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Lie Group Classification for a Generalised Coupled Lane-Emden System in Dimension One

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Abstract. In this article, we discuss the generalised coupled Lane-Emden system $u''+H(v)=0$, $v''+G(u)=0$ that applies to several physical phenomena. The Lie group classification of the underlying system shows that it admits a ten-dimensional equivalence Lie algebra. We also show that the principal Lie algebra in one dimension has several possible extensions, and obtain an exact solution for an interesting particular case via Noether integrals.

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Key words: Lie group classification, generalised Lane-Emden system, principal Lie algebra, equivalence transformations, Lie point symmetries, Noether integrals.

1. Introduction

For various forms of the function $f(y)$ and different values of the integer n, the generalised Lane-Emden equation

$$\frac{d^2y}{dx^2}+\frac{n}{x}\frac{dy}{dx}+f(y)=0 \tag{1.1}$$

has been used to model many phenomena in mathematical physics. When $n=2$ and $f(y)=y^r$ where r is a constant, Eq. (1.1) models the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [1–3]. Methods such as Adomian decomposition, numerical, perturbation and homotopy analysis, power series and a variational approach have been used to obtain solutions for this generalised equation — e.g. see [4] and references therein.

Leach [5] studied a modified Emden equation, which led to the symmetry based approach to equations of Lane-Emden-Fowler type. Recently, Lane-Emden systems have been

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used by various researchers to model other physical phenomena, including *inter alia* pattern formation, population evolution and chemical reactions. Existence and uniqueness results have been presented for Lane-Emden systems [7,8], and for other related systems [9–11].

Lie point symmetries of the Lane-Emden system were presented in Ref. [12], where it was shown that a Lie point symmetry of the radial Lane-Emden system

$$\frac{d^2u}{dx^2}+\left(\frac{n-1}{x}\right)\frac{du}{dx}+v^q=0\,,\qquad \frac{d^2v}{dx^2}+\left(\frac{n-1}{x}\right)\frac{dv}{dx}+u^p=0$$

is a Noether symmetry if and only if its parameters belong to the critical hyperbola

$$\frac{n}{p+1}+\frac{n}{q+1}=n-2\,.$$

A complete group classification of the nonlinear Lane-Emden system in dimension one

$$-\frac{d^2u}{dx^2}=v^q\,,\qquad -\frac{d^2v}{dx^2}=u^p \tag{1.2}$$

determined its Lie point symmetries, Noether symmetries and corresponding first integrals [13].

In this article, we discuss a generalisation of the system (1.2) where more general functions replace the power functions v^q and u^p, respectively. Thus we consider the generalised Lane-Emden system

$$\frac{d^2u}{dx^2}+H(v)=0,\qquad \frac{d^2v}{dx^2}+G(u)=0\,, \tag{1.3}$$

where $H(v)$ and $G(u)$ are arbitrary functions of v and u, respectively. Our main objective is to perform Lie group classification of the system (1.3). Group classification was first discussed by Lie [17], and many researchers have since applied Lie's methods to a wide range of problems. The group classification of the system (1.3) involves finding the Lie point symmetries of this system with arbitrary functions $H(v)$ and $G(u)$, and then determining all possible forms of $H(v)$ and $G(u)$ for which the symmetry group can be extended — cf. Ref. [18], and for applications of Lie group analysis to differential equations see Refs. [19–21] for example.

Although the group classification of the generalised Lane-Emden system

$$\frac{d^2u}{dx^2}+\frac{n}{x}\frac{du}{dx}+H(v)=0\,,\qquad \frac{d^2v}{dx^2}+\frac{n}{x}\frac{dv}{dx}+G(u)=0 \tag{1.4}$$

was performed in Ref. [14], and this system (1.3) can formally be viewed as a particular case (by setting $n=0$) of the generalised Lane-Emden system (1.4), the group classification of (1.4) was found with the restriction $n\neq 0$. Our discussion of the system (1.3) here is thus complementary to the previous discussion of the generalised Lane-Emden system (1.4), and it emerges that the principal Lie algebra for (1.3) is one-dimensional whereas it is trivial for (1.4). For completeness, we also present results on the Noether cases, which have been obtained before [15,16]. In addition, in the present article we use first integrals to obtain an exact solution of a special case of the system (1.3) — which is new for such systems, as far as we know. The equivalence transformations of the Lane-Emden system (1.3) are calculated in Section 2, and the principal Lie algebra and the group classification of the system (1.3) are discussed in Section 3. Results concerning Noether symmetries and first integrals are recalled in Section 4, and an exact solution of (1.3) for the special case $H(v) = 1/v$, $G(u) = 1/u$ using first integrals is presented in Section 5.

2. Equivalence Transformations

An equivalence transformation of the system (1.3) is an invertible transformation of the variables x, u and v that maps system (1.3) into a system of the same form, where the form of the transformed functions can differ from the form of the original functions $H(v)$ and $G(u)$ [22]. We rewrite the system (1.3) as

$$\begin{aligned} &\frac{d^2u}{dx^2} + H(v) = 0\,, \qquad \frac{d^2v}{dx^2} + G(u) = 0\,, \\ &H_x = 0\,, \quad H_u = 0\,, \quad G_x = 0\,, \quad G_v = 0\,, \end{aligned} \tag{2.1}$$

where u and v are differentiable with respect to an independent variable x, H is differentiable in x and v, and G is differentiable in x and u. The generators of the required continuous group of equivalence transformations are

$$\begin{aligned} Y =&\xi(x,u,v)\frac{\partial}{\partial x} + \eta^1(x,u,v)\frac{\partial}{\partial u} + \eta^2(x,u,v)\frac{\partial}{\partial v} \\ &+ \mu^1(x,u,v,H,G)\frac{\partial}{\partial H} + \mu^2(x,u,v,H,G)\frac{\partial}{\partial G}\,. \end{aligned} \tag{2.2}$$

We apply Lie's infinitesimal technique, using the prolongation of Y to the derivatives involved in the system (2.1) as follows (e.g. see [19]):

$$\tilde{Y} = Y + \zeta^1_x\frac{\partial}{\partial u'} + \zeta^2_x\frac{\partial}{\partial v'} + \zeta^1_{xx}\frac{\partial}{\partial u''} + \zeta^2_{xx}\frac{\partial}{\partial v''} + \mu^1_x\frac{\partial}{\partial H_x} + \mu^1_u\frac{\partial}{\partial H_u} + \mu^2_x\frac{\partial}{\partial G_x} + \mu^2_v\frac{\partial}{\partial G_v}\,,$$

where $\zeta^1_x, \zeta^2_x, \zeta^1_{xx}$ and ζ^2_{xx} are given by the usual prolongation formulas, and

$$\begin{aligned} \mu^1_x &= \tilde{D}_x(\mu^1) - H_x\tilde{D}_x(\xi) - H_u\tilde{D}_x(\eta^1)\,, \\ \mu^1_u &= \tilde{D}_u(\mu^1) - H_x\tilde{D}_u(\xi) - H_u\tilde{D}_u(\eta^1)\,, \\ \mu^2_x &= \tilde{D}_x(\mu^2) - G_x\tilde{D}_x(\xi) - G_v\tilde{D}_x(\eta^2)\,, \\ \mu^2_v &= \tilde{D}_v(\mu^2) - G_x\tilde{D}_v(\xi) - G_v\tilde{D}_v(\eta^2)\,, \end{aligned}$$

with

$$\begin{aligned}\tilde{D}_x &= \frac{\partial}{\partial x} + H_x \frac{\partial}{\partial H} + G_x \frac{\partial}{\partial G} + \cdots ,\\ \tilde{D}_u &= \frac{\partial}{\partial u} + H_u \frac{\partial}{\partial H} + G_u \frac{\partial}{\partial G} + \cdots ,\\ \tilde{D}_v &= \frac{\partial}{\partial v} + G_v \frac{\partial}{\partial G} + H_v \frac{\partial}{\partial H} + \cdots .\end{aligned}$$

The invariance test for the system (2.1) yields

$$\begin{aligned}&\tilde{Y}\left(\frac{d^2u}{dx^2} + H(v)\right)\bigg|_{(2.1)} = 0\,, \quad \tilde{Y}\left(\frac{d^2v}{dx^2} + G(u)\right)\bigg|_{(2.1)} = 0\,,\\ &\tilde{Y}(H_x)\bigg|_{(2.1)} = 0\,, \quad \tilde{Y}(H_u)\bigg|_{(2.1)} = 0\,, \quad \tilde{Y}(G_x)\bigg|_{(2.1)} = 0\,, \quad \tilde{Y}(G_v)\bigg|_{(2.1)} = 0\,, \end{aligned} \tag{2.3}$$

and on solving these equations we conclude that the system (2.1) has the ten-dimensional equivalence Lie algebra spanned by the equivalence generators

$$\begin{aligned}&X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = x\frac{\partial}{\partial x} - 2H\frac{\partial}{\partial H} - 2G\frac{\partial}{\partial G}\,, \quad X_3 = u\frac{\partial}{\partial u} + H\frac{\partial}{\partial H}\,,\\ &X_4 = v\frac{\partial}{\partial v} + G\frac{\partial}{\partial G}\,, \quad X_5 = \frac{\partial}{\partial u}, \quad X_6 = \frac{\partial}{\partial v}\,, \quad X_7 = x\frac{\partial}{\partial u}\,, \quad X_8 = x\frac{\partial}{\partial v}\,,\\ &X_9 = x^2\frac{\partial}{\partial u} - 2\frac{\partial}{\partial H}\,, \quad X_{10} = x^2\frac{\partial}{\partial v} - 2\frac{\partial}{\partial G}\,.\end{aligned}$$

Thus the ten-parameter equivalence group is given by

$$\begin{aligned}X_1 &: \ \bar{x} = x + a_1\,,\ \bar{u} = u\,,\ \bar{v} = v\,,\ \bar{H} = H\,,\ \bar{G} = G\,,\\ X_2 &: \ \bar{x} = e^{a_2}x\,,\ \bar{u} = u\,,\ \bar{v} = v\,,\ \bar{H} = e^{-2a_2}H\,,\ \bar{G} = e^{-2a_2}G\,,\\ X_3 &: \ \bar{x} = x\,,\ \bar{u} = e^{a_3}u\,,\ \bar{v} = v\,,\ \bar{H} = e^{a_3}H\,,\ \bar{G} = G\,,\\ X_4 &: \ \bar{x} = x\,,\ \bar{u} = u\,,\ \bar{v} = e^{a_4}v\,,\ \bar{H} = H\,,\ \bar{G} = e^{a_4}G\,,\\ X_5 &: \ \bar{x} = x\,,\ \bar{u} = u + a_5\,,\ \bar{v} = v\,,\ \bar{H} = H\,,\ \bar{G} = G\,,\\ X_6 &: \ \bar{x} = x\,,\ \bar{u} = u\,,\ \bar{v} = v + a_6\,,\ \bar{H} = H\,,\ \bar{G} = G\,,\\ X_7 &: \ \bar{x} = x\,,\ \bar{u} = u + a_7x\,,\ \bar{v} = v\,,\ \bar{H} = H\,,\ \bar{G} = G\,,\\ X_8 &: \ \bar{x} = x\,,\ \bar{u} = u\,,\ \bar{v} = v + a_8x\,,\ \bar{H} = H\,,\ \bar{G} = G\,,\\ X_9 &: \ \bar{x} = x\,,\ \bar{u} = u + a_9x^2\,,\ \bar{v} = v\,,\ \bar{H} = H - 2a_9\,,\ \bar{G} = G\,,\\ X_{10} &: \ \bar{x} = x\,,\ \bar{u} = u\,,\ \bar{v} = v + a_{10}x^2\,,\ \bar{H} = H\,,\ \bar{G} = G - 2a_{10}\,,\end{aligned}$$

and the composition of the above transformations gives

$$
\begin{aligned}
\bar{x} &= e^{a_2}(x + a_1)\,,\\
\bar{u} &= e^{a_3}(u + a_9 x^2 + a_7 x + a_5)\,,\\
\bar{v} &= e^{a_4}(v + a_{10} x^2 + a_8 x + a_6)\,,\\
\bar{H} &= e^{a_3 - 2a_2}(H - 2a_9)\,,\\
\bar{G} &= e^{a_4 - 2a_2}(G - 2a_{10})\,.
\end{aligned}
\tag{2.4}
$$

3. Principal Lie Algebra and Lie Group Classification

The generalised Lane-Emden system (1.3) admits a Lie point symmetry

$$X = \xi(x,u,v)\frac{\partial}{\partial x} + \eta^1(x,u,v)\frac{\partial}{\partial u} + \eta^2(x,u,v)\frac{\partial}{\partial v}$$

if

$$X^{[2]}\left(\frac{d^2u}{dx^2} + H(v)\right)\bigg|_{(1.3)} = 0\,, \quad X^{[2]}\left(\frac{d^2v}{dx^2} + G(u)\right)\bigg|_{(1.3)} = 0\,. \tag{3.1}$$

Then expanding (3.1) and separating the resulting PDE system with respect to the derivatives of u and v, we obtain an overdetermined system of fifteen PDE — viz.

$$\xi_{uu} = 0\,, \tag{3.2}$$
$$\xi_{vu} = 0\,, \tag{3.3}$$
$$\xi_{vv} = 0\,, \tag{3.4}$$
$$\eta^1_{uv} - \xi_{vx} = 0\,, \tag{3.5}$$
$$\eta^1_{uu} - 2\xi_{ux} = 0\,, \tag{3.6}$$
$$\eta^1_{vv} = 0\,, \tag{3.7}$$
$$2\eta^1_{ux} + 3H\xi_u - \xi_{xx} + G\xi_v = 0\,, \tag{3.8}$$
$$\eta^1_{xv} + H\xi_v = 0\,, \tag{3.9}$$
$$H'\eta^2 + \eta^1_{xx} - H\eta^1_u - G\eta^1_v + 2H\xi_x = 0\,, \tag{3.10}$$
$$\eta^2_{uv} - \xi_{ux} = 0\,, \tag{3.11}$$
$$\eta^2_{vv} - 2\xi_{vx} = 0\,, \tag{3.12}$$
$$\eta^2_{uu} = 0\,, \tag{3.13}$$
$$2\eta^2_{vx} + 3G\xi_v - \xi_{xx} + H\xi_u = 0\,, \tag{3.14}$$
$$\eta^2_{ux} + G\xi_u = 0\,, \tag{3.15}$$
$$G'\eta^1 + \eta^2_{xx} - H\eta^2_u - G\eta^2_v + 2G\xi_x = 0\,. \tag{3.16}$$

Solving Eqs. (3.2)-(3.4), we have

$$\xi = a(x)u + b(x)v + e(x)\,; \tag{3.17}$$

and from Eqs. (3.5)-(3.7)

$$\eta^1 = a'(x)u^2 + b'(x)vu + k(x)u + m(x)v + l(x)\,. \tag{3.18}$$

Inserting ξ and η^1 from Eqs. (3.17) and (3.18) into Eqs. (3.8)-(3.9) and solving the resulting equations yields

$$\xi = e(x)\,, \tag{3.19}$$
$$\eta^1 = c_1 v + k(x)u + l(x)\,, \tag{3.20}$$
$$2k'(x) - e''(x) = 0\,, \tag{3.21}$$

where c_1 is an arbitrary constant, and from Eqs. (3.11)-(3.13) we obtain

$$\eta^2 = E(x)u + c(x)v + d(x)\,. \tag{3.22}$$

On substituting ξ from Eq. (3.19) and η^2 from Eq. (3.22) into Eqs. (3.14)-(3.15), we then have

$$\eta^2 = c_2 u + c(x)v + d(x), \tag{3.23}$$
$$2c'(x) - e''(x) = 0\,, \tag{3.24}$$

where c_2 is an arbitrary constant. Finally, from Eq. (3.10) and Eq. (3.16) we obtain $c_1 = c_2 = 0$ and

$$\begin{aligned} &(ku + l)G'(u) + (2e' - c)G(u) + d'' = 0\,, \\ &(cv + d)H'(v) + (2e' - k)H(v) + l'' = 0\,. \end{aligned} \tag{3.25}$$

Thus in summary we have

$$\begin{aligned} &\xi = e(x)\,, \\ &\eta^1 = k(x)u + l(x)\,, \\ &\eta^2 = c(x)v + d(x)\,, \\ &e''(x) - 2k'(x) = 0\,, \\ &e''(x) - 2c'(x) = 0\,, \end{aligned}$$

and

$$\begin{aligned} &(ku + l)G'(u) + (2e' - c)G(u) + d'' = 0\,, \\ &(cv + d)H'(v) + (2e' - k)H(v) + l'' = 0\,. \end{aligned} \tag{3.26}$$

Consequently, we conclude that the principal Lie algebra of the system (1.3) is

$$X = \frac{\partial}{\partial x} \,,$$

and the classifying relations are given by

$$\begin{aligned} &(\alpha u + \beta) G'(u) + \gamma G(u) + \delta = 0 \,, \\ &(\theta v + \lambda) H'(v) + \varphi H(v) + \omega = 0 \,, \end{aligned} \tag{3.27}$$

where α, β, γ, δ, θ, λ, φ and ω are all constants.

The above classifying relations are invariant under the equivalence transformation (2.4) provided

$$\begin{aligned} &\bar{\alpha} = \alpha \,, \\ &\bar{\beta} = \alpha(a_9 x^2 + a_7 x + a_5) + \beta e^{-a_3} \,, \\ &\bar{\gamma} = \gamma \,, \\ &\bar{\delta} = \delta e^{2a_2 - a_4} - 2\gamma a_{10} \,, \\ &\bar{\theta} = \theta \,, \\ &\bar{\lambda} = \theta(a_{10} x^2 + a_8 x + a_6) + \lambda e^{-a_4} \,, \\ &\bar{\varphi} = \varphi \,, \\ &\bar{\omega} = \omega e^{2a_2 - a_3} - 2\varphi a_9 \,. \end{aligned}$$

Using the above relations, one can find the non-equivalent forms of the functions $H(v)$ and $G(u)$, prompting the seven cases presented in Table 1.

Remark 3.1. We note that all cases where the function forms of the arbitrary elements do not extend the principal Lie algebra are excluded in the preceding classification — including *inter alia* the logarithmic case. The linear case has been excluded, since one then obtains the 1-dimensional form of the biharmonic equation with known symmetries [23], and cases where the functions are constants are also excluded.

4. Noether Symmetries and First Integrals

For completeness, we now present results concerning Noether symmetries and first integrals for the system (1.3) found earlier— cf. Refs. [15,16].

Case 1. $H(v)$, $G(u)$ arbitrary.
This case provides us with a single Noether symmetry $X = \partial_x$, and the associated first integral is $I = u'v' + \int H(v)dv + \int G(u)du$.

Case 2. $H(v) = bv + a$, $G(u) = du + c$.
In this case, we obtain five Noether symmetries with corresponding first integrals — cf. Ref. [15].

Table 1: Classification results: b,d,m,n,p,q are constants, with $m,n,p,q \neq 0$.

$H(v)$	$G(u)$	Condition on const.	Symmetry operator(s)
$H(v)$ arbitrary	$G(u)$ arbitrary		$X_1 = \partial_x$.
$H(v) = bv^{-q}$	$G(u) = du^{-p}$	$b,d \neq 0$	$X_1 = \partial_x$, $X_2 = (1-qp)x\partial_x + 2u(1-q)\partial_u + 2v(1-p)\partial_v$. We recover the results obtained in Ref. [13].
$H(v) = bv^{-1}$	$G(u) = du^{-1}$	$b = d$	$X_1 = \partial_x$, $X_2 = u\partial_u - v\partial_v$, $X_3 = x\partial_x + 2u\partial_u$. We recover the results obtained in Ref. [13].
$H(v) = bv^{-3}$	$G(u) = du^{-3}$	$b = d$	$X_1 = \partial_x$, $X_2 = 2x\partial_x + u\partial_u + v\partial_v$, $X_3 = x^2\partial_x + ux\partial_u + vx\partial_v$. We recover the results obtained in Ref. [13].
$H(v) = bv^{-q}$	$G(u) = de^{-nu}$	$b,d \neq 0$	$X_1 = \partial_x$, $X_2 = qnx\partial_x + 2(q-1)\partial_u + 2nv\partial_v$.
$H(v) = be^{-mv}$	$G(u) = du^{-p}$	$b,d \neq 0$	$X_1 = \partial_x$, $X_2 = mpx\partial_x + 2mu\partial_u + 2(p-1)\partial_v$.
$H(v) = be^{-mv}$	$G(u) = de^{-nu}$	$b,d \neq 0$	$X_1 = \partial_x$, $X_2 = mnx\partial_x + 2m\partial_u + 2n\partial_v$.

Case 3. $H(v) = bv^{-q}$, $G(u) = du^{-p}$.

3.1 $p,q \neq 1$ with $2(p+q) - pq - 3 = 0$.
In this case, we have a single Noether symmetry

$$X = (1-pq)x\partial_x + 2(1-q)u\partial_x + 2(1-p)v\partial_v \ ;$$

and the associated first integral is

$$I = (1-pq)xu'v' + \frac{1-pq}{1-p}dxu^{1-p} + \frac{1-pq}{1-q}bxv^{1-q} + \frac{1-pq}{1-p}uv' + \frac{1-pq}{1-q}vu'$$

— cf. Case 5.1 in Ref. [15]. (For the special case $b = d = 1$, this result was also obtained in Ref. [13].)

3.2 $p,q = 1$ and $b = d$.
This case corresponds to two Noether symmetries — viz.

$$X_1 = \partial_x, \quad X_2 = u\partial_u - v\partial_v;$$

and the corresponding first integrals

$$I_1 = u'v - uv', \quad I_2 = u'v' + b\ln u + b\ln v$$

[15]. (For the special case $b = d = 1$, this result was also obtained in Ref. [13].)

3.3 $p, q = 3$.

In this case, we obtain a single Noether symmetry — viz.

$$X = x^2\partial_x + xu\partial_u + xv\partial_v;$$

and the corresponding Noether first integral is

$$I = x^2u'v' - x(uv' + vu') - \frac{x^2}{2}(du^{-2} + bv^{-2}) + uv$$

— cf. Ref. [16]. (For the special case $b = d = 1$, this result was also obtained in Ref. [13].)

Case 4. $H(v) = be^{-mv}$, $G(u) = de^{-nu}$.

This reduces to Case 1.

Case 5. $H(v) = a + b\ln v$, $G(u) = c + d\ln u$.

This also reduces to Case 1.

5. Exact Solution of (1.3) for a Special Case

Making use of first integrals, we can obtain an exact solution for a special case of the system (1.3) — viz. Case 3.2 of Section 4, with $H(v) = v^{-1}$ and $G(u) = u^{-1}$. There are then two Noether point symmetries and consequently two Noetherian integrals —

$$I_1 = u'v - uv' , \qquad I_2 = u'v' + \ln u + \ln v .$$

Thus the corresponding reduced equations are

$$u'v - uv' = A , \tag{5.1}$$

$$u'v' + \ln u + \ln v = B , \tag{5.2}$$

where A and B are arbitrary constants. If we set $A = B = 0$ for example, integration of Eqs. (5.1) and (5.2) yields the exact solution

$$u(x) = \frac{1}{\sqrt{C_1}} \exp\left\{-\mathrm{erf}^{-1}\left(\sqrt{\frac{2}{\pi}}(x + iC_2)\right)^2\right\} ,$$

$$v(x) = \sqrt{C_1} \exp\left\{-\mathrm{erf}^{-1}\left(\sqrt{\frac{2}{\pi}}(x + iC_2)\right)^2\right\}$$

for the system (1.3), where C_1 and C_2 are arbitrary constants and erf^{-1} is the inverse error function [24].

6. Concluding Remarks

In this work we performed the Lie group classification of the generalised coupled Lane-Emden system (1.3). This system admitted ten-dimensional equivalence Lie algebra. The principal Lie algebra was also obtained and several possible extensions of the principal Lie algebra were presented. Finally an exact solution for a special case of the system (1.3) was obtained.

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