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REAL PIECEWISE ALGEBRAIC VARIETY *1)

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Abstract

We give definitions of real piecewise algebraic variety and its dimension. By using the techniques of real radical ideal, P-radical ideal, affine Hilbert polynomial, *Bernstein-net form* of polynomials on simplex, and decomposition of semi-algebraic set, etc., we deal with the dimension of the real piecewise algebraic variety and real Nullstellensatz in C^{μ} spline ring.

Key words: Dimension, Real piecewise algebraic variety, C^{μ} spline ring.

1. Introduction

Multivariate splines as piecewise polynomials have been studied intensively in the past 20 years, and have become a kind of fundamental tool for computational geometry, numerial analysis, approximation, and optimization, etc.([8]). The interpolation of scattered date by multivariate splines is an important topic in computational geometry. It is concerned with several practical areas such as CAD,CAM, CAE, and Image processing. However, the construction of explicit interpolation schemes (especially Lagrange interpolation schemes) for spline spaces on given partition leads to complex problems. In principle, to solve an interpolation problem, one has to deal with the properties of piecewise algebraic variety and piecewise algebraic curves. Piecewise algebraic variety is the set of common zeros of the multivariate splines. Therefore, a key problem on interpolation by multivariate splines is to study the piecewise algebraic variety and piecewise algebraic curves. Piecewise algebraic variety, as a generalization of algebraic variety, is a new and important concept in algebraic geometry and computational geometry, and has great significance in theory and application (cf [8]). Some fundamental properties of piecewise algebraic variety were given in ([8],[9]). A generalization of Bezout theorem of piecewise algebraic curves has been obtained in ([10]).

In this paper, we give definitions of real piecewise algebraic variety and its dimension. By applying the techniques of real radical ideal, P-radical ideal(P be a cone), affine Hilbert polynomial, *Bernstein-net form* of polynomials on simplex, decomposition of semi-algebraic set, etc, ([1],[3],[5],[6]), we deal with the dimension of real piecewise algebraic variety and real Nullstellensatz in C^{μ} spline ring.

2. Definitions and Preliminaries

Let R be the real number field. We define n-space over R, denoted by R^n, to be the set of all n-tuples of R. An element $x = (x_1, \ldots, x_n) \in R^n$ will be called a point. Denote by $R[x]$ the polynomial ring in n variable over R.

By finite hyperplane patches in R^n, we subdivide a simply connected basic closed semi-algebraic domain $D \subseteq R^n$ with dimension n into finite simply connected subdomains with dimensions n(cf[5]), and each of them is homeomorphic to a hypercube, which is called a

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cell. So we get a partition of the domain D. Denote by Δ the partition of the domain D which consists of all partition cells $\delta_1, \dots, \delta_T$ and their faces $S_1, \dots, S_E$. Where $S_1, \dots, S_E$ are algebraic hypersurface patches or algebraic varieties with dimensions $< n$. The face of each cell $\delta_i \in \Delta$ consists of finite partition faces. It is well known that every cell δ_i can be written as the intersection of a collection of affine halfspaces, namely,

$$\delta_i = \{x \in R^n |\ S_1^{(i)}(x) \geq 0, \dots, S_{q_i}^{(i)}(x) \geq 0, \quad S_\alpha^{(i)} \in R[x], \alpha = 1, \quad \dots, q_i\}, \qquad 1^*$$
$$i = 1, 2, \dots, T.$$

Denote by $P(\Delta)$ the collection of functions f on D such that for every cell δ_i the restriction of f on δ_i, $f|_{\delta_i}$, is a polynomial function. $f|_{\delta_i}$ refers also to a polynomial corresponding to f on cell δ_i if no confusion can arise, i.e., $f|_{\delta_i} \in R[x]$. It is obvious that

$$S^\mu(\Delta) = \{f |\ f \in C^\mu(D) \cap P(\Delta)\}, \quad \mu \geq 0$$

is a ring over R, which is called C^μ spline ring. It is clear $R[x] \subseteq S^\mu(\Delta)$. The degree of $f \in S^\mu(\Delta)$ denoted by $degf$ is the maximal degree of polynomials corresponding to f on all cells of Δ. We say that

$$S_m^\mu(\Delta) := \{f |\ degf \leq m, \quad f \in S^\mu(\Delta)\}$$

is a multivariate spline space with degree m and smoothness μ. $S_m^\mu(\Delta)$ is a finite dimensional linear vector space on $m, \mu \geq 0$.

Now we discuss real C^μ piecewise algebraic variety on a partition Δ of a domain D in R^n.

Definition 2.1. *Let Δ be a partition of a domain $D \subseteq R^n$, Denote by δ_i, $i = 1, \dots, T$, all the cells of Δ. If there exist $f_1, \dots, f_s \in S^\mu(\Delta)$, $\mu \geq 0$ such that*

$$\mathcal{Z} = \mathcal{Z}(f_1, \dots, f_s) = \{x \in D : \quad f_i(x) = 0, \quad i = 1, \dots, s\},$$

then $\mathcal{Z}$ is called a **real C^μ piecewise algebraic variety** *defined by $f_1 \dots, f_s$ on the partition Δ.*

Thus a real C^μ piecewise algebraic variety $\mathcal{Z} = \mathcal{Z}(f_1, \dots, f_s)$ is the set of all real solutions of the system of equations $f_1 = \dots = f_s = 0$, i.e., the set of all real common zeros of C^μ multivariate splines $f_1, \dots, f_s$.

Let J be an ideal of $S^\mu(\Delta)$. Denote by $\mathcal{Z}(J)$ the set

$$\mathcal{Z}(J) = \{x \in D : \quad f(x) = 0, \quad f \in J\}.$$

Since $S^\mu(\Delta)$ is a Nöther ring([8]), J has a finite set of generators $f_1, ..., f_s \in S^\mu(\Delta)$ such that

$$\mathcal{Z}(J) = \mathcal{Z}(f_1, \dots, f_s).$$

Thus $\mathcal{Z}(J)$ is a real C^μ piecewise algebraic variety on the partition Δ of the domain D.

If J is an ideal of $R[x]$, then

$$\mathcal{Z}(J) = \{x \in R^n | f(x) = 0, \quad f \in J\}$$

is a real algebraic variety.

Let $V \subset R^n$ be a real algebraic variety. Denote by $\mathcal{P}(V) = R[x_1, ..., x_n]/\mathcal{I}(V)$ the ring of polynomial functions on V, where

$$\mathcal{I}(V) = \{f(x) \in R[x] \ : f(a) = 0, \quad a \in V\}.$$

Denote by $dim(V)$ the dimension of V. $dim(V)$ is equal to the dimension of the ring $\mathcal{P}(V)$, i.e., the maximal length of chains of prime ideal of $\mathcal{P}(V)$([4],[5],[7]).

Let $\mathcal{Z}$ be a real C^μ piecewise algebraic variety. $clos_{zar}(\mathcal{Z} \cap \delta_i)$ denotes the Zariski closure of $\mathcal{Z} \cap \delta_i, i = 1, \dots, T$. If for some $\mathcal{Z} \cap \delta_i = \emptyset$, then $dim(clos_{zar}(\mathcal{Z} \cap \delta_i)) = -1$.(cf. [4],[7])

Definition 2.2. *Let Δ be a partition with T cells δ_i of a domain $D \subseteq R^n$. If $\mathcal{Z}$ is a real C^μ piecewise algebraic variety on the partition Δ, we define the* **dimension** *of $\mathcal{Z}$ (denoted by $dim\mathcal{Z}$) to be maximum dimension of the Zariski closure of $\mathcal{Z} \cap \delta_i, i = 1, \ldots, T$.*

Denote by $R[x]_{\leq s}$ the set of polynomials of total degree $\leq s$ in $R[x]$. Given an ideal $I \subset R[x]$, we let

$$I_{\leq s} = I \cap R[x]_{\leq s}$$

denote the set of polynomials in I of total degree $\leq s$.

Definition 2.3. ([1]) *Let I be an ideal in $R[x]$. The affine Hilbert polynomial of I is the function on the non-negative integer s defined by*

$$^{a}HP_I(s) = dimR[x]_{\leq s}/I_{\leq s} = dimR[x]_{\leq s} - dimI_{\leq s}.$$

Definition 2.4. ([5]). *Let A be a commutative ring, and I be an ideal of A. The real radical ideal of I is defined by*

$$\sqrt[R]{I} = \{a \in A | \exists m \in N, \exists b_1, \cdots, b_p \in A, a^{2m} + b_1^2 + \cdots + b_p^2 \in I\}.$$

A cone P of A is a subset of A satisfying the following properties:

$$a \in P, b \in P \rightarrow a + b \in P,$$

$$a \in P, b \in P \rightarrow ab \in P,$$

$$a \in A \rightarrow a^2 \in P.$$

Denote by $\sum A^2$ the set of sums of squares of elements of A, namely,

$$\sum A^2 = \{\sum_{i=1}^{n} a_i^2 | a_1, \ldots, a_n \in A, n \in N\}.$$

The set $\sum A^2$ is the smallest cone of A. Let P be a cone of A, and $(a_i)_{i\in\beta}$ a family of elements of A. Denote by $M_{(a_i)_{i\in\beta}}$ the multiplicative monoid generated by $(a_i)_{i\in\beta}$, i.e., the set of finite products of elements of $(a_i)_{i\in\beta}$. Denote by $P[(a_i)_{i\in\beta}]$ the smallest cone of A containing P and $(a_i)_{i\in\beta}$. We have

$$P_{[(a_i)_{i\in\beta}]} = \{p + \sum_{i=1}^{r} q_i b_i | \ \ p, q_1, \ldots, q_r \in P, \ \ b_1, \cdots, b_r \in M[(a_i)_{i\in\beta}]\}.$$

In particular, $\sum A^2[(a_i)_{i\in\beta}]$ is a cone generated by $(a_i)_{i\in\beta}$.

Let I be an ideal of the commutative ring A, and P a cone of A, The P-radical ideal of I is defined by ([5])

$$\sqrt[P]{I} = \{a \in A | \ \exists m \in N, \ \exists p \in P, a^{2m} + p \in I\}.$$

In the next section, P_i denotes a cone of $R[x]$ generated by $(S_\alpha^{(i)})_{\alpha\in\{1,\cdots,q_i\}}$ (1*) , i.e.,

$$P_i = \sum R^2[x][(S_\alpha^{(i)})_{\alpha\in\{1,\ldots,q_i\}}], \ \ i = 1, \cdots, T.$$

$MDCS(\mathcal{Z}(< LT(\sqrt[R]{I}) >))$ denotes the maximum dimension of a coordinate subspace in the algebraic variety $\mathcal{Z}(< LT(\sqrt[R]{I}) >)$, where $LT(\sqrt[R]{I})$ is the set of leading terms of elements of $\sqrt[R]{I}$, and $< LT(\sqrt[R]{I}) >$ is the ideal generated by the elements of $LT(\sqrt[R]{I})$.

3. Main Results

Lemma 3.1. *If I is an ideal in $R[x]$, $\mathcal{Z} = \mathcal{Z}(I)$ is an irreducible real algebraic variety, then*

$$dim\mathcal{Z} = deg^a HP_{<LT(\sqrt[R]{I})>} = MDCS(\mathcal{Z}(< LT(\sqrt[R]{I}) >)).$$

Proof. By real Nullstellensatz ([5]) $\mathcal{I}(\mathcal{Z}(I)) = \sqrt[R]{I}$ and, by hypothesis, $\sqrt[R]{I}$ is a prime ideal. Let $\mathcal{P}(\mathcal{Z}) = R[x]/\sqrt[R]{I}$ be the ring of polynomial functions on $\mathcal{Z}(I)$, then the ring of polynomial functions $\mathcal{P}(\mathcal{Z})$ is an integral domain. If $\mathcal{K}(\mathcal{Z})$ is the field of fraction of $\mathcal{P}(\mathcal{Z})$, then $dim\mathcal{Z}$ is equal to the transcendence degree of $\mathcal{K}(\mathcal{Z})$ over R. Hence we have

$$\begin{aligned} dim\mathcal{Z} &= deg^a HP_{\mathcal{I}(\mathcal{Z})} = deg^a HP_{\sqrt[R]{I}} \\ &= deg^a HP_{<LT(\sqrt[R]{I})>} \\ &= MDCS(\mathcal{Z}(< LT(\sqrt[R]{I}) >)). \end{aligned}$$

This proves Lemma 3.1.

Theorem 3.1. *Let $\mathcal{Z} = \mathcal{Z}(J)$ be a real C^μ piecewise algebraic variety on a partition Δ with T cells δ_i $(i = 1, 2, \ldots, T)$ of a domain D, where $J \subseteq S^\mu(\Delta)$ is an ideal generated by $f_1, \ldots, f_s$. For each $i \in \{1, \ldots, T\}$, suppose that $I_i \subseteq R[x]$ is an ideal generated by $f_1|_{\delta_i}, \ldots, f_s|_{\delta_i}$, P_i is a cone of $R[x]$ generated by $(S_\alpha^{(i)})_{\alpha\in\{1,\ldots,q_i\}}$, and $\sqrt[P_i]{I_i}$ is the P_i-radical ideal of I_i. $M^{(i)}$ denotes the multiplicative monoid generated by $(S_\alpha^{(i)})_{\alpha\in\{1,\ldots,q_i\}}$. Then*

I. *$\mathcal{Z} \cap \delta_i = \emptyset$ if and only if there exist $l \in P_i$, and $h \in I_i$ such that $l + h + 1 = 0$;*

II. *$\mathcal{Z} \cap In(\delta_i) = \emptyset$ if and only if there exist $l \in P_i$, $h \in I_i$, and $g \in M^{(i)}$ such that $l + h + g^2 = 0$, where $In(\delta_i)$ denotes the interior of a cell δ_i;*

III. *If $\mathcal{Z} \cap \delta_i \neq \emptyset$, then*

$$clos_{zar}(\mathcal{Z} \cap \delta_i) = \mathcal{Z}(\sqrt[P_i]{I_i}),$$

where $clos_{zar}(\mathcal{Z} \cap \delta_i), \mathcal{Z}(\sqrt[P_i]{I_i}) \subseteq R^n$ are the algebraic varieties.

Proof.

I. Apply Positivstellensatz ([5],[6]) to the set

$$\begin{aligned} \mathcal{Z} \cap \delta_i &= \{x \in R^n |\ S_\alpha^{(i)}(x) \geq 0, \alpha = 1, \ldots, q_i, f_1|_{\delta_i}(x) = \cdots = f_s|_{\delta_i}(x) = 0\ \} \\ &= \{x \in R^n | S_\alpha^{(i)}(x) \geq 0, \alpha = 1, \ldots, q_i,\ 1 \neq 0,\ f_1|_{\delta_i}(x) = \cdots = f_s|_{\delta_i}(x) = 0\ \}, \end{aligned}$$

obtaining $\mathcal{Z} \cap \delta_i = \emptyset$ iff

$$\exists l \in P_i, \text{ and } h \in I_i \text{ such that } l + h + 1 = 0.$$

II. Apply Positivstellensatz to the set $\mathcal{Z} \cap In(\delta_i)$, namely,

$$\{x \in R^n |\ S_\alpha^{(i)}(x) \geq 0,\ S_\alpha^{(i)}(x) \neq 0, \alpha = 1, \ldots, q_i, f_1|_{\delta_i}(x) = \cdots = f_s|_{\delta_i}(x) = 0\ \},$$

obtaining $\mathcal{Z} \cap In(\delta_i) = \emptyset$ iff

$$\exists l \in P_i,\ \ h \in I_i,\ \ and\ \ g \in M^i \text{ such that } l + h + g^2 = 0.$$

III. If $\mathcal{Z} \cap \delta_i \neq \emptyset$, then

$$\mathcal{I}(\mathcal{Z} \cap \delta_i) = \mathcal{I}(\delta_i \cap \mathcal{Z}(I_i)) = \{f \in R[x] | f(x) = 0, \forall x \in \delta_i \cap \mathcal{Z}(I_i)\}.$$

Since

$$\forall x \in \delta_i \cap \mathcal{Z}(I_i),\quad f(x) = 0$$

is equivalent to the following property(i):
the set

$$\{x \in R^n | S_\alpha^i \geq 0,\ \alpha = 1, \ldots, q_i,\ f(x) \neq 0,\ f_1|_{\delta_i}(x) = \cdots = f_s|_{\delta_i}(x) = 0\}$$

is empty.

By Positivstellensatz, the property (i) is equivalent to the following fact that

$$\exists m \in N, g \in P_i, \text{ such that } f^{2m} + g \in I_i$$
$$\Leftrightarrow f \in \sqrt[P_i]{I_i}.$$

Hence we have

$$\forall x \in \delta_i \cap \mathcal{Z}(I_i), \quad f(x) = 0 \quad \Leftrightarrow \quad f \in \sqrt[P_i]{I_i}.$$

Thus

$$\mathcal{I}(\mathcal{Z} \cap \delta_i) = \sqrt[P_i]{I_i},$$
$$clos_{zar}(\mathcal{Z} \cap \delta_i) = \mathcal{Z}(\mathcal{I}(\mathcal{Z} \cap \delta_i)) = Z(\sqrt[P_i]{I_i}).$$

This proves Theorem 3.1.

Denote

$$\Gamma = \{i \in \{1, \dots, T\} | \text{there are no } l \in P_i \text{ and } h \in I_i \text{ such that } l + h + 1 = 0\}.$$

We can get a generalization of real Nullstellensatz in C^μ spline ring from the proof of theorem 3.1 as follows

Theorem 3.2. *Let $\mathcal{Z} = \mathcal{Z}(J)$ be a real C^μ piecewise algebraic variety on a partition Δ with T cells δ_i $(i = 1, 2, \dots, T)$ of a domain D, where $J \subset S^\mu(\Delta)$ is an ideal generated by $f_1, f_2, \dots, f_s$. For each $i \in \{1, \dots, T\}$, suppose that $I_i \subseteq R[x]$ is an ideal generated by $f_1|_{\delta_i}, f_2|_{\delta_i}, \dots, f_s|_{\delta_i}$, P_i is a cone of $R[x]$ generated by $(S_\alpha^{(i)})_{\alpha \in \{1, \dots, q_i\}}$, and $\sqrt[P_i]{I_i}$ is the P_i-radical ideal of I_i. Then $f \in S_m^\mu(\Delta)$ vanishes on $\mathcal{Z}$ if and only if $f|_{\delta_i} \in \sqrt[P_i]{I_i}$, for any $i \in \Gamma$.*

Let D be a polyhedral domain in R^n, and every cell δ_i of Δ be a simplex in R^n. It is well known that for any given $f \in S_m^\mu(\Delta)$, the polynomials $p^{[i]} = f|_{\delta_i}, i = 1, \dots, T$, can be represented in *Bernstein-net form* on the simplex δ_i as follows

$$p^{[i]}(w_0, \dots, w_n) = \sum_{|\lambda| = m} b_\lambda^{[i]} \prod_{j=0}^{n} w_j^{\lambda_j},$$

where $b_\lambda^{[i]} = p_\lambda^{[i]} \dfrac{m!}{\lambda_0! \lambda_1! \cdots \lambda_n!}$, and $w = (w_0, \dots, w_n)$ is the barycentric coordinates of x on δ_i, $\lambda = (\lambda_0, \dots, \lambda_n)$, $|\lambda| = \sum_{j=0}^n \lambda_j$, $\lambda_i \in \{0, 1, \dots, m\}$.

Theorem 3.3. *Let $\mathcal{Z} = \mathcal{Z}(J)$ be a real C^μ piecewise algebraic variety on a partition Δ with T simplices δ_i $(i = 1, \dots, T)$ of a polyhedral D, where $J \subseteq S^\mu(\Delta)$ is an ideal generated by $f_1, \dots, f_s$. For each $v \in \{1, \dots, s\}$, suppose that the Bernstein-net form of f_v on the simplex δ_i is $p_v^{[i]}(w) = f_v|_{\delta_i} = \sum_{|\lambda| = m_v} b_{v\lambda}^{[i]} \prod_{j=0}^{n} w_j^{\lambda_j}$. $I_i \subseteq R[w]$ is an ideal generated by $p_1^{[i]}(w), \dots, p_s^{[i]}(w)$, $\prod_{j=0}^n w_j - 1$, $i = 1, \dots, T$. Then*

I. *$\mathcal{Z} \cap \delta_i = \emptyset$ if and only if there exists $h \in I_i$ which has the form*

$$h = 1 + t_0 + \sum_{k=1}^{r} t_k \prod_{j \in Q_k} w_j,$$

where t_k, $k = 0, \dots, r$, are sums of squares of polynomials in $R[w]$, $Q_k \subseteq \{0, \dots, n\}$;

II. *$\mathcal{Z} \cap In(\delta_i) = \emptyset$ if and only if there exists $h \in I_i$ which has the form*

$$h = (\prod_{j \in Q_0} w_j)^2 + t_0 + \sum_{k=1}^{r} t_k \prod_{j \in Q_k} w_j,$$

where $t_k,\ k=0,\dots,r$, *are sums of squares of polynomials in* $R[w]$, $Q_k \subseteq \{0,\dots,n\}$;

III. *Let* $\Gamma_0 = \{i \in \{1,\dots,T\} | \mathcal{Z} \cap \delta_i \neq \emptyset\}$. *Then* $f \in S_m^\mu(\Delta)$, *of which Bernstein-net form on* δ_i, $p^{[i]}(w)$, *vanishes on* $\mathcal{Z}$ *if and only if for any given* $i \in \Gamma_0$, *there exist* $h \in I_i$ *and* $l \in N$ *such that* $h + (p^{[i]})^{2l}$ *can be written in the form*

$$h + (p^{[i]})^{2l} = t_0 + \sum_{k=1}^{r} t_k \prod_{j \in Q_k} w_j,$$

where $t_k,\ k=0,\dots,r$, *are sums of squares of polynomials in* $R[w]$, $Q_k \subseteq \{0,\dots,n\}$.

In short, for any given $i \in \Gamma_0$, $p^{[i]}(w) \in \sqrt[P_i]{I_i}$, *where* P_i *is a cone of* $R[w]$ *generated by* $w_0,\dots,w_n$;

VI. *For any* $i \in \Gamma_0$, $clos_{zar}(\mathcal{Z} \cap \delta_i) = \mathcal{Z}(\sqrt[P_i]{I_i})$, *where* $clos_{zar}(\mathcal{Z} \cap \delta_i), \mathcal{Z}(\sqrt[P_i]{I_i}) \subseteq R^{n+1}$ *are algebraic varieties.*

Proof.

I. Apply Positivstellensatz to the set

$$\mathcal{Z} \cap \delta_i = \{w \in R^{n+1} |\ w_j \geq 0, j=0,\dots,n, p_1^{[i]}(w) = \ \cdots\ = p_s^{[i]}(w) = 0, \prod_{j=0}^{n} w_j - 1 = 0\ \}$$

$$= \{w \in R^{n+1} |\ w_j \geq 0, j=0,\dots,n, 1 \neq 0, p_1^{[i]}(w) = \ \cdots\ = p_s^{[i]}(w) = 0, \prod_{j=0}^{n} w_j - 1 = 0\ \},$$

obtaining $\mathcal{Z} \cap \delta_i = \emptyset$ if and only if there exists $h \in I_i$ which has the form

$$h = 1 + t_0 + \sum_{k=1}^{r} t_k \prod_{j \in Q_k} w_j,$$

where $t_k,\ k=0,\dots,r$, are sums of squares of polynomials in $R[w]$, $Q_k \subseteq \{0,\dots,n\}$.

II. Apply Positivstellensatz to the set $\mathcal{Z} \cap In(\delta_i)$, i.e.,

$$\{w \in R^{n+1} |\ w_j \geq 0, w_j \neq 0, j=0,\dots,n, p_1^{[i]}(w) = \ \cdots\ = p_s^{[i]}(w) = 0, \prod_{j=0}^{n} w_j - 1 = 0\ \},$$

we get that $\mathcal{Z} \cap In(\delta_i) = \emptyset$ if and only if there exists $h \in I_i$ which has the form

$$h = (\prod_{j \in Q_0} w_j)^2 + t_0 + \sum_{k=1}^{r} t_k \prod_{j \in Q_k} w_j,$$

where $t_k,\ k=0,\dots,r$, are sums of squares of polynomials in $R[w]$, $Q_k \subseteq \{0,\dots,n\}$.

III. For any $i \in \Gamma_0$, since

$$\forall w \in \delta_i \cap \mathcal{Z}(J), \quad p^{[i]}(w) = 0$$

is equivalent to the following fact that the set

$$\{w \in R^{n+1} |\ w_j \geq 0, j=0,\dots,n, p^{[i]}(w) \neq 0, p_1^{[i]}(w) = \ \cdots\ = p_s^{[i]}(w) = 0, \prod_{j=0}^{n} w_j - 1 = 0\ \}$$

is empty.

Hence f vanishes on $\mathcal{Z}$ if and only if for any $i \in \Gamma_0$, there exist $h \in I_i$, and $l \in N$ such that $h + (p^{[i]})^{2l}$ can be written in the form

$$h + (p^{[i]})^{2l} = t_0 + \sum_{k=1}^{r} t_k \prod_{j \in Q_k} w_j,$$

where $t_k,\ k = 0, \dots, r$, are sums of squares of polynomials in $R[w]$, $Q_k \subseteq \{0, \dots, n\}$. In short, $p^{[i]}(w) \in \sqrt[P_i]{I_i}$, where P_i is a cone of $R[w]$ generated by w_j, $j = 0, \dots, n$.

VI. Obviously, we can get immediately the conclusion from **III**. This proves Theorem 3.3.

Theorem 3.4. *Let $\mathcal{Z} = \mathcal{Z}(J)$ be a real C^μ piecewise algebraic variety on a partition Δ with T cells δ_i $(i = 1, 2, \dots, T)$ of a domain D, where $J \subset S^\mu(\Delta)$ is an ideal generated by $f_1, f_2, \dots, f_s$. For each $i \in \{1, \dots, T\}$, suppose that $I_i \subseteq R[x]$ is an ideal generated by $f_1|_{\delta_i}, f_2|_{\delta_i}, \dots, f_s|_{\delta_i}$, P_i is a cone of $R[x]$ generated by $(S_\alpha^{(i)})_{\alpha \in \{1, \dots, q_i\}}$, and $\sqrt[P_i]{I_i}$ is the P_i-radical ideal of I_i.*

I. *If $\Gamma = \emptyset$, then $dim \mathcal{Z} = -1$;*

II. *If $\Gamma \neq \emptyset$, $\mathcal{Z}_{(\alpha,j)} = \mathcal{Z}(I_{(\alpha,j)}), j = 1, \dots, b_\alpha$, are all irreducible components of the real algebraic variety $\mathcal{Z}(\sqrt[P_\alpha]{I_\alpha})$, where $I_{(\alpha,j)}$ is an ideal in $R[x]$, $\alpha \in \Gamma$. Then*

$$\begin{aligned} dim \mathcal{Z} &= max_{\alpha \in \Gamma, j \in \theta_\alpha} (deg^a HP_{<LT(\sqrt[R]{I_{(\alpha,j)}})>}) \\ &= max_{\alpha \in \Gamma, j \in \theta_\alpha} (MDCS(\mathcal{Z}(< LT(\sqrt[R]{I_{(\alpha,j)}}) >))), \end{aligned}$$

where $\theta_\alpha = \{1, \dots, b_\alpha\}$.

Proof.

I. If $\Gamma = \emptyset$, then for any $i \in \{1, \dots, T\}$, there exist $l \in P_i$, and $h \in I_i$ such that $l + h + 1 = 0$. By Theorem 3.1, we get $\mathcal{Z} \cap \delta_i = \emptyset$. Hence

$$dim(clos_{zar}(\mathcal{Z} \cap \delta_i)) = -1 \quad for \ \ all\ i \in \{1, \dots, T\}.$$

So

$$dim \mathcal{Z} = -1$$

II. If $\Gamma \neq \emptyset$, then by Theorem 3.1, for any $\alpha \in \Gamma$, $\mathcal{Z}(\sqrt[P_\alpha]{I_\alpha})$ is a nonempty real algebraic variety, and

$$clos_{zar}(\mathcal{Z} \cap \delta_\alpha) = \mathcal{Z}(\sqrt[P_\alpha]{I_\alpha}) = \cup_{j \in \theta_\alpha} \mathcal{Z}_{(\alpha,j)} \quad \alpha \in \Gamma.$$

Hence

$$dim(clos_{zar}(\mathcal{Z} \cap \delta_\alpha)) = max_{j \in \theta_\alpha}(dim \mathcal{Z}_{(\alpha,j)}).$$

It follows the Lemma 3.1, and the Definition 2.2 that

$$\begin{aligned} dim \mathcal{Z} &= max_{\alpha \in \Gamma} dim(clos_{zar}(\mathcal{Z} \cap \delta_\alpha)) \\ &= max_{\alpha \in \Gamma, j \in \theta_\alpha} dim(\mathcal{Z}_{(\alpha,j)}) \\ &= max_{\alpha \in \Gamma, j \in \theta_\alpha} (deg^a HP_{<LT(\sqrt[R]{I_{(\alpha,j)}})>}) \\ &= max_{\alpha \in \Gamma, j \in \theta_\alpha} (MDCS(\mathcal{Z}(< LT(\sqrt[R]{I_{(\alpha,j)}}) >))). \end{aligned}$$

So we have proved Theorem 3.4.

When $\Gamma \neq \emptyset$, for arbitrary $\alpha \in \Gamma$,the set $\mathcal{Z} \cap \delta_\alpha$ is nonempty. Hence the set $\mathcal{Z} \cap \delta_\alpha$ is a real semi-algebraic set([5]). By the decomposition theorem of semi-algebraic sets ([5]), the set $\mathcal{Z} \cap \delta_\alpha$ can be written as the disjoint union of a finite number of semi-algebraic sets $A_{(\alpha,j)}$,i.e., $\mathcal{Z} \cap \delta_\alpha = \cup_{j=i}^{e_\alpha} A_{(\alpha,j)}$, each of them is semi-algebraically homeomorphic to an open hypercube $]0,1[^d \subset R^d$,for some $d \in N$ (with $]0,1[^0$ being a point). Hence we have

Theorem 3.5. *Let $\mathcal{Z}[J]$ be a real C^μ piecewise algebraic variety on a partition Δ of D, and J be an ideal of $S^\mu(\Delta)$. If $\Gamma \neq \emptyset$ and for any $\alpha \in \Gamma$, the set $\mathcal{Z} \cap \delta_\alpha$ is the union of a finite number of semi-algebraic sets $A_{(\alpha,j)}$ in which each of them is a semi-algebraically homeomorphic to an open hypercube $]0,1[^{d_{(\alpha,j)}}$, $d_{(\alpha,j)} \in N \cup \{0\}$. Then*

$$dim\mathcal{Z} = max_{\alpha\in\Gamma, j\in w_\alpha} d_{(\alpha,j)},$$

where $w_\alpha = \{1, 2, \ldots, e_\alpha\}$.

Proof. Because of $A_{(\alpha,j)}$ is a semi-algebraically homeomorphic to an open hypercube $]0,1[^{d_{(\alpha,j)}}$, there exists a continous semi-algebraical mapping

$$g_{(\alpha,j)}: \quad A_{(\alpha,j)} \to R^{d_{(\alpha,j)}}$$

such that $g_{(\alpha,j)}$ maps bijectively the semi-algebraic set $A_{(\alpha,j)}$ onto the set $g_{(\alpha,j)}(A_{(\alpha,j)})$, i.e. $]0,1[^{d_{(\alpha,j)}}$. Hence

$$dimA_{(\alpha,j)} = dim(g_{(\alpha,j)}(A_{(\alpha,j)}) = dim(]0,1[^{d_{(\alpha,j)}}) = d_{(\alpha,j)},$$

where $\alpha \in \Gamma, \quad j \in w_\alpha$.

Moreover, in view of

$$dim(clos_{zar}(\mathcal{Z} \cap \delta_\alpha)) = max_{j\in w_\alpha} dimA_{(\alpha,j)},$$

we have

$$dim\mathcal{Z} = max_{\alpha\in\Gamma} dim(\mathcal{Z} \cap \delta_\alpha) = max_{\alpha\in\Gamma, j\in w_\alpha} d_{(\alpha,j)}.$$

This proves theorem 3.5.

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