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MORTAR FINITE VOLUME METHOD WITH ADINI ELEMENT FOR BIHARMONIC PROBLEM *1)

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Abstract

In this paper, we construct and analyse a mortar finite volume method for the discretization for the biharmonic problem in R^2. This method is based on the mortar-type Adini nonconforming finite element spaces. The optimal order H^2-seminorm error estimate between the exact solution and the mortar Adini finite volume solution of the biharmonic equation is established.

Mathematics subject classification: 65N30, 65N15.
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1. Introduction

In recent years, the mortar finite element method as a special nonconforming domain decomposition technique has attracted many researchers' attention. More and more papers on this method have appeared. We refer to [3] and [19] for the general presentation of the mortar element method and [2], [7],[12], [15], and [20] for details.

In the mortar finite element method, the computational domain is first decomposed into a coarse sub-domain partition. The triangulations on different sub-domains need not match across sub-domain interfaces. The basic idea of this method is to replace the strong continuity condition by a weaker suitable constraint on the interfaces between different sub-domains. Suitable constraint, i.e., the mortar condition, guarantees the optimal discretization schemes.

On the other hand, the finite volume method (also called the box method, generalized difference method) is popular in computational fluid mechanics due to their conservation properties of the original problems. In the past several decades, many researchers have analysed the finite volume method for the selfadjoint (or non-selfadjoint and nondefinite) elliptic partial differential equations using the finite element spaces. Professors Ronghua Li *et al* have systematically studied the finite volume method and obtained many significant results, we refer to the monograph [18] for the general presentation of the finite volume method and [1], [5] [6], [8], [9] [13], [16], [17], [21], and [22] for details.

Recently, Ewing, Lazarov and Lin [11] consider the mortar finite volume element approximations of second order elliptic equations on non-matching grids. The discretization is based on the Petrov-Galerkin method with a solution space of continuous piecewise linear functions over each sub-domain and a test space of piecewise constant functions. They use finite volume element approximations on the sub-domains and finite element on the interfaces for Lagrange multipliers and get an optimal order convergence in energy norm.

In the paper [14], we extend the mortar finite element method to the mortar finite volume method, construct and study a mortar finite volume method which is based on the mortar

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Crouzeix-Raviart finite element space. The optimal order error estimates in broken H^1-norm and in L^2-norm have been developed.

In this paper, we construct and analyse the mortar finite volume method with Adini nonconforming element which is used to solve the biharmonic problem. The restriction of the mortar finite element space to any sub-domain is the Adini nonconforming finite element space. In this paper, we will prove the optimal order error estimate in broken H^2-seminorm.

The remainder of this paper is organized as follows. In Section 2 we introduce notation, construct a triangulation $\mathcal{T}_h$ of Ω and give the corresponding dual partition. In Section 3, we consider the mortar finite volume method, and get some lemmas which will be used in later convergence proof. In Section 4, we estimate the difference between the exact solution and the mortar finite volume approximation in H^2 broken seminorm.

2. Notation and Preliminaries

In this section, we provide some preliminaries and notation. In this paper, we suppose the boundary of the multi-rectangular domain Ω parallel to the OX_1 and OX_2 axises. Consider a geometrically conforming version of the mortar finite volume method, i.e., Ω is divided into non-overlapping rectangular sub-domains Ω_i

$$\overline{\Omega} = \cup_{i=1}^{N} \overline{\Omega}_i,$$

where $\overline{\Omega}_i \cap \overline{\Omega}_j$ is an empty set or a vertex or an edge for $i \neq j$.

Each sub-domain Ω_i is triangulated to produce an rectangular quasi-uniform mesh $\mathcal{T}_{h_i} = \{K\}$ with mesh parameter h_i, where h_i is the largest diameter of the elements in $\mathcal{T}_{h_i}$. The triangulations of sub-domains generally do not align at the sub-domain interfaces. Let Γ_{ij} denote the open straight line segment which is common to Ω_i and Ω_j and Γ denote the union of all interfaces between the sub-domains, i.e., $\Gamma = \cup \partial\Omega_i \backslash \partial\Omega$. We assume that the endpoints of each interface segment in Γ are vertices of $\mathcal{T}_{h_i}$ and $\mathcal{T}_{h_j}$. Let $\mathcal{T}_h$ denote the global mesh $\cup_i \mathcal{T}_{h_i}$ which is assumed quasi-uniform in this paper and $h = \max_{1\leq i \leq N} h_i$.

Since the triangulation $\mathcal{T}_h$ is independent over the sub-domains, each side $\Gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$ is provided with two different and independent 1-D meshes, denoted by $\mathcal{T}_{h_i}(\Gamma_{ij})$ and $\mathcal{T}_{h_j}(\Gamma_{ij})$, respectively. We define one of the sides of Γ_{ij} as a mortar one, the other as a non-mortar one, which are denoted by γ_i and δ_j, respectively. The sets of vertices belonging to $\overline{\Omega}$, $\overline{\Omega}_i$, $\partial\Omega_i$, $\partial\Omega$, γ_i, δ_j and K are denoted by Ω_h, $\Omega_{i,h}$, $\partial\Omega_{i,h}$, $\partial\Omega_h$, $\gamma_{i,h}$, $\delta_{j,h}$ and K_h, respectively.

Define the Adini nonconforming finite element space on sub-domain Ω_i:

$$\begin{aligned} \widetilde{V}_{h,i} = \widetilde{V}_{h,i}(\Omega_i) = \{v \in L^2(\Omega_i) : v|_K \in P_3(K) \oplus \ \operatorname{span}\{x_1^3 x_2, x_1 x_2^3\} \ \text{for} \ K \in \mathcal{T}_{h_i}, \\ v, v_{x_1}, v_{x_2} \ \text{are continuous at the vertices and} \\ v(a) = v_{x_1}(a) = v_{x_2}(a) = 0, \ \forall a \in \partial\Omega_{i,h} \cap \partial\Omega_h\}. \end{aligned}$$

We can now introduce the global space $\widetilde{V}_h$:

$$\widetilde{V}_h = \prod_{i=1}^{N} \widetilde{V}_{h,i}(\Omega_i)$$

with the so called broken H^2-seminorm:

$$|v|_{2,h} = |v|_{2,h,\Omega} = \left(\sum_{i=1}^{N} |v|^2_{2,h,\Omega_i} \right)^{\frac{1}{2}}, \quad |v|_{2,h_i,\Omega_i} = \left(\sum_{K \in \mathcal{T}_{h_i}} |v|^2_{H^2(K)} \right)^{\frac{1}{2}}.$$

Let $W(\delta_j)$ be the subspace of the space $L^2(\Gamma_{ij})$:

$$W(\delta_j) = \{v \in C^0(\overline{\delta}_j), v|_{\overline{K}\cap\delta_j} \in P_1(\overline{K} \cap \delta_j), \quad \forall K \in \mathcal{T}_{h_j}\}.$$

Let the subspace of $W(\delta_j)$ denoted by $\widehat{W}(\delta_j)$ be the space formed by continuous piecewise linear functions which are constants on two elements which touch the ends of the non-mortar side δ_j.

Let a and b be the extremities of the non-mortar side δ_j. We denote by π_h^* the operator from $C^0(\overline{\delta}_j)$ into $W_h(\delta_j)$ defined for any function v of $C^0(\overline{\delta}_j)$ by

$$(\pi_h^* v)(a) = v(a), \quad \text{and} \quad (\pi_h^* v)(b) = v(b),$$
$$\int_{\delta_j} (v - \pi_h^* v)\varphi \mathrm{d}s = 0, \quad \varphi \in \widehat{W}_h(\delta_j).$$

The following stability property of this operator is given in the proof of Lemma 4.3 in [3]:

$$||\pi_h^* v||_{L^2(\delta_j)} \leq C||v||_{L^2(\delta_j)}. \tag{2.1}$$

We introduce the orthogonal projection operator π_h from $L^2(\delta_j)$ onto $\widehat{W}_h(\delta_j)$:

$$(\pi_h v, \varphi)_{L^2(\delta_j)} = (v, \varphi)_{L^2(\delta_j)}, \quad \forall \varphi \in \widehat{W}_h(\delta_j).$$

The following property of this operator is given in Lemma 4.1 in [3]:

Lemma 2.1. *For any function $v \in H^s(\delta_j), s = 0, 1/2, 1,$ we have*

$$||v - \pi_h v||_{L^2(\delta_j)} \leq Ch_j^s ||v||_{H^s(\delta_j)}.$$

In this paper, as a auxiliary tool, we introduce the bilinear interpolation operator I_{h_i} : $\widetilde{V}_{h,i} \to \widetilde{S}_{h,i}, i = 1, ..., N$ and $I_h = \prod_{i=1}^N I_{h_i}$, where $\widetilde{S}_{h,i}$ is the piecewise bilinear finite element space on the sub-domain Ω_i. For $\omega_i \in \widetilde{V}_{h,i}$ and $K \in \mathcal{T}_{h_i}$,

$$I_{h_i}\omega_i(P_0) = \omega_i(P_0), \quad P_0 \in K_h, \quad I_{h_i}\omega_i|_K \in Q_1(K).$$

We say that the functions $\omega \in \widetilde{V}_h$ satisfies the mortar conditions if

$$\omega|_{\gamma_i}(p) = \omega|_{\delta_j}(p), \quad \forall p \in \delta_{j,h}. \tag{2.2}$$

$$\int_{\delta_j} (I_{h_i}\partial_n \omega|_{\gamma_i} - I_{h_j}\partial_n \omega|_{\delta_j})\varphi \mathrm{d}s = 0, \quad \varphi \in \widehat{W}(\delta_j), \tag{2.3}$$

$$\int_{\delta_j} (I_{h_i}\partial_\tau \omega|_{\gamma_i} - I_{h_j}\partial_\tau \omega|_{\delta_j})\varphi \mathrm{d}s = 0, \quad \varphi \in \widehat{W}(\delta_j), \tag{2.4}$$

where n denotes the unit outside normal along the interface $\gamma_i = \delta_j = \Gamma_{ij}$ with the direction from γ_i to δ_j and τ is the tangential unit corresponding to n.

We define the mortar-type Adini nonconforming finite element space V_h as the subspace of $\widetilde{V}_h$, in which the function v satisfies the mortar conditions (2.2) –(2.4) and is continuous at all crosspoints till its first derivatives.

For each non-mortar $\delta_j = \Gamma_{ij} \subset \Gamma$, we introduce the L^2 orthogonal projection Q^{δ_j} : $L^2(\Gamma_{ij}) \to M(\delta_j)$ defined by

$$(Q^{\delta_j} u, \psi)_{L^2(\delta_j)} = (u, \psi)_{L^2(\delta_j)}, \quad \forall \psi \in M(\delta_j),$$

where $M(\delta_j)$ is the piecewise constant function space defined on the non-mortar side δ_j. Similarly we can define $M(\gamma_i)$ and Q^{γ_i}.

From the definition of Q^{γ_i} and Q^{δ_j}, the trace theorem, we get

Lemma 2.2. *Assume that $s \subset \partial\Omega_i$ is a side of Ω_i, (s may be a mortar or non-mortar side). For any $v \in \widetilde{V}_{h,i}$ and $u \in H^1(\Omega_i)$, we have*

$$||v - Q^s v||_{L^2(s)} \leq Ch_i^{\frac{1}{2}} |v|_{1,h,\Omega_i}, \qquad ||u - Q^s u||_{L^2(s)} \leq Ch_i^{\frac{1}{2}} |u|_{1,\Omega_i}.$$

In order to give the descriptions of mortar finite volume method, we build the dual partition of the original triangulation $\mathcal{T}_h$, which has a one to one corresponding with the vertices of the original mesh. In each sub-domain Ω_i, we construct the dual partition of $\mathcal{T}_{h_i}$ as follows. Choose an interior vertex P of K, there are four elements surrounding it. We suppose that they are K_1, K_2, K_3 and K_4. Taking the barycenter Z_{K_i} of the element $K_i, (i = 1, 2, 3, 4)$, and connecting $Z_{K_1}, Z_{K_2}, Z_{K_3}$ and Z_{K_4}, we get the covolume K_P^* corresponding to the vertex P. Moreover, we also associate a corresponding covolume with each nodal point $P \in \partial\Omega_{i,h}$. Thus we obtain a group of covolumes covering the domain Ω, which is the dual triangulation of the original one. This procedure is illustrated in Figure 2.1.

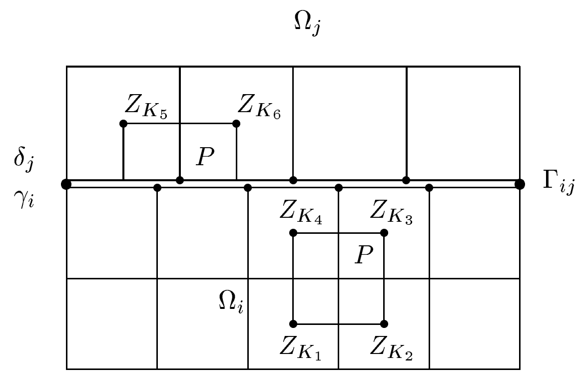

Fig.2.1: Non-matching meshes on the interface Γ_{ij}

We shall denote the dual partition as $\mathcal{T}_h^* = \cup K_P^*$ and associate with it the *test function space* U_h :

$$U_h = \{v \in L^2(\Omega) : v|_{K_P^*} \in P_1(K_P^*), \quad v|_{K_P^*} = 0 \text{ on any boundary dual element } K_P^*\}.$$

Given a function $\omega_h \in V_h$, the corresponding function $\Pi_h^*\omega_h$ in the test function space U_h is defined as

$$\Pi_h^*\omega_h = \sum_{P_0 \in \Omega_h} \left(\omega_h(P_0)\Psi_{P_0}^{(0)} + \frac{\partial\omega_h(P_0)}{\partial x_1}\Psi_{P_0}^{(1)} + \frac{\partial\omega_h(P_0)}{\partial x_2}\Psi_{P_0}^{(2)} \right), \quad \forall \omega_h \in V_h,$$

where $\Psi_{P_0}^{(0)}, \Psi_{P_0}^{(1)}$ and $\Psi_{P_0}^{(2)}$ are three basis functions of U_h with respect to the vertex $P_0(x_1^0, x_2^0)$:

$$\Psi_{P_0}^{(0)}(P) = \begin{cases} 1, & P_0 \in K_{P_0}^*, \\ 0, & P_0 \notin K_{P_0}^*, \end{cases} \qquad \Psi_{P_0}^{(1)}(P_0) = \begin{cases} x_1 - x_1^0, & P_0 \in K_{P_0}^*, \\ 0, & P_0 \notin K_{P_0}^*, \end{cases}$$

$$\Psi_{P_0}^{(2)}(P_0) = \begin{cases} x_2 - x_2^0, & P_0 \in K_{P_0}^*, \\ 0, & P_0 \notin K_{P_0}^*. \end{cases}$$

In this paper, the notation of Sobolev spaces and associated norms are the same as those in Ciarlet [10], and C denotes the positive constant independent of h and the associated functions and may be different at different occurrence.

3. Mortar Finite Volume Method

Let Ω be a multi-rectangular domain in R^2 and $f \in L^2(\Omega)$. Consider the following boundary value problem of the biharmonic equation

$$\begin{cases} \Delta^2 u = f, & \text{in} \quad \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

The equivalent variational form of (3.1) is: Find $u \in H_0^2(\Omega)$ such that

$$A(u,v) = (f,v), \quad \forall v \in H_0^2(\Omega), \tag{3.2}$$

where

$$A(u,v) = \int_\Omega \left(\Delta u \Delta v + (2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}) \right) \mathrm{d}x,$$

$$(f,v) = \int_\Omega f v \mathrm{d}x.$$

We know that

$$A(\omega,v) \le C|\omega|_{2,\Omega}|v|_{2,\Omega}, \quad C|v|_{2,\Omega}^2 \le A(v,v), \quad \forall \omega, v \in H_0^2(\Omega). \tag{3.3}$$

Then the problem (3.2) has a unique solution $u \in H_0^2(\Omega)$.

Multiplying a function $v_h \in U_h$ on both sides of (3.1), integrating over the associated box K_P^* and using the following Green's formulae (cf. (1.2.7) and (1.2.9) in [10]),

$$\int_{K^*} \Delta^2 u v \mathrm{d}x = \int_{K^*} \Delta u \Delta v \mathrm{d}x + \int_{\partial K^*} \frac{\partial \Delta u}{\partial n} v \mathrm{d}s - \int_{\partial K^*} \Delta u \frac{\partial v}{\partial n} \mathrm{d}s,$$

$$\int_{K^*} (2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}) \mathrm{d}x = \int_{\partial K^*} (\frac{\partial^2 u}{\partial n \partial \tau}\frac{\partial v}{\partial \tau} - \frac{\partial^2 u}{\partial \tau^2}\frac{\partial v}{\partial n}) \mathrm{d}s,$$

we get

$$\begin{aligned} &\int_{\partial K^*} \left(\frac{\partial \Delta u}{\partial n} v_h - \Delta u \frac{\partial v_h}{\partial n} + \frac{\partial^2 u}{\partial n \partial \tau}\frac{\partial v_h}{\partial \tau} - \frac{\partial^2 u}{\partial \tau^2}\frac{\partial v_h}{\partial n} \right) \mathrm{d}s \\ = &\int_{K^*} f v_h \mathrm{d}x, \quad v_h \in U_h, \end{aligned} \tag{3.4}$$

where $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial \tau}$ denote respectively the derivatives along the outer normal and the tangent directions.

Based on the above equality, we construct the mortar finite volume method as follows: Find $u_B \in V_h$ such that for any $v_h \in V_h$,

$$a_h(u_B, \Pi_h^* v_h) = \sum_{K^* \in \mathcal{T}_h^*} \int_{K^*} f \Pi_h^* v_h \mathrm{d}x, \tag{3.5}$$

where

$$a_h(u_B, \Pi_h^* v_h) = \sum_{K \in \mathcal{T}_h} I_K(u_B, \Pi_h^* v_h), \tag{3.6}$$

$$\begin{aligned} I_K(u_B, \Pi_h^* v_h) = \sum_{P \in K_h} \Bigg(&\int_{\partial K_P^* \cap K} (\frac{\partial \Delta u_B}{\partial n}\Pi_h^* v_h - \Delta u_B \frac{\partial \Pi_h^* v_h}{\partial n}) \mathrm{d}s \\ &+ \int_{\partial K_P^* \cap K} (\frac{\partial^2 u_B}{\partial n \partial \tau}\frac{\partial \Pi_h^* v_h}{\partial \tau} - \frac{\partial^2 u_B}{\partial \tau^2}\frac{\partial \Pi_h^* v_h}{\partial n}) \mathrm{d}s \Bigg). \end{aligned}$$

From (3.4)-(3.5), we obtain for any function $v_h \in V_h$,

$$\begin{aligned} a_h(u-u_B,\Pi_h^* v_h) \quad - & \sum_{\delta_j \subset \Gamma} \Bigg(\int_{\delta_j} (\frac{\partial \Delta u}{\partial n}[\Pi_h^* v_h] - \Delta u \left[\frac{\partial \Pi_h^* v_h}{\partial n}\right]) \mathrm{d}s \\ & + \int_{\delta_j} (\frac{\partial^2 u}{\partial \tau^2}\left[\frac{\partial \Pi_h^* v_h}{\partial n}\right] - \frac{\partial^2 u}{\partial n \partial \tau}\left[\frac{\partial \Pi_h^* v_h}{\partial \tau}\right]) \mathrm{d}s \Bigg) \\ = \quad & 0, \end{aligned} \tag{3.7}$$

where $[\cdot]$ denotes the jump of the associated functions along the interface δ_j.

Given a rectangle $K \in \mathcal{T}_h$ with vertices $P_m(x_1^m, x_2^m)(m = i,j,k,l)$ where the left-below, right-below, right-above and left-above vertex is P_i, P_j, P_k and P_l respectively, we set $\Delta x_1 = |P_iP_j|$, $\Delta x_2 = |P_iP_k|$ and $\lambda_K = (\Delta x_2/\Delta x_1)^2$. For the rectangle $K \in \mathcal{T}_h$, there exists a invertible affine transformation: $\xi = (x_1 - x_1^i)/\Delta x_1$, $\eta = (x_2 - x_2^i)/\Delta x_2$ which maps K onto the reference element $\widehat{K} = [0,1;0,1]$. We introduce the following norm in the space V_h :

$$||u_h||_h = \left(\sum_{K\in\mathcal{T}_h} \frac{1}{\mathrm{meas}(K)} \mu_K(u_h)^T \mu_K(u_h) \right)^{\frac{1}{2}}, \tag{3.8}$$

where

$$\begin{aligned} \mu_K(v) \quad = \quad & \Big[v_i - v_j + v_k - v_l, (\frac{\partial v}{\partial \xi})_i + v_i - v_j, (\frac{\partial v}{\partial \xi})_j + v_i - v_j, \\ & (\frac{\partial v}{\partial \xi})_k + v_l - v_k, (\frac{\partial v}{\partial \xi})_l + v_l - v_k, (\frac{\partial v}{\partial \eta})_i + v_i - v_l, \\ & (\frac{\partial v}{\partial \eta})_j + v_j - v_k, (\frac{\partial v}{\partial \eta})_k + v_j - v_k, (\frac{\partial v}{\partial \eta})_l + v_i - v_l \Big]^T. \end{aligned}$$

The notation $v_i, (\frac{\partial v}{\partial \xi})_i$ denote $v(P_i), (\frac{\partial v}{\partial \xi})(P_i)$ respectively and the others can be understood in the same way.

Lemma 3.1[9]**.** *There exists a constant independent of V_h and $K \in \mathcal{T}_h$ such that*

$$|\Pi_h^* \omega_h - I_h \omega_h| \le C h_K |\omega_h|_{2,K}, \quad x \in K, \tag{3.9}$$

$$\max_{x\in K} \left(|\frac{\partial}{\partial x_1}(\Pi_h^* \omega_h - I_h \omega_h)|, |\frac{\partial}{\partial x_2}(\Pi_h^* \omega_h - I_h \omega_h)| \right) \le C|\omega_h|_{2,K}. \tag{3.10}$$

By means of the same method as that in [9], we know that the following lemmas also hold for the mortar Adini finite element space V_h.

Lemma 3.2. *There exist constants C_1 and C_2 independent of the function $u_h \in V_h$ such that*

$$C_1||u_h||_h \le |u_h|_{2,h} \le C_2||u_h||_h, \quad \forall u_h \in V_h. \tag{3.11}$$

The seminorm $|v|_{2,h}$ is a norm indeed over the space V_h. $|v|_{2,h} = 0$ means that v is linear in all elements of Ω_i, then from the continuity of $v, \partial v/\partial x_1, \partial v/\partial x_2$ at all vertices of $\overline{\Omega}_i$ we know that v linear in Ω_i and from the mortar conditions (2.2)-(2.4) follows that v linear in Ω. The boundary condition yield $v = 0$.

Lemma 3.3. *Assume that $\frac{2}{3} \le \lambda_K \le \frac{3}{2}$, $\forall K \in \mathcal{T}_h$. Then the bilinear form $a_h(u_h, \Pi_h^* u_h)$ is V_h-elliptic, i.e., there exists a constant α independent of u_h such that*

$$\alpha |u_h|_{2,h}^2 \le a_h(u_h, \Pi_h^* u_h), \quad \forall u_h \in V_h.$$

Then, from Lemma 3.3, we know that the problem (3.5) has a unique solution.

The following Lemma 3.4 is proved in [8]. We formulate it here which will be used in our convergence proof.

Lemma 3.4. *There exists a constant C independent of h_K such that $v|_K \in H^1(K)$ for every $K \in \mathcal{T}_h$*

$$\int_{\partial K} v^2 \mathrm{d}s \le C(h_K^{-1}||v||_{0,K}^2 + h_K|v|_{1,K}^2), \quad \forall K \in \mathcal{T}_h. \tag{3.12}$$

From Lemma 3.1 and Lemma 3.4, we get

Lemma 3.5. *Assume that $s \subset \partial\Omega_k$ is a side of Ω_k, (s may be a mortar or non-mortar side). For any $\omega_h \in \widetilde{V}_{h,k}$, we have*

$$||\Pi_h^*\omega_h - I_h\omega_h||_{L^2(s)} \le Ch_i^{3/2}|\omega_h|_{2,h,\Omega_k},$$

$$||\partial_n(\Pi_h^*\omega_h - I_h\omega_h)||_{L^2(s)} \le Ch_i^{1/2}|\omega_h|_{2,h,\Omega_k},$$

$$||\partial_\tau(\Pi_h^*\omega_h - I_h\omega_h)||_{L^2(s)} \le Ch_i^{1/2}|\omega_h|_{2,h,\Omega_k}.$$

4. Error Estimate in H^2-seminorm

In this section, we prove the error estimate for the mortar finite volume method presented in the previous section. First, we give the proof of approximation error.

Let $\tilde{u}_i$ be the interpolation of u defined at all degrees of freedom of Adini element at all nodal points of $\Omega_{i,h}$. We have $\tilde{u}_i \in \widetilde{V}_{h,i}$ (see e.g. [10]),

$$|u - \tilde{u}_i|_{H^s(\Omega_i)} \le Ch_i^{3-s}|u|_{3,\Omega_i}, \quad s = 0, 1, 2. \tag{4.1}$$

Let $\tilde{u}|_{\Omega_i} = \tilde{u}_i, i = 1, ..., N$. The function $\tilde{u} \in \widetilde{V}_h$ may not satisfy the mortar conditions across the interface. In the following, we define a function ω such that $u^I = \tilde{u} + \omega$ satisfies the mortar conditions.

To do it, given each side $\Gamma_{ij} = \gamma_i = \delta_j$, we first define three functions on the non-mortar side δ_j :

$$\omega_1 = I_{h_j}(\tilde{u}_i|_{\gamma_i} - \tilde{u}_j|_{\delta_j}), \quad \omega_2 = \pi_h^*(I_{h_i}\partial_n\tilde{u}_i|_{\gamma_i} - I_{h_j}\partial_n\tilde{u}_j|_{\delta_j}),$$
$$\omega_3 = \pi_h^*(I_{h_i}\partial_\tau\tilde{u}_i|_{\gamma_i} - I_{h_j}\partial_\tau\tilde{u}_j|_{\delta_j}).$$

From the mortar conditions (2.2)-(2.4) and the definition of π_h^*, we know

$$\omega_1(p) = \tilde{u}_i|_{\gamma_i}(p) - \tilde{u}_j|_{\delta_j}(p), \quad p \in \delta_{j,h}, \tag{4.2}$$

$$\int_{\delta_j} \omega_2\varphi \mathrm{d}s = \int_{\delta_j}(I_{h_i}\partial_n\tilde{u}_i|_{\gamma_i} - I_{h_j}\partial_n\tilde{u}_j|_{\delta_j})\varphi \mathrm{d}s, \quad \varphi \in \widehat{W}(\delta_j), \tag{4.3}$$

$$\int_{\delta_j} \omega_3\varphi \mathrm{d}s = \int_{\delta_j}(I_{h_i}\partial_\tau\tilde{u}_i|_{\gamma_i} - I_{h_j}\partial_\tau\tilde{u}_j|_{\delta_j})\varphi \mathrm{d}s, \quad \varphi \in \widehat{W}(\delta_j), \tag{4.4}$$

We now define a global function $\omega \in \widetilde{V}_h$ by setting the values of all degree of freedom at all nodal points of all sub-domains. We first set to zero all degree of ω at nodal points which are not in any non-mortar side δ_j. For the vertex p on non-mortar side δ_j, we set $\omega(p) = \omega_1(p), \partial_n\omega(p) = \omega_2(p), \partial_\tau\omega(p) = \omega_3(p)$. We define $u^I = \tilde{u} + \omega$ as the interpolation of u. Obviously, the function $u^I \in V_h$ and satisfies the mortar conditions (2.2)–(2.4).

Lemma 4.1. *For any $u \in H_0^2(\Omega) \cap H^3(\Omega)$, there exists a constant C such that*

$$|u - u^I|_{2,h} \leq C \left(\sum_{i=1}^{N} h_i^2 |u|_{3,\Omega_i}^2 \right)^{\frac{1}{2}}. \tag{4.5}$$

Proof. From (4.1) and the triangle inequality, we have

$$|u - u^I|_{2,h}^2 \leq C \left(|u - \tilde{u}|_{2,h}^2 + |\omega|_{2,h}^2 \right) \leq C \left(\sum_{i=1}^{N} h_i^2 |u|_{3,\Omega_i}^2 + |\omega|_{2,h}^2 \right). \tag{4.6}$$

Let the non-mortar side δ_j be parallel to the axis OX_1. Then $\partial_n \omega|_{\delta_j} = \partial_{x_2}\omega|_{\delta_j}$ or $\partial_n \omega|_{\delta_j} = -\partial_{x_2}\omega|_{\delta_j}$. By means of the reference rectangle and scaling argument, we obtain

$$\begin{aligned} |\omega|_{2,h}^2 &= \sum_{\delta_j \subset \Gamma} |\omega|_{2,h,\Omega_j}^2 \\ &\leq C \sum_{\delta_j \subset \Gamma} \sum_{p \in \delta_{j,h}} \left(h_j^{-2} \omega^2(p) + (\partial_n \omega(p))^2 + (\partial_\tau \omega(p))^2 \right) \\ &\leq C \sum_{\delta_j \subset \Gamma} \left(h_j^{-3} ||\omega_1||_{L^2(\delta_j)}^2 + h_j^{-1} ||\omega_2||_{L^2(\delta_j)}^2 + h_j^{-1} ||\omega_3||_{L^2(\delta_j)}^2 \right). \end{aligned} \tag{4.7}$$

From the triangle inequality, the standard interpolation theory and the trace theorem, we get

$$\begin{aligned} ||\omega_1||_{L^2(\delta_j)} &\leq Ch_j |\tilde{u}_i|_{\gamma_i} - \tilde{u}_j|_{\delta_j}|_{H^1(\delta_j)} + ||\tilde{u}_i|_{\gamma_i} - \tilde{u}_j|_{\delta_j}||_{L^2(\delta_j)} \\ &\leq Ch_j (|\tilde{u}_i|_{\gamma_i} - u|_{H^1(\delta_j)} + |u - \tilde{u}_j|_{\delta_j}|_{H^1(\delta_j)}) \\ &\quad + ||\tilde{u}_i|_{\gamma_i} - u|_{L^2(\delta_j)} + |u - \tilde{u}_j|_{\delta_j}|_{L^2(\delta_j)}) \\ &\leq C(h_i^{5/2} |u|_{3,\Omega_i} + h_j^{5/2} |u|_{3,\Omega_j}), \end{aligned} \tag{4.8}$$

From (2.1), the triangle inequality, the standard interpolation theory and the trace theorem, we get the estimate for $||\omega_2||_{L^2(\delta_j)}$

$$\begin{aligned} ||\omega_2||_{L^2(\delta_j)} &\leq C||I_{h_i} \partial_n \tilde{u}_i|_{\gamma_i} - I_{h_j} \partial_n \tilde{u}_j|_{\delta_j}||_{L^2(\delta_j)} \\ &\leq C(||I_{h_i} \partial_n \tilde{u}_i|_{\gamma_i} - \partial_n \tilde{u}_i|_{\gamma_i}||_{L^2(\delta_j)} + ||\partial_n \tilde{u}_i|_{\gamma_i} - \partial_n u||_{L^2(\delta_j)} \\ &\quad + ||\partial_n u - \partial_n \tilde{u}_j|_{\delta_j}||_{L^2(\delta_j)} + ||\partial_n \tilde{u}_j|_{\delta_j} - I_{h_j} \partial_n \tilde{u}_j|_{\delta_j}||_{L^2(\delta_j)}) \\ &\leq C(h_i^{3/2} |u|_{3,\Omega_i} + h_j^{3/2} |u|_{3,\Omega_j}). \end{aligned} \tag{4.9}$$

The term $||\omega_3||_{L^2(\delta_j)}$ can be estimated in a similar way. Consequently, combining (4.7)-(4.9) with (4.6), we obtain the desired result (4.5).

Next, we define a *local equivalent mapping* $\mathcal{M}_i^A : \widetilde{V}_{h,i}(\Omega_i) \to \widetilde{V}_{h,i}^B(\Omega_i)$, (see [19] and [4]), where $\widetilde{V}_{h_i}^B(\Omega_i)$ is a C^1 smooth functions that are bicubic in each rectangular element of $\mathcal{T}_{h_i}$, known as Bogner-Fox-Schmit finite element space.

Definition 4.1. *We define $\mathcal{M}_i^A : \widetilde{V}_{h,i}(\Omega_i) \to \widetilde{V}_{h,i}^B(\Omega_i)$ by setting their values of all respective degrees of freedom at all nodal points of $\overline{\Omega}_i$, as follows, let p be a nodal point of $\overline{\Omega}_i, u \in \widetilde{V}_{h,i}$, then*

$$\mathcal{M}_i^A u(p) = u(p),$$
$$\partial_{x_j} \mathcal{M}_i^A u(p) = u_{x_j}(p), \quad j = 1, 2,$$
$$\partial_{x_1 x_2} \mathcal{M}_i^A u(p) = 0.$$

In the following lemma, we state some properties of the operator $\mathcal{M}_i^A$. The proof of this lemma can be found in [4], see Lemma 5.1 there.

Lemma 4.2. *Suppose $u \in \widetilde{V}_{h,i}(\Omega_i)$. Then we have*

$$|\mathcal{M}_i^A u|_{H^s(\Omega_i)} \approx |u|_{s,h,\Omega_i}, \quad s = 0,1,2,$$

$$||u - \mathcal{M}_i^A u||_{L^2(\Omega_i)} + h_i|u - \mathcal{M}_i^A u|_{H^1(\Omega_i)} \leq Ch_i^2 |u|_{2,h,\Omega_i}.$$

Lemma 4.3. *Let $s \subset \partial\Omega_k$ be a side of Ω_k, (s may be a mortar or non-mortar side). For any $\omega_h \in V_h$, we have*

$$||I_{h_k}\omega_h|_s - \omega_h|_s||_{L^2(s)} \leq Ch_k^{3/2}|\omega_h|_{2,h,\Omega_k}; \tag{4.10}$$

$$||I_{h_k}\partial_n\omega_h|_s - \partial_n\omega_h|_s||_{L^2(s)} \leq Ch_k^{1/2}|\omega_h|_{2,h,\Omega_k}; \tag{4.11}$$

$$||I_{h_k}\partial_\tau\omega_h|_s - \partial_\tau\omega_h|_s||_{L^2(s)} \leq Ch_k^{1/2}|\omega_h|_{2,h,\Omega_k}. \tag{4.12}$$

Proof. We only prove (4.10) holds. The other estimates can be proved in a similar way. From Lemma 3.4, we get

$$\begin{aligned} ||I_{h_k}\omega_h|_s - \omega_h|_s||^2_{L^2(\partial K \cap s)} &\leq C(h_K^{-1}||I_{h_k}\omega_h|_K - \omega_h|_K||^2_{L^2(K)} \\ &\quad + h_K|I_{h_k}\omega_h|_K - \omega_h|_K|^2_{H^1(K)}) \\ &\leq Ch_K^3|\omega_h|^2_{2,K}. \end{aligned} \tag{4.13}$$

Summing all these ∂K in s, we obtain (4.10).

Lemma 4.4. *Suppose $\omega_h \in V_h$. Then we have*

$$||I_{h_i}\omega_h|_{\gamma_i} - I_{h_j}\omega_h|_{\delta_j}||_{0,\delta_j} \leq C(h_i^{3/2}|\omega_h|_{2,h,\Omega_i} + h_j^{3/2}|\omega_h|_{2,h,\Omega_j}). \tag{4.14}$$

Proof. From the mortar condition (2.2) and the definition of the operator $\mathcal{M}_i^A$ we have $I_{h_j}\omega_h|_{\gamma_i} = I_{h_j}\omega_h|_{\delta_j}$ and

$$\begin{aligned} ||I_{h_i}\omega_h|_{\gamma_i} - I_{h_j}\omega_h|_{\delta_j}||_{0,\delta_j} &= ||I_{h_i}(\mathcal{M}_i^A\omega_h)|_{\gamma_i} - I_{h_j}\omega_h|_{\delta_j}||_{0,\delta_j} \\ &\leq ||I_{h_i}(\mathcal{M}_i^A\omega_h)|_{\gamma_i} - \mathcal{M}_i^A\omega_h|_{\gamma_i}||_{0,\delta_j} \\ &\quad + ||\mathcal{M}_i^A\omega_h|_{\gamma_i} - I_{h_j}(\mathcal{M}_i^A\omega_h)|_{\gamma_i}||_{0,\delta_j} \\ &\quad + ||I_{h_j}((\mathcal{M}_i^A\omega_h)|_{\gamma_i} - \omega_h|_{\gamma_i})||_{0,\delta_j}. \end{aligned} \tag{4.15}$$

From the interpolation theory, the trace theorem and Lemma 4.2, we have

$$||I_{h_i}(\mathcal{M}_i^A\omega_h)|_{\gamma_i} - \mathcal{M}_i^A\omega_h|_{\gamma_i}||_{0,\delta_j} \leq Ch_i^{3/2}|\omega_h|_{2,h,\Omega_i}; \tag{4.16}$$

$$||\mathcal{M}_i^A\omega_h|_{\gamma_i} - I_{h_j}(\mathcal{M}_i^A\omega_h)|_{\gamma_i}||_{0,\delta_j} \leq Ch_i^{3/2}|\omega_h|_{2,h,\Omega_i}. \tag{4.17}$$

Using the triangle inequality and the interpolation estimate for I_{h_j}, we get

$$\begin{aligned} ||I_{h_j}((\mathcal{M}_i^A\omega_h)|_{\gamma_i} - \omega_h|_{\gamma_i})||_{0,\delta_j} &\leq ||\mathcal{M}_i^A\omega_h|_{\gamma_i} - \omega_h|_{\gamma_i}||_{0,\delta_j} \\ &\quad + Ch_j|\mathcal{M}_i^A\omega_h|_{\gamma_i} - \omega_h|_{\gamma_i}|_{1,\delta_j}. \end{aligned} \tag{4.18}$$

From Lemma 3.4 and Lemma 4.2, we obtain

$$||\mathcal{M}_i^A\omega_h|_{\gamma_i} - \omega_h|_{\gamma_i}||^2_{0,\delta_j} = \sum_{K \in \mathcal{T}_{h_i}} ||\mathcal{M}_i^A\omega_h|_{\gamma_i} - \omega_h|_{\gamma_i}||^2_{0,\overline{K}\cap\delta_j}$$

$$\begin{aligned} &\le C\sum_{K\in\mathcal{T}_{h_i}}(h_i^{-1}||\mathcal{M}_i^A\omega_h-\omega_h||^2_{0,K} \\ &\qquad +h_i|\mathcal{M}_i^A\omega_h-\omega_h|^2_{1,K}) \\ &\le Ch_i^3|\omega_h|^2_{2,h,\Omega_i}. \end{aligned} \tag{4.19}$$

$$|\mathcal{M}_i^A\omega_h|_{\gamma_i}-\omega_h|_{\gamma_i}|^2_{1,\delta_j}\le Ch_i|\omega_h|^2_{2,h,\Omega_i}. \tag{4.20}$$

Combining all above inequalities yield the desired result (4.14).

Lemma 4.5. *Suppose $\omega_h\in V_h$. Then*

$$||I_{h_i}\partial_n\omega_h|_{\gamma_i}-I_{h_j}\partial_n\omega_h|_{\delta_j}||_{L^2(\gamma_i)}\le C(h_i^{1/2}|\omega_h|_{2,h,\Omega_i}+h_j^{1/2}|\omega_h|_{2,h,\Omega_j}); \tag{4.21}$$

$$||I_{h_i}\partial_\tau\omega_h|_{\gamma_i}-I_{h_j}\partial_\tau\omega_h|_{\delta_j}||_{L^2(\delta_j)}\le C(h_i^{1/2}|\omega_h|_{2,h,\Omega_i}+h_j^{1/2}|\omega_h|_{2,h,\Omega_j}). \tag{4.22}$$

Proof. We only need to prove (4.21) holds. (4.22) can be proved in a similar way. From the mortar condition (2.3), we get for any $v\in L^2(\delta_j)$

$$\begin{aligned} & |(I_{h_i}\partial_n\omega_h|_{\gamma_i}-I_{h_j}\partial_n\omega_h|_{\delta_j},v)_{0,\delta_j}| \\ = & |(I_{h_i}\partial_n\omega_h|_{\gamma_i}-I_{h_j}\partial_n\omega_h|_{\delta_j},v-\pi_hv)_{0,\delta_j}| \\ = & |(I_{h_i}\partial_n\omega_h|_{\gamma_i}-\pi_h\mathcal{M}_i^A\partial_n\omega_h|_{\gamma_i}-I_{h_j}\partial_n\omega_h|_{\delta_j}+\pi_h\mathcal{M}_j^A\partial_n\omega_h|_{\delta_j},v-\pi_hv)_{0,\delta_j}| \\ \le & C||v||_{0,\delta_j}\left(||I_{h_i}\partial_n\omega_h|_{\gamma_i}-\partial_n\omega_h|_{\gamma_i}||_{0,\gamma_i}+||\partial_n\omega_h|_{\gamma_i}-\mathcal{M}_i^A\partial_n\omega_h|_{\gamma_i}||_{0,\gamma_i}\right. \\ & +||\mathcal{M}_i^A\partial_n\omega_h|_{\gamma_i}-\pi_h\mathcal{M}_i^A\partial_n\omega_h|_{\gamma_i}||_{0,\gamma_i}+||I_{h_j}\partial_n\omega_h|_{\delta_j}-\partial_n\omega_h|_{\delta_j}||_{0,\delta_j} \\ & \left.+||\partial_n\omega_h|_{\delta_j}-\mathcal{M}_j^A\partial_n\omega_h|_{\delta_j}||_{0,\delta_j}+||\mathcal{M}_j^A\partial_n\omega_h|_{\delta_j}-\pi_h\mathcal{M}_j^A\partial_n\omega_h|_{\delta_j}||_{0,\delta_j}\right). \end{aligned}$$

From Lemma 4.2 and Lemma 3.4, we obtain

$$||\partial_n\omega_h|_{\gamma_i}-\mathcal{M}_i^A\partial_n\omega_h|_{\gamma_i}||_{0,\gamma_i}\le Ch_i^{1/2}|\omega_h|_{2,h,\Omega_i}. \tag{4.23}$$

From Lemma 2.1, the trace theorem and Lemma 4.2, we have

$$||\mathcal{M}_i^A\partial_n\omega_h|_{\gamma_i}-\pi_h\mathcal{M}_i^A\partial_n\omega_h|_{\gamma_i}||_{0,\gamma_i}\le Ch_j^{1/2}|\omega_h|_{2,h,\Omega_j}. \tag{4.24}$$

The terms $||\partial_n\omega_h|_{\delta_j}-\mathcal{M}_j^A\partial_n\omega_h|_{\delta_j}||_{0,\delta_j}$ and $||\mathcal{M}_j^A\partial_n\omega_h|_{\delta_j}-\pi_h\mathcal{M}_j^A\partial_n\omega_h|_{\delta_j}||_{0,\delta_j}$ can be also estimated as above. Then, combining all inequalities above with Lemma 4.3, we have

$$|(I_{h_i}\partial_n\omega_h|_{\gamma_i}-I_{h_j}\partial_n\omega_h|_{\delta_j},v)_{0,\delta_j}|\le C||v||_{0,\delta_j}(h_i^{\frac12}|\omega_h|_{2,h,\Omega_i}+h_j^{\frac12}|\omega_h|_{2,h,\Omega_j}).$$

From the definition of $L^2(\delta_j)$-norm, we obtain the desired result (4.21).

Theorem 4.1. *Suppose that u and u_B are the solutions of (3.2) and (3.5), respectively. Then, there exists a constant C such that*

$$|u-u_B|_{2,h}\le C\left(\sum_{i=1}^N h_i^2(|u|^2_{3,\Omega_i}+h_i^2|u|^2_{4,\Omega_i})\right)^{\frac12}. \tag{4.25}$$

Proof. From Lemma 3.3, we have for $u^I\in V_h$

$$|u_B-u^I|_{2,h}\le C\sup_{0\ne\omega_h\in V_h}\frac{a_h(u_B-u^I,\Pi_h^*\omega_h)}{|\omega_h|_{2,h}}. \tag{4.26}$$

From (3.7), we get

$$\begin{aligned} & a_h(u_B - u^I, \Pi_h^*\omega_h) \\ = \; & a_h(u - u^I, \Pi_h^*\omega_h) - \sum_{\delta_j \subset \Gamma} \left(\int_{\delta_j} (\frac{\partial \Delta u}{\partial n}[\Pi_h^*\omega_h] - \Delta u \left[\frac{\partial \Pi_h^*\omega_h}{\partial n}\right]) \mathrm{d}s \right. \\ & \left. - \int_{\delta_j} (\frac{\partial^2 u}{\partial \tau^2} \left[\frac{\partial \Pi_h^*\omega_h}{\partial n}\right] - \frac{\partial^2 u}{\partial n \partial \tau} \left[\frac{\partial \Pi_h^*\omega_h}{\partial \tau}\right]) \mathrm{d}s \right). \end{aligned} \tag{4.27}$$

In order to estimate $a_h(u - u^I, \Pi_h^*\omega_h)$, from (3.6), we may rewrite it as

$$\begin{aligned} a_h(u - u^I, \omega_h) &= \sum_{K \in \mathcal{T}_h} I_K(u - u^I, \Pi_h^*\omega_h) \\ &= \sum_{K \in \mathcal{T}_h} (E_K^1 + E_K^2 + E_K^3 + E_K^4 + E_K^5), \end{aligned} \tag{4.28}$$

where

$$E_K^1 = \sum_{P \in K_h} \int_{\partial K_P^* \cap K} \frac{\partial \Delta u}{\partial n} \Pi_h^*\omega_h \mathrm{d}s,$$

$$E_K^2 = \sum_{P \in K_h} \int_{\partial K_P^* \cap K} -\frac{\partial \Delta u^I}{\partial n} \Pi_h^*\omega_h \mathrm{d}s,$$

$$E_K^3 = \sum_{P \in K_h} -\int_{\partial K_P^* \cap K} \Delta(u - u^I) \frac{\partial \Pi_h^*\omega_h}{\partial n} \mathrm{d}s,$$

$$E_K^4 = -\sum_{P \in K_h} \int_{\partial K_P^* \cap K} \frac{\partial^2 (u - u^I)}{\partial \tau^2} \frac{\partial \Pi_h^*\omega_h}{\partial n} \mathrm{d}s,$$

$$E_K^5 = \sum_{P \in K_h} \int_{\partial K_P^* \cap K} \frac{\partial^2 (u - u^I)}{\partial n \partial \tau} \frac{\partial \Pi_h^*\omega_h}{\partial \tau} \mathrm{d}s.$$

Using Lemma 4.1 and the technique given in the proof of theorem 3 in Chen [9], we obtain the estimation of E_K^2, E_K^3, E_K^4 and E_K^5 :

$$|E_K^2|, \quad |E_K^3|, \quad |E_K^4|, \quad |E_K^5| \le C h_K |u|_{3,K} |\omega_h|_{2,K},$$

In the following, we estimate E_K^1. Since $I_h\omega_h \in C^0(\Omega_i)$, we have

$$\begin{aligned} |E_K^1| &= \sum_{P \in K_h} \left| \int_{\partial K_P^* \cap K} \frac{\partial \Delta u}{\partial n} (\Pi_h^*\omega_h - I_h\omega_h) \mathrm{d}s \right| \\ &\le C h_K |\omega_h|_{2,K} \sum_{P \in K_h} \int_{\partial K_P^* \cap K} \left| \frac{\partial \Delta u}{\partial n} \right| \mathrm{d}s, \end{aligned} \tag{4.29}$$

where Lemma 3.1 is used.

From the Cauchy-Schwarz inequality and Lemma 3.4, we obtain

$$\begin{aligned} \sum_{P \in K_h} \int_{\partial K_P^* \cap K} \left| \frac{\partial \Delta u}{\partial n} \right| \mathrm{d}s &\le C h_K^{1/2} \sum_{P \in K_h} \left(\int_{\partial K_P^* \cap K} \left| \frac{\partial \Delta u}{\partial n} \right|^2 \mathrm{d}s \right)^{\frac{1}{2}} \\ &\le C(|u|_{3,K} + h_K |u|_{4,K}). \end{aligned} \tag{4.30}$$

From (4.29) and (4.30), we get

$$|E_K^1| \le C h_K (|u|_{3,K} + h_K |u|_{4,K}) |\omega_h|_{2,K}.$$

From the inequalities as above, we have

$$|a_h(u-u^I,\Pi_h^*\omega_h)|\le C\left(\sum_{i=1}^N h_i^2(|u|_{3,\Omega_i}^2+h_i^2|u|_{4,\Omega_i}^2)\right)^{\frac{1}{2}}|\omega|_{2,h}. \tag{4.31}$$

Next, we estimate the remainder terms of right side of (4.27). First, we have

$$\begin{aligned}&\int_{\delta_j}\frac{\partial\Delta u}{\partial n}[\Pi_h^*\omega_h]\mathrm{d}s\\ =&\int_{\gamma_i}\frac{\partial\Delta u}{\partial n}(\Pi_h^*\omega_h|_{\gamma_i}-I_{h_i}\omega_h)\mathrm{d}s+\int_{\delta_j}\frac{\partial\Delta u}{\partial n}(\Pi_h^*\omega_h|_{\delta_j}-I_{h_j}\omega_h)\mathrm{d}s\\ &+\int_{\delta_j}\frac{\partial\Delta u}{\partial n}(I_{h_i}\omega_h|_{\gamma_i}-I_{h_j}\omega_h|_{\delta_j})\mathrm{d}s.\end{aligned} \tag{4.32}$$

From Cauchy-Schwarz inequality, Lemma 3.5, we get

$$\begin{aligned}\int_{\delta_j}\frac{\partial\Delta u}{\partial n}[\Pi_h^*\omega_h]\mathrm{d}s\quad\le\quad & C(h_i^{3/2}|u|_{3,\gamma_i}|\omega_h|_{2,h,\Omega_i}+h_j^{3/2}|u|_{3,\delta_j}|\omega_h|_{2,h,\Omega_j})\\ &+|\int_{\delta_j}\frac{\partial\Delta u}{\partial n}(I_{h_i}\omega_h|_{\gamma_i}-I_{h_j}\omega_h|_{\delta_j})\mathrm{d}s|.\end{aligned} \tag{4.33}$$

From Lemma 3.4, we can derive that

$$|u|_{3,\delta_j}\le C(h_j^{-\frac{1}{2}}|u|_{3,\Omega_j}+h_j^{\frac{1}{2}}|u|_{4,\Omega_j}). \tag{4.34}$$

Therefore, from Cauchy-Schwarz inequality and Lemma 4.4, we get

$$\begin{aligned}|\int_{\delta_j}\frac{\partial\Delta u}{\partial n}(I_{h_i}\omega_h|_{\gamma_i}-I_{h_j}\omega_h|_{\delta_j})\mathrm{d}s|\quad\le\quad & C\big((h_i|u|_{3,\Omega_i}+h_i^2|u|_{4,\Omega_i})|\omega_h|_{2,h,\Omega_i}\\ &+(h_j|u|_{3,\Omega_j}+h_j^2|u|_{4,\Omega_j})|\omega_h|_{2,h,\Omega_j}\big).\end{aligned} \tag{4.35}$$

Combining (4.33) with (4.35), we have

$$|\sum_{\delta_j\subset\Gamma}\int_{\delta_j}\frac{\partial\Delta u}{\partial n}[\Pi_h^*\omega_h]\,\mathrm{d}s|\le C\left(\sum_{i=1}^N(h_i^2|u|_{3,\Omega_i}^2+h_i^4|u|_{4,\Omega_i})\right)^{\frac{1}{2}}|\omega_h|_{2,h}. \tag{4.36}$$

Since $\partial_n\Pi_h^*\omega_h$, $\partial_\tau\Pi_h^*\omega_h$ are piecewise constants on the dual meshes of γ_i and δ_j, $I_{h_i}\partial_n\omega_h|_{\gamma_i}$, $I_{h_j}\partial_n\omega_h|_{\delta_j}$ $I_{h_i}\partial_\tau\omega_h|_{\gamma_i}$ and $I_{h_j}\partial_\tau\omega_h|_{\delta_j}$ are continuous piecewise linear functions, we have

$$\int_s(\partial_n\Pi_h^*\omega_h|_s-I_{h_k}\partial_n\omega_h|_s)\varphi\mathrm{d}s=0,\quad\forall\varphi\in M(s), \tag{4.37}$$

$$\int_s(\partial_\tau\Pi_h^*\omega_h|_s-I_{h_k}\partial_\tau\omega_h|_s)\varphi\mathrm{d}s=0,\quad\forall\varphi\in M(s), \tag{4.38}$$

where s is a mortar side or a non-mortar side of the sub-domain Ω_k and $M(s)$ is the piecewise constant function space defined on the original mesh of the side s, which is given in section 2.

From (4.37), we have

$$\begin{aligned}&\int_{\delta_j}\Delta u\,[\partial_n\Pi_h^*\omega_h]\,\mathrm{d}s\\ =&\int_{\gamma_i}\Delta u(\partial_n\Pi_h^*\omega_h|_{\gamma_i}-I_{h_i}\partial_n\omega_h)\mathrm{d}s+\int_{\delta_j}\Delta u(\partial_n\Pi_h^*\omega_h|_{\delta_j}-I_{h_j}\partial_n\omega_h)\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
& +\int_{\delta_j} \Delta u(I_{h_i}\partial_n\omega_h - I_{h_j}\partial_n\omega_h)\mathrm{d}s \\
= & \int_{\gamma_i} (\Delta u - Q^{\gamma_i}\Delta u)(\partial_n\Pi_h^*\omega_h|_{\gamma_i} - I_{h_i}\partial_n\omega_h)\mathrm{d}s + \int_{\delta_j} \Delta u(I_{h_i}\partial_n\omega_h - I_{h_j}\partial_n\omega_h)\mathrm{d}s \\
& +\int_{\delta_j} (\Delta u - Q^{\delta_j}\Delta u)(\partial_n\Pi_h^*\omega_h|_{\delta_j} - I_{h_j}\partial_n\omega_h)\mathrm{d}s \\
= & \quad T_1 + T_2 + T_3,
\end{aligned} \tag{4.39}
$$

where $Q^{\gamma_i}(\Delta u) \in M(\gamma_i)$ and $Q^{\delta_j}(\Delta u) \in M(\delta_j)$.

From Cauchy-Schwarz inequality, Lemma 2.2, Lemma 3.5 and the standard interpolation theory, we have

$$
\begin{aligned}
|T_1| \quad \leq \quad & ||\Delta u - Q^{\gamma_i}\Delta u||_{0,\delta_j)}(||\partial_n\Pi_h^*\omega_h - \partial_n I_{h_i}\omega_h||_{0,\delta_j)} \\
& +||\partial_n I_{h_i}\omega_h - \partial_n\omega_h||_{0,\delta_j)} + ||\partial_n\omega_h - I_{h_i}\partial_n\omega_h||_{0,\delta_j)}) \\
\leq \quad & Ch_i|u|_{3,\Omega_i}|\omega_h|_{2,h,\Omega_i}.
\end{aligned} \tag{4.40}
$$

Similarly, we get the estimation for $|T_3|$.

$$
|T_3| \leq Ch_j|u|_{3,\Omega_j}|\omega_h|_{2,h,\Omega_j}. \tag{4.41}
$$

From the mortar condition (2.3), Lemma 2.1 and Lemma 4.5, we get

$$
\begin{aligned}
|T_2| \quad = \quad & |\int_{\delta_j} (\Delta u - \pi_h(\Delta u))(I_{h_i}\partial_n\omega_h|_{\gamma_i} - I_{h_j}\partial_n\omega_h|_{\delta_j})\mathrm{d}s| \\
\leq \quad & Ch_j^{1/2}|u|_{3,\Omega_j}(h_i^{1/2}|\omega_h|_{2,h,\Omega_i} + h_j^{1/2}|\omega_h|_{2,h,\Omega_j}) \\
\leq \quad & Ch_j|u|_{3,\Omega_j}(|\omega_h|_{2,h,\Omega_i} + |\omega_h|_{2,h,\Omega_j})
\end{aligned} \tag{4.42}
$$

From (4.39)-(4.42), we obtain

$$
\left|\sum_{\delta_j\subset\Gamma}\int_{\delta_j} \Delta u\,[\partial_n\Pi_h^*\omega_h]\,\mathrm{d}s\right| \leq C\left(\sum_{i=1}^N h_i^2|u|_{3,\Omega_i}^2\right)^{\frac{1}{2}}|\omega_h|_{2,h}. \tag{4.43}
$$

Similar estimates can be obtained:

$$
\left|\sum_{\delta_j\subset\Gamma}\int_{\delta_j} \frac{\partial^2 u}{\partial\tau^2}\left[\frac{\partial\Pi_h^*\omega_h}{\partial n}\right]\mathrm{d}s\right| \leq C\left(\sum_{i=1}^N h_i^2|u|_{3,\Omega_i}^2\right)^{\frac{1}{2}}|\omega_h|_{2,h}. \tag{4.44}
$$

$$
|\sum_{\delta_j\subset\Gamma}\int_{\delta_j} \frac{\partial^2 u}{\partial n\partial\tau}\left[\frac{\partial\Pi_h^*\omega_h}{\partial\tau}\right]\mathrm{d}s| \leq C\left(\sum_{i=1}^N h_i^2|u|_{3,\Omega_i}^2\right)^{\frac{1}{2}}|\omega_h|_{2,h}. \tag{4.45}
$$

From (4.26), (4.27), (4.31), (4.36), (4.43)-(4.45), and Lemma 4.1 we get the desired result (4.25).

Remark. Theorem 4.1 is proved under H^4 smoothness hypothesis and the regularity assumption underlying the original partial differential equation is not used.

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