k+1)}) - F(x^{(k)}) - F'(x^{(k)}) \\ &\quad \cdot (x^{(k+1)} - x^{(k)}) + R_k\} + R_{k+1}.\end{aligned} \tag{3.10}$$

Since

$$\begin{aligned}\|R_{k+1}\| &\le C_3\Gamma\|F(x^{(k+1)})\| \le C_3\Gamma(\|F(x^{(k+1)}) - F(x^{(k)})\| + \|F(x^{(k)})\|) \\ &\le C_2C_3\Gamma\|x^{(k+1)} - x^{(k)}\| + C_3\Gamma\|F(x^{(k)})\|,\end{aligned}$$

then

$$\|\Delta_{k+1}\| \le \frac{1}{1-\zeta_k}[\|F'(x^{(k)})^{-1}\|\|F(x^{(k+1)}) - F(x^{(k)}) - F'(x^{(k)})(x^{(k+1)} - x^{(k)})\| + \|R_k\|]$$

$$+ \|R_{k+1}\| \le \frac{1}{1-\zeta_k}[\beta_k(1/2)L\|x^{(k+1)} - x^{(k)}\|^2 + C_3\Gamma\|F(x^{(k)})\|]$$

$$+ C_2C_3\Gamma\|\Delta_k\| + C_3\Gamma\|F(x^{(k)})\| \le \frac{1}{1-\zeta_k}\left[(1/2)\zeta_k\|\Delta_k\| + C_3\Gamma\frac{C_2}{1-C_2C_3\Gamma}\|\Delta_k\|\right]$$

$$+ C_2C_3\Gamma\|\Delta\| + C_3\Gamma\frac{C_2}{1-C_2C_3\Gamma}\|\Delta_k\| = [\frac{\zeta_k}{2(1-\zeta_k)} + \frac{2-\zeta_k}{1-\zeta_k}\cdot\frac{C_2C_3\Gamma}{1-C_2C_3\Gamma}$$

$$+ C_2C_3\Gamma]\|\Delta_k\| \le [\frac{\tau}{2(1-\tau)} + C_2C_3\Gamma + \frac{2-\tau}{1-\tau}\cdot\frac{C_2C_3\Gamma}{1-C_2C_3\Gamma}]\|\Delta_k\|. \tag{3.11}$$

Since

$$C_2C_3\Gamma \le (1/4)C_7C_4\Gamma < (1/4)(1-2\tau),$$

by Lemma 3.2, we obtain

$$\|\Delta_{k+1}\| < (1-\tau)\|\Delta_k\|.$$

Therefore the lemma is true.

Now we prove Theorem 3.1.

Proof. First,we prove by induction that for any k, (i) $\zeta_k \le \tau$, (ii) $x^{(k+1)} \in S_1(x^{(0)}, r_1)$.

For $k = 0$, (i) obviously holds. Since

$$\|\Delta_0\| = \| - F'(x^{(0)})^{-1}F(x^{(0)}) + R_0\| \le \|F'(x^{(0)})^{-1}\|\|F(x^{(0)})\| + C_3\Gamma\|F(x^{(0)})\|$$

$$\le \beta_0\eta_0 + C_3\Gamma\|F(x^{(0)})\| < \beta_0\eta_0 + C_7C_4\Gamma\eta_0 \le (\beta_0 + 1 - 2\tau)\eta_0 < \tau r_1$$

$$< r_1, \tag{3.12}$$

(ii) holds for $k = 0$.

Suppose that (i) and (ii) hold for $k \le m$. Then for $k = m+1$, by the inductive assumptions and Lemma 3.4, we have

$$\|\Delta_{s+1}\| < (1-\tau)\|\Delta\|, \quad 1 \le s \le m. \tag{3.13}$$

It follows that

$$\zeta_{m+1} = \beta_{m+1}L\|\Delta_{m+1}\| \le \frac{\beta_m}{1-\zeta_m}L(1-\tau)\|\Delta_m\| \le \beta_m L\|\Delta_m\| = \zeta_m \le \tau.$$

Moreover, from (3.13) and (3.12), we have

$$\|x^{(m+1)} - x^{(0)}\| \le \sum_{s=0}^{m+1}\|x^{(s+1)} - x^{(s)}\| = \sum_{s=0}^{m+1}\|\Delta_s\| < \sum_{s=0}^{m+1}(1-\tau)^s\|\Delta_0\|$$

$$< \sum_{s=0}^{\infty}(1-\tau)^s\|\Delta_0\| = \frac{1}{\tau}\|\Delta_0\| \le \frac{1}{\tau}\tau r_1 = r_1. \tag{3.14}$$

It implies that $x^{m+2} \in S_1(x^{(0)}, r_1)$. Thus, (i) and (ii) hold for any k. Then, by Lemma 3.4, we have

$$\|\Delta_{k+1}\| < (1-\tau)\|\Delta_k\|, \quad \forall k.$$

Hence,

$$\|\Delta_k\| \le (1-\tau)\|\Delta_{k-1}\| \le \cdots \le (1-\tau)^k\|\Delta_0\|.$$

It follows that $\|\Delta_k\| \to 0$ as $k \to \infty$, which implies that $x^{(k)}$ is a Cauchy sequence. Therefore, $\{x^{(k)}\}$ is convergent.

Suppose that $x^{(k)} \to x^*$. Letting $m \to \infty$ on both sides of (3.14), and making use of (3.12), we have

$$\|x^* - x^{(0)}\| \le \sum_{s=0}^{m} \|\Delta_s\| < \sum_{s=0}^{m}(1-\tau)^s\|\Delta_0\| = \frac{1}{\tau}\|\Delta_0\| \le r_1.$$

It implies that $x^* \in S_1(x^{(0)}, r_1)$.

From Lemma 3.1, we have

$$\|F(x^{(k)})\| \le \frac{C_2}{1 - C_2C_5\Gamma}\|\Delta_k\|.$$

Letting $k \longrightarrow \infty$, we have

$$\lim_{k\to\infty} \|F(x^{(k)})\| = 0.$$

By the continuity of $F(x)$, $F(x^*) = 0$, that is, x^* is a solution of (1.1).

Suppose that there exists another solution $\tilde{x} \in \bar{S}_1(x^{(0)}, r_1)$. By the mean value theorem, we have

$$f_j(x^{(k)}) = f_j(x^{(k)}) - f_j(\tilde{x}) = \nabla f_j(x^{(k)} + \theta_j^{(k)}(\tilde{x} - x^{(k)}))^T(x^{(k)} - \tilde{x}), \quad \forall j,$$

where $\theta_j^{(k)} \in (0,1)$. It follows that

$$F(x^{(k)}) = \begin{bmatrix} \nabla f_1(x^{(k)} + \theta_1^{(k)}(\tilde{x} - x^{(k)}))^T \\ \vdots \\ \nabla f_n(x^{(k)} + \theta_n^{(k)}(\tilde{x} - x^{(k)}))^T \end{bmatrix} (x^{(k)} - \tilde{x}). \tag{3.15}$$

Since $x^{(k)} \in S_1(x^{(0)}, r_1) \subset S_0(x^{(0)}, r_0)$, then $x^{(k)} + \theta_j^{(k)}(\tilde{x} - x^{(k)}) \in S_1(x^{(0)}, r_0), \forall j$. Thus, by Lemma 2.2,

$$\begin{aligned} B_k &= \begin{pmatrix} \nabla f_1(x^{(k)} + \theta_1^{(k)}(\tilde{x} - x^{(k)}))^T \\ \vdots \\ \nabla f_n(x^{(k)} + \theta_n^{(k)}(\tilde{x} - x^{(k)}))^T \end{pmatrix} \\ &= [\nabla f_1(x^{(k)} + \theta_1^{(k)}(\tilde{x} - x^{(k)})), \cdots, \nabla f_n(x^{(k)} + \theta_n^{(k)}(\tilde{x} - x^{(k)}))]^T \end{aligned}$$

is invertible, and satisfies $\|B_k^{-1}\| \le \alpha$. Hence, from (4.10), we have

$$x^{(k)} - \tilde{x} = B_k^{-1}F(x^{(k)}).$$

It follows that
$$\|x^{(k)} - \tilde{x}\| \le \|B_1^{-1}\|\|F(x^{(k)})\| \le \alpha\|F(x^{(k)})\|.$$
Letting $k \to \infty$, we obtain
$$\|x^* - \tilde{x}\| \le \lim_{k\to\infty} \alpha\|F(x^{(k)})\| = \alpha\|F(x^*)\| = 0.$$
It implies that $\tilde{x} = x^*$. Therefore, x^* is the unique solution of (1.1) in $\bar{S}_1(x^{(0)}, r_1)$.

4. Choices of the Step Length h_k

In the above discussion, we assume that the step length satisfies
$$0 < |h_k| < \delta, \quad \forall k.$$
Such an assumption brings certain convenience to the discussion of the semi-local convergence of the Brent method. But it does not reveal the real convergence rate of the method. In fact, the choice of the step length plays an important role in the convergence rate of the method. In this section, we discuss several choices of the step length for practical applications of the method.

Choice 1. Brent Step length [1]
$$h_k = \begin{cases} -\sigma_1^{(k-1)^{-1}} f_1(x^{(k)}), & \text{if } f_1(x^{(k)}) \ne 0, \\ \bar{\varepsilon}(\bar{\varepsilon} \text{ is sufficiently small}, & \text{otherwise} \end{cases}$$
where $\bar{\varepsilon}$ is the termination condition, that is, if $\|F(x^{(k)})\| < \bar{\varepsilon}$, the computation is terminated.

Choice 2. Self-adaptive step length
$$h_k = \begin{cases} O(f_1(x^{(k)})), & \text{if } f_1(x^{(k)}) \ne 0, \\ O(f_j(x^{(k)})), & \text{if } f_1(x^{(k)}) = \cdots = f_{j-1}(x^{(k)}) = 0, f_j(x^{(k)}) \ne 0. \end{cases}$$
If $f_1(x^{(k)}) = \cdots = f_n(x^{(k)}) = 0$, then $x^{(k)}$ is a solution of (1.1). If $f_1(x^{(k)}) = \cdots = f_{j-1}(x^{(k)}) = 0$, and $f_j(x^{(k)}) = 0$, then by the algorithm structure of the Brent method, we have $y_1^{(k)} = \cdots = y_j^{(k)} = x^{(k)}$. Thus, $f_j(y_j^{(k)}) = f_j(x^{(k)})$. Hence, it is easy to see that such a choice does not increase the amount of function evaluations. In practical computation, for a sufficiently small $\bar{\varepsilon} > 0$, if $\max\{|f_i(x^{(k)})|, i = 1, 2, \cdots, j\} < \bar{\varepsilon}$, and $|f_{j+1}(x^{(k)})| \ge \bar{\varepsilon}$, we take $h_k = O(|f_{j+1}(x^{(k)})|)$, and $y_1^{(k)} = \cdots = y_{j+1}^{(k)} = x^{(k)}$. It is of self-test property, and is superior over the Brent step length.

Choice 3. Steffenson step length
$$h_k = O(\|F(x^{(k)})\|).$$

Theorem 4.1. *Suppose that the assumptions of Theorem* 3.1 *hold. Then for choices* 1–3, *there exists a constant* $C_8 > 0$ *such that*
$$\|R_k\| \le C_8\|F(x^{(k)})\|^2.$$

Proof. For choice 1, if $f_1(x^{(k)}) \neq 0$, then

$$|h_k| \leq |\sigma_1^{(k-1)^{-1}} f_1(x^{(k)})| \leq C_1^{-1} \|F(x^{(k)})\|. \tag{4.1}$$

If $f_1(x^{(k)}) = 0$, since the computation is not terminated, then,

$$h_k = \bar{\varepsilon} \leq \|F(x^{(k)})\|. \tag{4.2}$$

For choices 2 and 3, it is easy to prove that there exists a constant $k > 0$ such that

$$|h_k| \leq K\|F(x^{(k)})\|. \tag{4.3}$$

Thus, for choices 1–3, we have

$$|h_k| \leq \max\{C_1^{-1}, K, 1\}\|F(x^{(k)})\|. \tag{4.4}$$

Hence, from (2.20), we obtain

$$\begin{aligned}\|R_k\| &\leq C_3\{[(1 + C_2C_1^{-1})^{n-1} - 1]\|F(x^{(k)})\| + C_2\max\{C_1^{-1}, K, 1\}\|F(x^{(k)})\|\}\|F(x^{(k)})\| \\ &= C_3[(1 + C_2C_1^{-1})^{n-1} - 1 + C_2\max\{C_1^{-1}, K, 1\}]\|F(x^{(k)})\|^2.\end{aligned}$$

Let $C_8 = C_3[(1 + C_2C_1^{-1})^{n-1} - 1 + C_2\max\{C_1^{-1}, K, 1\}]$. Therefore, the theorem is proved.

Based on the above estimation, we can prove the local convergence rate of the Brent method under the regional estimation conditions.

Theorem 4.2. *Suppose that the assumptions of Theorem* 3.1 *hold. Then, starting from* $x^{(0)}$, *the sequence* $x^{(k)}$ *generated by the Brent method converges quadratically to the unique solution* x^* *of* (1.1) *in* $\bar{S}_1(x^{(0)}, r_1)$.

Proof. From Theorem 3.1, we know that the sequence $x^{(k)}$ converges to the unique solution x^* of (1.1) in $\bar{S}_1(x^{(0)}, r_1)$.

From the iterative procedure,we have

$$\begin{aligned}x^{(k+1)} - x^* &= x^{(k)} - x^* - F'(x)^{-1}F(x^{(k)}) + R_k = -F'(x^{(k)})^{-1}[F(x^{(k)}) - F(x^*) \\ &\quad - F'(x^{(k)})(x^{(k)} - x^*)] + R_k = -F'(x^{(k)})^{-1}[F(x^{(k)}) - F(x^*) \\ &\quad - F'(x^{(k)})(x^{(k)} - x^*)] + F'(x^{(k)})^{-1}[F'(x^*) - F'(x^{(k)})](x^{(k)} - x^*) + R_k.\end{aligned}$$

Thus

$$\begin{aligned}\|x^{(k+1)} - x^*\| &\leq \|F'(x^{(k)})^{-1}\|\|F(x^{(k)}) - F(x^*) - F'(x^*)(x^{(k)} - x^*)\| \\ &\quad + \|F'(x^{(k)})^{-1}\|\|F'(x^*) - F(x^{(k)})\|\|x^{(k)} - x^*\| + \|R_k\| \\ &\leq \alpha(1/2)L\|x^{(k)} - x^*\|^2 + \alpha L\|x^{(k)} - x^*\|^2 + C_8\|F(x^{(k)})\|^2 \\ &\leq \frac{3}{2}\alpha L\|x^{(k)} - x^*\|^2 + C_8\|F(x^{(k)}) - F(x^*)\|^2 \\ &\leq \frac{3}{2}\alpha L\|x^{(k)} - x^*\|^2 + C_8C_2^2\|x^{(k)} - x^*\|^2 \leq (\frac{3}{2}\alpha L + C_8C_2^2)\|x^{(k)} - x^*\|^2.\end{aligned}$$

It implies that $x^{(k)}$ converges quadratically to x^*.

References

[1] R.P. Brent, Some efficient algorithms for solving systems of nonlinear equations, *SIAM J.Numer.Anal.*, 10 (1973), 327–344.

[2] K.M. Brown, A quadratically convergent newton-like method upon gaussian-elimination, *SIAM J.Numer.Anal.*, 6 (1969), 560–569.

[3] R.S. Dembo, S.C. Eisenstat and T. Steihaug, Inexact newton methods, *SIAM J.Number. Anal.*, 19 (1982), 400–408.

[4] J.E.Jr. Dennis, On the convergence of broyden's methods for nonlinear systems of equations, *Math. Comp.*, 25 (1971), 559–569.

[5] J.E.Jr. Dennis and R.B. Schnabel, Numerical Method for Unconstrained Optimization and Nonlinear Equations, Prentice Hall, Englewood Cliffs, New Jersey, 1983.

[6] D.M. Gay, Brown's Method and Some Generalizations,with Applications to Minimization Problems, Cornell Univ., Comput. Sci. Techn. Rep., 1975, 75–225.

[7] L.V. Kantorovich, On Newton's method for functional equations, *Dokl. Akad. SSSR,* 59 (1948), 1237–1240.

[8] G.J. Miel, Unified error analysis for Newton-type methods, *Numer. Math.*, 33 (1979), 391–396.

[9] J.J. More and M.Y. Cosnard, Numerical solution of nonlinear equations, *ACM Trans. Math. Software,* 5 (1979), 64–85.

[10] I. Moret, On the behavior of approximate newton methods, *Computing,* 37, (1986), 185–193.

[11] J.M. Ortega and W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, 1970.

[12] T.J. Ypma, Local convergence of inexact Newton methods, *SIAM J. Numer. Anal.*, 21 (1984), 583–590.