Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. 3, No. 2, pp. 239-258

DOI: 10.4208/aamm.10-m1042
April 2011

A Stabilized Finite Element Method for Non-Stationary Conduction-Convection Problems

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Received 25 July 2010; Accepted (in revised version) 6 December 2010
Available online 5 January 2011

Abstract. This paper is concerned with a stabilized finite element method based on two local Gauss integrations for the two-dimensional non-stationary conduction-convection equations by using the lowest equal-order pairs of finite elements. This method only offsets the discrete pressure space by the residual of the simple and symmetry term at element level in order to circumvent the inf-sup condition. The stability of the discrete scheme is derived under some regularity assumptions. Optimal error estimates are obtained by applying the standard Galerkin techniques. Finally, the numerical illustrations agree completely with the theoretical expectations.

AMS subject classifications: 65N30, 65N12, 65Z05

Key words: Non-stationary conduction-convection equations, finite element method, stabilized method, stability analysis, error estimate.

1 Introduction

The non-stationary conduction-convection problems are the important dissipative nonlinear system in atmospheric dynamics. The governing equations couple viscous incompressible flow and heat transfer process [10], where the incompressible fluid is the Boussinesq approximation to the non-stationary Navier-Stokes equations. Christon [5] summarized some relevant results for the fluid dynamics of thermally driven cavity. A multigrid (MG) technique was applied for the conduction-convection problems [23, 24]. Luo [22] combined proper orthogonal decomposition (POD) with the Petrov-Galerkin least squares mixed finite element (PLSMFE) method for the problems. The mixed finite element (MFE) method is one of the important approaches for

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solving the non-stationary conduction-convection problems, where the combination of finite element subspaces satisfies the Babuska-Brezzi (BB) inequality [16, 26, 27]. It is well known that the simplest equal-order elements like the $P_1 - P_1$ triangular elements are not stable. However, the $P_1 - P_1$ elements are computationally convenient in a parallel processing and multigrid context with the simple logic and regular data structure.

Numerical experiments show that in the solution of the Stokes and Navier-Stokes problems, ensuring stability is essential if a reasonable rate of convergence of such iterations is to be achieved [19]. Some techniques that have been used to stabilize both the velocity and pressure belong to a class of residual-based methods. For example, the streaming upwind Petrov-Galerkin (SUPG) method [18], the Douglas-Wang method [11] and the well-known Galerkin least squares (GLS) method [12–14]. A common drawback in these stabilization techniques is, however, that stabilization parameters are necessarily incurred either explicitly or implicitly. Thus the development of mixed finite elements free from the stabilization parameters has become increasingly important. Stabilized mixed finite element methods are developed by using the pressure gradient projection (PGP) method [7–9] and the related local pressure gradient stabilization (LPS) method [1] in which the continuity equation is relaxed using the jumps of pressure across element interfaces. This stabilization strategy requires edge-based data structure and a subdivision of grids into patches.

Recently, based on polynomial pressure projection, a new family of stabilized methods has been proposed and studied in [2, 3]. The new method relaxes the continuity equation to enforce the BB condition in incompatible mixed spaces by using two local Gauss integral approximation [20, 21]. It does not require a specification of mesh-dependent parameters and edge-based data structure, and it always leads to symmetric problems. In addition, it is completely local at the element level. Consequently, the new stabilized method under consideration can be integrated in existing codes with very little additional coding effort.

In this paper, we extend the stabilized finite element method [21] to the non-stationary conduction-convection equations. Firstly, we use two local Gauss integral approximation in stabilized method for the lowest equal-order pairs of mixed finite elements, such as

$$P_1 - P_1 - P_1 \quad \text{or} \quad Q_1 - Q_1 - Q_1.$$

Then we derive the stability and the optimal error estimates. The results indicate that this method has the same convergence order as the usual Galerkin finite element method using the same pair of finite elements. Finally, we numerically compare this new method with other numerical method. Numerical results show that the new method takes less CPU time than the $P_1b - P_1b - P_1$ method. The numerical illustrations agree completely with the theoretical expectations.

The remainder of the paper is organized as follows. In the next section, abstract functional setting for the two-dimensional non-stationary conduction-convection problems is given with some basic statement. The stabilized finite element method is given in Section 3. Error estimates for the stabilized finite element solution are derived in Sections 4. In Section 5, numerical results completely confirm the accuracy of the stabilized method in this article. Finally, the conclusion is drawn in Section 6.

2 Function settings

Let Ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz-continuous boundary Γ, satisfying a condition stated in (A1) below. The non-stationary conduction-convection problems are:

$$u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \lambda jT, \quad \mathrm{div} u = 0, \qquad (x,t)\in\Omega\times(0,t_n), \tag{2.1a}$$
$$T_t - \lambda^{-1}\Delta T + u\cdot\nabla T = 0, \qquad (x,t)\in\Omega\times(0,t_n), \tag{2.1b}$$
$$u(x,t) = 0, \quad T(x,t) = T_0, \qquad (x,t)\in\partial\Omega\times(0,t_n), \tag{2.1c}$$
$$u(x,0) = 0, \quad T(x,0) = 0, \qquad x\in\Omega, \tag{2.1d}$$

where $u{=}u(x,t){=}(u_1(x,t),u_2(x,t))$ represents the velocity vector, $p{=}p(x,t)$ the pressure, $T{=}T(x,t)$ the temperature, $\nu{=}\sqrt{Pr/Re}$ the viscosity, $\lambda{=}\sqrt{PrRe}$ the Grashoff number, Re the Reynolds number, Pr the Prandtl number, $j{=}(0,1)$ is a 2-D vector, $t_n{>}0$ the final time, and $u_t{=}\partial u/\partial t$.

To introduce a variational formulation, let

$$\begin{aligned}
&X = H_0^1(\Omega)^2, \qquad Y = L^2(\Omega)^2, \qquad W = H^1(\Omega),\\
&W_0 = H_0^1(\Omega), \qquad M = L_0^2(\Omega) = \Big\{q\in L^2(\Omega)\Big|\int_\Omega q dx = 0\Big\},\\
&V = \{v\in X \mid \mathrm{div} v = 0\}, \qquad D(A) = \big(H^2(\Omega)\big)^2\cap V.
\end{aligned}$$

As noted, further assumptions on Ω are needed : (see [15,17,19,21,22,28])

(**A1**) $\partial\Omega$ is of C^2, or Ω is a two-dimensional convex polygon so that the unique solution $(v,q){\in}(X,M)$ of the steady Stokes problems

$$-\nu\Delta v + \nabla q = g, \quad \mathrm{div} v = 0, \quad v|_{\partial\Omega} = 0,$$

for prescribed $g{\in}Y$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \le c\|g\|_0,$$

where $c{>}0$ is a generic constant depending on the data (ν,Ω). We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm on $L^2(\Omega)$ or $(L^2(\Omega))^2$. $\|\cdot\|_i$ denotes the usual norm of Sobolev spaces $H^i(\Omega)$ or $H^i(\Omega)^2$, ($i{=}1,2$). The spaces $H_0^1(\Omega)$ and X are equipped with their usual scalar product and norm

$$((u,v)) = (\nabla u,\nabla v), \quad \|u\|_1^2 = ((u,u)).$$

We note the following Poincare inequality:

$$\|v\|_0 \le \gamma\|v\|_1, \quad \forall v\in X, \tag{2.2}$$

where γ is a positive constant depending only on Ω.

(**A2**) The extension of boundary data T_0 (still T_0) satisfy

$$\sup_{0<t\leq t_n}\big(\|T_0\|_2+\|T_{0t}\|_1\big)\leq C, \tag{2.3}$$

for some constant C. The continuous bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X\times X$, $X\times M$ and $W\times W$ respectively by

$$a(u,v)=\nu(\nabla u,\nabla v),\;\; b(v,q)=-(v,\nabla q)=(\mathrm{div}v,q),\;\; d(T,\phi)=\lambda^{-1}(\nabla T,\nabla\phi),$$

and the trilinear terms $a_1(\cdot,\cdot,\cdot)$ and $a_2(\cdot,\cdot,\cdot)$, on $X\times X\times X$ and $X\times W\times W_0$, by

$$\begin{aligned}
a_1(u,v,w)&=\big((u\cdot\nabla)v,w\big)+\frac{1}{2}\big((\mathrm{div}u)v,w\big)\\
&=\frac{1}{2}\big((u\cdot\nabla)v,w\big)-\frac{1}{2}\big((u\cdot\nabla)w,v\big),\quad \forall u,v,w\in X\times X\times X,\\
a_2(u,T,\phi)&=\big(u\cdot(\nabla T),\phi\big)+\frac{1}{2}\big((\mathrm{div}u)T,\phi\big),\qquad \forall u,T,\phi\in X\times W\times W_0.
\end{aligned}$$

(**A3**) Moreover, $a_1(\cdot,\cdot,\cdot)$ and $a_2(\cdot,\cdot,\cdot)$ satisfy the following properties:

$$\begin{aligned}
&a_1(u,v,w)=-a_1(u,w,v),\qquad a_2(u,T,\phi)=-a_2(u,\phi,T),\\
&|a_1(u,v,w)|+|a_1(v,u,w)|+|a_1(w,v,u)|\leq c\|u\|_1\|v\|_2\|w\|_0,\\
&|a_2(u,v,w)|+|a_2(v,u,w)|+|a_2(w,v,u)|\leq c\|u\|_1\|v\|_2\|w\|_0,\\
&|a_1(u,v,w)|+|a_1(w,v,u)|\leq c\|u\|_0^{\frac{1}{2}}\|u\|_1^{\frac{1}{2}}\Big(\|v\|_0^{\frac{1}{2}}\|v\|_1^{\frac{1}{2}}\|w\|_1+\|w\|_0^{\frac{1}{2}}\|w\|_1^{\frac{1}{2}}\|v\|_1\Big),\\
&|a_2(u,v,w)|+|a_2(w,v,u)|\leq c\|u\|_1\|w\|_1\|v\|_1.
\end{aligned}$$

Under the above notations, the mixed variational form of Eqs. (2.1a) and (2.1b) is to seek $(u,T,p)\in(X,W,M)$, $t\in(0,t_n)$, such that, for all $(v,\phi,q)\in(X,W_0,M)$,

$$(u_t,v)+B((u,p);(v,q))+a_1(u,u,v)=\lambda(jT,v), \tag{2.4a}$$

$$(T_t,\phi)+d(T,\phi)+a_2(u,T,\phi)=0, \tag{2.4b}$$

$$T(x,t)|_\Gamma=T_0,\quad T(x,0)=0,\quad u(x,0)=0, \tag{2.4c}$$

where $B((u,p);(v,q))=a(u,v)-b(v,p)+b(u,q)$.

For convenience, we recall the Gronwall Lemma that will be frequently used.

Lemma 2.1. ([25]) *Let $g(t),l(t)$ and $\xi(t)$ be three nonnegative functions satisfying, for $t\in[0,t_n]$,*

$$\xi(t)+G(t)\leq c+\int_0^t l ds+\int_0^t g\xi ds,$$

where $G(t)$ is a nonnegative function on $[0,t_n]$. Then

$$\xi(t)+G(t)\leq\Big(c+\int_0^t l ds\Big)\exp\Big(\int_0^t g ds\Big). \tag{2.5}$$

The following result concerning the existence, uniqueness, and regularity of a solution to the conduction-convection equations is presented under the Assumptions (A1)-(A3).

Lemma 2.2. *Assume that* (A1)-(A3) *hold. Then, for any given* $t_n>0$*, there exists a unique solution* (u, T, p) *satisfying the following regularities:*

$$\sup_{0<t\le t_n}\big(\|u(t)\|_2+\|T(t)\|_2+\|p(t)\|_1+\|u_t(t)\|_0+\|T_t(t)\|_0\big) \\ +\int_0^{t_n}\|u_t\|_1^2+\|T_t\|_1^2 ds\le C, \tag{2.6a}$$

$$\sup_{0<t\le t_n}\tau(t)\|u_t\|_1^2+\int_0^{t_n}\tau(t)\big(\|u_t\|_2^2+\|p_t\|_1^2+\|u_{tt}\|_0^2\big)dt\le C, \tag{2.6b}$$

where $\tau(t)=\min\{1,t\}$.

Proof. Eq. (2.6a) was given by DiBenedetto et al. in [10]. Similarly as the method in [17], we can obtain (2.6b). We omit the proof here. ☐

3 Stabilized finite element method

Let $h>0$, finite element subspaces $(X_h, W_h, M_h)\subseteq(X, W, M)$, which are associated with τ_h, a subdivision of Ω into triangles or quadrilaterals, assumed to be regular in the usual sense [4,6]. For all $K\in\tau_h$, $R_1(K)=Q_1(K)$ if K is quadrilateral, and $R_1(K)=P_1(K)$ if K is triangular. We assume that for the finite element spaces W_h, the following approximation properties hold: for $\phi\in H^2(\Omega)$, there exist approximation $r_h\phi\in W_h$ such that

$$\|\phi-r_h\phi\|_0+h\|\phi-r_h\phi\|_1\le ch^2\|\phi\|_2, \tag{3.1}$$

where the H_1-projection $r_h: W\to W_h$

$$(\nabla T-\nabla r_hT,\nabla\phi_h)=0,\qquad \forall T\in W,\ \ \phi_h\in W_h. \tag{3.2}$$

We also assume that the inverse inequality holds [4,6]

$$\|v_h\|_1\le ch^{-1}\|v_h\|_0,\quad \forall v_h\in X_h. \tag{3.3}$$

The finite element subspaces of this paper are defined by giving the continues piecewise (bi)linear velocity subspace:

$$X_h=\big\{v\in X;\ v_i|_K\in R_1(K),\ i=1,2,\ \forall K\in\tau_h\big\},$$

the continues piecewise (bi)linear temperature subspace:

$$W_h=\big\{T\in W;\ T|_K\in R_1(K),\ \forall K\in\tau_h\big\},$$

and the continues piecewise (bi)linear pressure subspace:

$$M_h = \{T \in M;\ p|_K \in R_1(K),\ \forall K \in \tau_h\}.$$

It is well known that this lowest equal-order finite element pair does not satisfy the inf-sup condition. We define the following local difference between a consistent and an under-integrated mass matrices the stabilized formulation [20]

$$G(p_h, q_h) = \bar{p}^T(M_k - M_1)\bar{q} = \bar{p}^T M_k \bar{q} - \bar{p}^T M_1 \bar{q}. \tag{3.4}$$

Here, let

$$\begin{aligned}
&\bar{p}^T = [p_0, p_1, \cdots, p_{N-1}]^T, \quad \bar{q} = [q_0, q_1, \cdots, q_{N-1}], \quad M_{ij} = (\varphi_i, \varphi_j),\\
&p_h = \sum_{i=0}^{N-1} p_i \varphi_i, \quad p_i = p_h(x_i), \quad \forall p_h \in M_h,\ \ i, j = 0, 1, \cdots, N-1,
\end{aligned}$$

where φ_i is the basis function of the pressure on the domain Ω such that its value is one at node x_i and zero at other nodes; the symmetric and positive M_k and M_1 are pressure mass matrix computed by using k-order and 1-order Gauss integrations in each direction, respectively; Also, p_i and $q_i, i = 0, 1, \cdots, N-1$ are the value of p_h and q_h at the node x_i, $\bar{p}^T$ is the transpose of the matrix $\bar{p}$.

Let $\Pi_h : M \to R_0$ be the L^2-projection with the following properties [20]:

$$(p - \Pi_h p, q_h) = 0, \qquad \forall p \in M,\ q_h \in R_0, \tag{3.5a}$$

$$\|\Pi_h p\|_0 \le c\|p\|_0, \qquad \forall p \in M, \tag{3.5b}$$

$$\|p - \Pi_h p\|_0 \le ch\|p\|_1, \qquad \forall p \in H^1(\Omega) \cap M, \tag{3.5c}$$

where

$$R_0 = \{q_h \in M;\ q_h|_K \text{ is a constant},\ \forall K \in \tau_h\}.$$

Then we can rewrite the bilinear form $G(\cdot,\cdot)$ by

$$G(p, q) = (p - \Pi_h p, q - \Pi_h q). \tag{3.6}$$

Using the above notation, the stabilized variational formulation of problems (2.1a) and (2.1b) reads: find $(u_h, T_h, p_h) \in (X_h, W_h, M_h)$, such that, for all $(v_h, \phi_h, q_h) \in (X_h, W_{0h}, M_h)$,

$$(u_{ht}, v_h) + B_h((u_h, p_h); (v_h, q_h)) + a_1(u_h, u_h, v_h) = \lambda(jT_h, v_h), \tag{3.7a}$$

$$(T_{ht}, \phi_h) + d(T_h, \phi_h) + a_2(u_h, T_h, \phi_h) = 0, \tag{3.7b}$$

$$T_h|_{\partial\Omega} = T_{0h}, \quad T_h(0) = 0, \quad u_h(0) = 0, \tag{3.7c}$$

where T_{0h} is an approximation of T_0, W_{0h}=$W_h \cap W_0$ and

$$B_h((u_h, p_h); (v_h, q_h)) = a(u_h, v_h) - b(v_h, p_h) + b(u_h, q_h) + G(p_h, q_h), \tag{3.8}$$

is the new stabilized bilinear form. The following theorem establishes the weak coercivity of (3.8) for the equal-order finite element pairs $R_1 - R_1 - R_1$.

Theorem 3.1. *(see [20]) Let (X_h, W_h, M_h) be defined as above. Then there exists a positive constant β, independent of h, such that*

$$|B_h((u,p);(v,q))| \leq c\big(\|u\|_1 + \|p\|_0\big)\big(\|v\|_1 + \|q\|_0\big), \qquad \forall (u,p),(v,q) \in (X,M), \tag{3.9a}$$

$$\beta\big(\|u_h\|_1 + \|p_h\|_0\big) \leq \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{B_h((u_h,p_h);(v_h,q_h))}{\|v_h\|_1 + \|q_h\|_0}, \quad \forall (u_h,p_h) \in (X_h,M_h), \tag{3.9b}$$

$$|G(p,q)| \leq c\|p - \Pi_h p\|_0 \|q - \Pi_h q\|_0, \qquad \forall p,q \in M. \tag{3.9c}$$

4 Error analysis

In order to derive error estimates of the finite element solution (u_h, T_h, p_h), we also define the Galerkin projection $(R_h, Q_h) : (X,M) \to (X_h, M_h)$ by requiring

$$B_h((R_h(u,p), Q_h(u,p));(v_h,q_h)) = B((u,p);(v_h,q_h)), \ \forall (u,p) \in (X,M),\ (v_h,q_h) \in (X_h,M_h),$$

which is well defined and satisfies the following approximation properties [21]:

$$\|R_h(u,p) - u\|_0 + h\big(\|R_h(u,p) - u\|_1 + \|Q_h(u,p) - p\|_0\big) \leq ch^2(\|u\|_2 + \|p\|_1), \tag{4.1}$$

for all $(u,p)\in(H^2(\Omega)\cap X, H^1(\Omega)\cap M)$.

Lemma 4.1. *Under the Assumptions of* (A1)-(A3)*, for $t\in[0,t_n]$, (u_h, T_h, p_h) satisfies*

$$\|u_h\|_0^2 + \nu\int_0^t \|u_h\|_1^2 + G(p_h,p_h)ds \leq c, \tag{4.2a}$$

$$\|T_h\|_0^2 + \lambda^{-1}\int_0^t \|T_h\|_1^2 ds \leq c. \tag{4.2b}$$

Proof. Taking $(v_h,q_h)=(u_h,p_h)$ in (3.7a), and using (2.2) and (A3), we have

$$\frac{d}{dt}\|u_h\|_0^2 + 2\nu\|u_h\|_1^2 + G(p_h,p_h) \leq \nu\|u_h\|_1^2 + c\|T_h\|_0^2. \tag{4.3}$$

Integrating (4.3) from 0 to t, and noting $u_h(0)=0$, which yields

$$\|u_h\|_0^2 + \nu\int_0^t \|u_h\|_1^2 + G(p_h,p_h)ds \leq c + c\int_0^t \|T_h\|_0^2 ds. \tag{4.4}$$

Choosing $\phi_h=T_h - r_hT_0$ in (3.7b), and using (3.2), we derive

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|T_h - r_hT_0\|_0^2 + \lambda^{-1}\|T_h\|_1^2 - a_2(u_h,T_h,r_hT_0)\\ =&\lambda^{-1}(\nabla T_h, \nabla T_0) - (r_hT_{0t}, T_h - r_hT_0).\end{aligned} \tag{4.5}$$

Due to (3.1) and (A3), one finds

$$|a_2(u_h,T_h,r_hT_0)| \leq c\|u_h\|_1\|T_h\|_1\|r_hT_0\|_1 \leq \frac{1}{8}\lambda^{-1}\|T_h\|_1^2 + c\|T_0\|_2^2\|u_h\|_1^2,$$

$$|(r_hT_{0t}, T_h - r_hT_0)| \leq \|r_hT_{0t}\|_0\|r_hT_0\|_0 + c\|r_hT_{0t}\|_0\|T_h\|_1 \leq \|T_0\|_2^2 + c\|T_{0t}\|_1^2 + \frac{1}{8}\lambda^{-1}\|T_h\|_1^2.$$

Combining the above estimates and integrating (4.5) for t from 0 to t, using (A2) and noting

$$\|T_h(0)-r_hT_0(0)\|_0^2\leq 2\|T_0(0)-r_hT_0(0)\|_0^2+2\|T_0(0)\|_0^2\leq c, \tag{4.6}$$

which implies

$$\|T_h-r_hT_0\|_0^2+\lambda^{-1}\int_0^t\|T_h\|_1^2ds\leq c+c\int_0^t\|u_h\|_1^2. \tag{4.7}$$

Combining (4.7) with (4.4), and applying the Gronwall Lemma, we obtain (4.2a) and (4.2b). □

Theorem 4.1. *Under the Assumptions of* (A1)-(A3), *for* $t\in[0,t_n]$, (u_h,T_h,p_h) *satisfies*

$$\nu\|u_h\|_1^2+G(p_h,p_h)+\int_0^t\|u_{ht}\|_0^2ds\leq c, \tag{4.8a}$$

$$\|u-u_h\|_0^2+\int_0^t\nu\|u-u_h\|_1^2+G(\mu_h,\mu_h)ds\leq ch^2, \tag{4.8b}$$

$$\lambda^{-1}\|T_h\|_1^2+\int_0^t\|T_{ht}\|_0^2ds\leq c, \tag{4.8c}$$

$$\|T-T_h\|_0^2+\lambda^{-1}\int_0^t\|T-T_h\|_1^2ds\leq ch^2, \tag{4.8d}$$

where $\mu_h=Q_h(u,p)-p_h$.

Proof. By differentiating the terms $b(u_h,q_h)+G(p_h,q_h)$ in (3.7a) with respect to t and taking $(v_h,q_h)=(u_{ht},p_h)$, we see that

$$\|u_{ht}\|_0^2+\frac{1}{2}\frac{d}{dt}\big(\nu\|u_h\|_1^2+G(p_h,p_h)\big)+a_1(u,u_h,u_{ht})-a_1(u-u_h,u_h,u_{ht})=\lambda(jT_h,u_{ht}). \tag{4.9}$$

Thanks to (A3) and (3.3), we arrive at

$$\begin{aligned}
&|a_1(u,u_h,u_{ht})|\leq c\|u\|_2\|u_h\|_1\|u_{ht}\|_0\leq\frac{1}{8}\|u_{ht}\|_0^2+c\|u\|_2^2\|u_h\|_1^2,\\
&|a_1(u-u_h,u_h,h_{ht})|\leq c\|u_{ht}\|_0^{\frac{1}{2}}\|u_{ht}\|_1^{\frac{1}{2}}\Big(\|u-u_h\|_0^{\frac{1}{2}}\|u-u_h\|_1^{\frac{1}{2}}\|u_h\|_1+\|u_h\|_0^{\frac{1}{2}}\|u_h\|_1^{\frac{1}{2}}\|u-u_h\|_1\Big)\\
&\qquad\qquad\leq ch^{-1}\|u_{ht}\|_0\Big(\|u-u_h\|_0^{\frac{1}{2}}\|u-u_h\|_1^{\frac{1}{2}}\|u_h\|_0^{\frac{1}{2}}\|u_h\|_1^{\frac{1}{2}}+\|u_h\|_0\|u-u_h\|_1\Big),\\
&\lambda(jT_h,u_{ht})\leq c\|T_h\|_0\|u_{ht}\|_0\leq\frac{1}{8}\|u_{ht}\|_0^2+c\|T_h\|_0^2.
\end{aligned}$$

Combining the above estimates with (4.9) and noting (2.6a), Lemma 4.1, we derive

$$\begin{aligned}
&\|u_{ht}\|_0^2+\frac{d}{dt}\big(\nu\|u_h\|_1^2+G(p_h,p_h)\big)\\
\leq& c+c\|u_h\|_1^2+ch^{-2}\|u-u_h\|_1^2+ch^{-2}\|u-u_h\|_0^2\|u_h\|_1^2.
\end{aligned} \tag{4.10}$$

Integrating (4.10) for t from 0 to t and noting

$$\nu\|u_h(0)\|_1^2+G\big(p_h(0),p_h(0)\big)\leq c\big(\|u_h(0)\|_1^2+\|p_h(0)\|_0^2\big)\leq c,$$

we obtain

$$\begin{aligned}&\nu\|u_h\|_1^2+G(p_h,p_h)+\int_0^t\|u_{ht}\|_0^2ds\\ \leq&c+ch^{-2}\int_0^t\|u-u_h\|_1^2ds+ch^{-2}\int_0^t\|u-u_h\|_0^2\|u_h\|_1^2ds.\end{aligned}\tag{4.11}$$

By subtracting (3.7a) from (2.4a), we find

$$\begin{aligned}&(u_t-u_{ht},v_h)+B_h((e_h,\mu_h);(v_h,q_h))+a_1(u,u-u_h,v_h)\\ &\quad+a_1(u-u_h,u,v_h)-a_1(u-u_h,u-u_h,v_h)\\ &=\lambda\big(j(T-T_h),v_h\big),\quad\forall(v_h,q_h)\in(X_h,M_h),\end{aligned}\tag{4.12}$$

where

$$e_h=R_h(u,p)-u_h,\quad\mu_h=Q_h(u,p)-p_h,\quad u-u_h=w_h+e_h.$$

Then setting (v_h,q_h)=(e_h,μ_h) in (4.12), it follows that

$$\begin{aligned}&(u_t-u_{ht},e_h)+\nu\|e_h\|_1^2+G(\mu_h,\mu_h)+a_1(u-u_h,u,e_h)\\ &\quad+a_1(u_h,w_h,e_h)=\lambda(j(T-T_h),e_h).\end{aligned}\tag{4.13}$$

Due to (A3), we have

$$\begin{aligned}&|a_1(u-u_h,u,e_h)|\leq c\|u\|_2\|e_h\|_1\|u-u_h\|_0\leq\frac{1}{8}\nu\|e_h\|_1^2+c\|u\|_2^2\|u-u_h\|_0^2,\\ &|a_1(u_h,w_h,e_h)|\leq c\|u_h\|_1\|e_h\|_1\|w_h\|_1\leq\frac{1}{8}\nu\|e_h\|_1^2+c\|u_h\|_1^2\|w_h\|_1^2,\\ &|\lambda(j(T-T_h),e_h)|\leq\frac{1}{8}\nu\|e_h\|_1^2+c\|T-T_h\|_0^2.\end{aligned}$$

Combining the above estimates with (4.13), and using (2.6a) and (4.1), which implies

$$\begin{aligned}&\frac{d}{dt}\|u-u_h\|_0^2+\nu\|e_h\|_1^2+G(\mu_h,\mu_h)\\ \leq&ch^2\|u_t-u_{ht}\|_0+c\|T-T_h\|_0^2+ch^2\|u_h\|_1^2+c\|u-u_h\|_0^2.\end{aligned}\tag{4.14}$$

Integrating (4.14) for t from 0 to t, using Lemma 4.1 and noting $u(0)$=$u_h(0)$=0, which yields

$$\begin{aligned}&\|u-u_h\|_0^2+\int_0^t\nu\|e_h\|_1^2+G(\mu_h,\mu_h)ds\\ \leq&ch^2+ch^2\Big(\int_0^t\|u_t\|_0^2+\|u_{ht}\|_0^2ds\Big)^{\frac{1}{2}}+c\int_0^t\Big(\|u-u_h\|_0^2+\|T-T_h\|_0^2\Big)ds.\end{aligned}\tag{4.15}$$

By subtracting (3.7b) from (2.4b), we obtain

$$\begin{aligned}&(T_t - T_{ht}, \phi_h) + d(T - T_h, \phi_h) + a_2(u - u_h, T, \phi_h) + a_2(u_h, T - T_h, \phi_h)\\ =&0, \quad \forall \phi_h \in W_{0h}.\end{aligned} \tag{4.16}$$

Choosing $\phi_h = r_h T - T_h$ in (4.16), using (A3) and (3.2), we see that

$$\begin{aligned}&(T_t - T_{ht}, r_h T - T_h) + \lambda^{-1}\|r_h T - T_h\|_1^2 + a_2(u - u_h, T, r_h T - T_h)\\ &\quad + a_2(u_h, T - r_h T, r_h T - T_h) = 0.\end{aligned} \tag{4.17}$$

Applying (A3), one finds

$$\begin{aligned}|a_2(u - u_h, T, r_h T - T_h)| \le& c\|T\|_1\|u - u_h\|_1\|r_h T - T_h\|_1\\ \le& \frac{1}{8}\lambda^{-1}\|r_h T - T_h\|_1^2 + c\|T\|_1^2\|u - u_h\|_1^2,\\ |a_2(u_h, T - r_h T, r_h T - T_h)| \le& c\|u_h\|_1\|r_h T - T_h\|_1\|T - r_h T\|_1\\ \le& \frac{1}{8}\lambda^{-1}\|r_h T - T_h\|_1^2 + c\|u_h\|_1^2\|T - r_h T\|_1^2,\\ |(T_t - r_h T_t, r_h T - T_h)| \le& \frac{1}{8}\lambda^{-1}\|r_h T - T_h\|_1^2 + c\|T_t - r_h T_t\|_0^2\\ \le& \frac{1}{8}\lambda^{-1}\|r_h T - T_h\|_1^2 + ch^2\|T_t\|_1^2.\end{aligned}$$

Combining the above estimates with (4.17), and using (2.6a), (3.1), we have

$$\begin{aligned}&\frac{d}{dt}\|r_h T - T_h\|_0^2 + \lambda^{-1}\|r_h T - T_h\|_1^2\\ \le& ch^2\|T_t\|_1^2 + ch^2\|u_h\|_1^2 + c\|u - u_h\|_1^2.\end{aligned} \tag{4.18}$$

Integrating (4.18) for t from 0 to t, using Lemma 4.1 and noting $T_h(0) = r_h T(0)$, we obtain

$$\|T - T_h\|_0^2 + \int_0^t \lambda^{-1}\|T - T_h\|_1^2 ds \le ch^2 + c\int_0^t \|u - u_h\|_1^2 ds. \tag{4.19}$$

Using (4.15), (4.19), and applying Gronwall Lemma, we derive

$$\|u - u_h\|_0^2 + \int_0^t \nu\|u - u_h\|_1^2 ds \le ch^2 + ch^2\Big(\int_0^t \|u_{ht}\|_0^2 ds\Big)^{\frac{1}{2}}, \tag{4.20a}$$

$$\|T - T_h\|_0^2 + \int_0^t \nu\|T - T_h\|_1^2 ds \le ch^2 + ch^2\Big(\int_0^t \|u_{ht}\|_0^2 ds\Big)^{\frac{1}{2}}. \tag{4.20b}$$

Combining (4.20a), (4.20b) with (4.11), which yields (4.8a), (4.8b) and (4.8d). Choosing $\phi_h = T_{ht} - r_h T_{0t}$ in (3.7b), and using (3.2) we derive

$$\begin{aligned}&\|T_{ht}\|_0^2 + \frac{1}{2}\frac{d}{dt}\|T_h\|_1^2 + a_2(u_h, T_h - T, T_{ht} - r_h T_{0t})\\ &\quad + a_2(u_h, T, T_{ht} - r_h T_{0t}) = (T_{ht}, r_h T_{0t}) + d(T_h, r_h T_{0t}).\end{aligned} \tag{4.21}$$

Due to (A3), (3.1) and (3.3), one finds

$$
\begin{aligned}
a_2(u_h, T_h - T, T_{ht} - r_h T_{0t}) &\le ch^{-1}\|u_h\|_1\|T - T_h\|_1\|T_{ht} - r_h T_{0t}\|_0 \\
&\le \frac{1}{8}\|T_{ht}\|_0^2 + c\|T_{0t}\|_1^2 + ch^{-2}\|u_h\|_1^2\|T - T_h\|_1^2, \\
a_2(u_h, T, T_{ht} - r_h T_{0t}) &\le c\|u_h\|_1\|T\|_2\|T_{ht} - r_h T_{0t}\|_0 \\
&\le \frac{1}{8}\|T_{ht}\|_0^2 + c\|T_{0t}\|_1^2 + c\|T\|_2^2\|u_h\|_1^2, \\
|(T_{ht}, r_h T_{0t})| \le \frac{1}{8}\|T_{ht}\|_0^2 + c\|T_{0t}\|_1^2, &\qquad |d(T_h, r_h T_{0t})| \le c\|T_h\|_1\|T_{0t}\|_1.
\end{aligned}
$$

Combining the above estimates and integrating (4.21) for t from 0 to t, using (A2) and noting

$$\|T_h(0)\|_1 = \|r_h T(0)\|_1 \le c,$$

we have

$$\|T_h\|_1^2 + \int_0^t \|T_{ht}\|_0^2 ds \le ch^{-2}\int_0^t \|T - T_h\|_1^2 ds + c\int_0^t \|T\|_2^2\|u_h\|_1^2 ds + c\int_0^t \|T_h\|_1^2 ds. \tag{4.22}$$

Combining (4.22) with (4.8d), and applying Lemma 4.1, we obtain (4.8c). $\square$

Lemma 4.2. *Under the Assumptions of* (A1)-(A3)*, for* $t\in[0, t_n]$*,* (u_h, T_h, p_h) *satisfies*

$$\|u_{ht}\|_0^2 + \int_0^t \nu\|u_{ht}\|_1^2 + G(p_{ht}, p_{ht}) ds \le c, \tag{4.23a}$$

$$\tau(t)(\nu\|u_{ht}\|_1^2 + G(p_{ht}, p_{ht})) + \int_0^t \tau(s)\|u_{htt}\|_0^2 ds \le c. \tag{4.23b}$$

Proof. First, by differentiating (3.7a) with respect to t, we see that

$$(u_{htt}, v_h) + B_h((u_{ht}, p_{ht}); (v_h, q_h)) + a_1(u_{ht}, u_h, v_h) + a_1(u_h, u_{ht}, v_h) = \lambda(jT_{ht}, v_h). \tag{4.24}$$

Then taking (v_h, q_h)=(u_{ht}, p_{ht}), we have

$$\frac{1}{2}\frac{d}{dt}\|u_{ht}\|_0^2 + \nu\|u_{ht}\|_1^2 + a_1(u_{ht}, u_h, u_{ht}) + G(p_{ht}, p_{ht}) = \lambda(jT_{ht}, u_{ht}). \tag{4.25}$$

Due to (A3) and Theorem 4.1, we find

$$
\begin{aligned}
&|a_1(u_{ht}, u_h, u_{ht})| \le c\|u_h\|_0^{\frac{1}{2}}\|u_h\|_1^{\frac{1}{2}}\|u_{ht}\|_0^{\frac{1}{2}}\|u_{ht}\|_1^{\frac{3}{2}} \le \frac{1}{8}\nu\|u_{ht}\|_1^2 + c\|u_{ht}\|_0^2, \\
&|\lambda(jT_{ht}, u_{ht})| \le \lambda\|T_{ht}\|_0\|u_{ht}\|_0 \le \frac{1}{8}\nu\|u_{ht}\|_1^2 + c\|T_{ht}\|_0^2.
\end{aligned}
$$

Integrating (4.25) for t from 0 to t, using Theorem 4.1 we obtain (4.23a).

By differentiating terms $b(u_{ht}, q_h) + G(p_{ht}, q_h)$ in (4.24), and choosing (v_h, q_h) $=(u_{htt}, p_{ht})$, which implies

$$
\begin{aligned}
\|u_{htt}\|_0^2 + \frac{1}{2}\frac{d}{dt}(\nu\|u_{ht}\|_1^2 + G(p_{ht}, p_{ht})) + a_1(u_{ht}, u_h, u_{htt}) \\
+ a_1(u_h, u_{ht}, u_{htt}) = \lambda(jT_{ht}, u_{htt}).
\end{aligned} \tag{4.26}
$$

Thanks to (A3) and (3.3), we arrive at

$$\begin{aligned}
a_1(u, u_{ht}, u_{htt}) &\le c\|u\|_2\|u_{ht}\|_1\|u_{htt}\|_0 \le \frac{1}{8}\|u_{htt}\|_0^2 + c\|u\|_2^2\|u_{ht}\|_1^2,\\
a_1(u-u_h, u_{ht}, u_{htt}) &\le \|u_{htt}\|_0^{\frac12}\|u_{htt}\|_1^{\frac12}\big(\|u_{ht}\|_0^{\frac12}\|u_{ht}\|_1^{\frac12}\|u-u_h\|_1\\
&\quad + \|u-u_h\|_0^{\frac12}\|u-u_h\|_1^{\frac12}\|u_{ht}\|_1\big)\\
&\le \frac{1}{8}\|u_{htt}\|_0^2 + ch^{-2}\|u_{ht}\|_0^2\|u-u_h\|_1^2\\
&\quad + ch^{-1}\|u-u_h\|_0\|u_{ht}\|_1^2(\|u\|_1 + \|u_h\|_1),\\
|\lambda(jT_{ht}, u_{htt})| &\le \frac{1}{8}\|u_{htt}\|_0^2 + c\|T_{ht}\|_0^2.
\end{aligned}$$

Combining the above estimates with (4.26), and noting (2.6a), Theorem 4.1, (4.23a), we derive

$$\|u_{htt}\|_0^2 + \frac{d}{dt}\big(\nu\|u_{ht}\|_1^2 + G(p_{ht}, p_{ht})\big) \le c\|T_{ht}\|_0^2 + c\|u_{ht}\|_1^2 + ch^{-2}\|u-u_h\|_1^2. \tag{4.27}$$

Multiplying (4.27) by $\tau(t)$, integrating t from 0 to t, and using Theorem 4.1 and (4.23a), we obtain (4.23b). □

Lemma 4.3. *Under the Assumptions of* (A1)-(A3), *for* $t\in[0, t_n]$, (u_h, T_h, p_h) *satisfies*

$$\tau(t)\nu\|u-u_h\|_1^2 + \int_0^t \tau(s)\|u_t - u_{ht}\|_0^2 ds \le ch^2, \tag{4.28a}$$

$$\tau(t)\lambda^{-1}\|T-T_h\|_1^2 + \int_0^t \tau(s)\|T_t - T_{ht}\|_0^2 ds \le ch^2. \tag{4.28b}$$

Proof. Differentiating the term $b(e_h, q_h) + G(\mu_h, q_h)$ in (4.12), and taking (v_h, q_h) $=(e_{ht}, \mu_h)$, it follows that

$$\begin{aligned}
&\|e_{ht}\|_0^2 + (w_{ht}, e_{ht}) + \frac{1}{2}\frac{d}{dt}(\nu\|e_h\|_1^2 + G(\mu_h, \mu_h)) + a_1(u-u_h, u, e_{ht})\\
&+ a_1(u, u-u_h, e_{ht}) + a_1(u-u_h, u-u_h, e_{ht}) = \lambda(j(T-T_h), e_{ht}).
\end{aligned} \tag{4.29}$$

Due to (A3) and (3.3), one finds

$$\begin{aligned}
|a_1(u-u_h, u, e_{ht})| &\le \|u-u_h\|_1\|u\|_2\|e_{ht}\|_0 \le \frac{1}{8}\|e_{ht}\|_0^2 + c\|u\|_2^2\|u-u_h\|_1^2,\\
|a_1(u-u_h, u-u_h, e_{ht})| &\le \|e_{ht}\|_0^{\frac12}\|e_{ht}\|_1^{\frac12}\|u-u_h\|_0^{\frac12}\|u-u_h\|_1^{\frac32}\\
&\le \frac{1}{8}\|e_{ht}\|_0^2 + ch^{-1}\|u-u_h\|_0\|u-u_h\|_1^3,\\
|\lambda(j(T-T_h), e_{ht})| &\le \frac{1}{8}\|e_{ht}\|_0^2 + c\|T-T_h\|_0^2,\\
|(w_{ht}, e_{ht})| &\le \frac{1}{8}\|e_{ht}\|_0^2 + c\|w_{ht}\|_0^2.
\end{aligned}$$

Combining the above estimates with (4.29), and noting (2.6a), (4.1), Lemma 4.1, Theorem 4.1, which implies

$$\begin{aligned}&\|u_t-u_{ht}\|_0^2+\frac{d}{dt}(\nu\|e_h\|_1^2+G(\mu_h,\mu_h))\\ \le& ch^2+ch^2(\|u_t\|_2^2+\|p_t\|_1^2)+c\|u-u_h\|_1^2.\end{aligned}\tag{4.30}$$

Multiplying (4.30) by $\tau(t)$ integrating it for t from 0 to t, using (2.6b), (4.1), Theorem 4.1 which yields (4.28a). Then setting $\phi_h{=}r_hT_t-T_{ht}$ in (4.16), using (A3) and (3.2), it follows that

$$\begin{aligned}&(T_t-T_{ht},r_hT_t-T_{ht})+\frac{\lambda^{-1}}{2}\frac{d}{dt}\|r_hT-T_h\|_1^2+a_2(u-u_h,T,r_hT_t-T_{ht})\\&+a_2(u,T-T_h,r_hT_t-T_{ht})-a_2(u-u_h,T-T_h,r_hT_t-T_{ht})=0.\end{aligned}\tag{4.31}$$

Thanks to (A3) and (3.3), we arrive at

$$\begin{aligned}&|a_2(u-u_h,T,r_hT_t-T_{ht})|\le c\|u-u_h\|_1\|T\|_2\|r_hT_t-T_{ht}\|_0\\ \le&\frac18\|r_hT_t-T_{ht}\|_0^2+c\|u-u_h\|_1^2\|T\|_2^2,\\&|a_2(u,T-T_h,r_hT_t-T_{ht})|\le c\|u\|_2\|T-T_h\|_1\|r_hT_t-T_{ht}\|_0\\ \le&\frac18\|r_hT_t-T_{ht}\|_0^2+c\|T-T_h\|_1^2\|u\|_2^2,\\&|a_2(u-u_h,T-T_h,r_hT_t-T_{ht})|\le c\|u-u_h\|_1\|r_hT_t-T_{ht}\|_1\|T-T_h\|_1\\ \le&\frac18\|r_hT_t-T_{ht}\|_0^2+ch^{-2}\|u-u_h\|_1^2\|T-T_h\|_1^2,\\&(T_t-r_hT_t,r_hT_t-T_{ht})\le\frac18\|r_hT_t-T_{ht}\|_0^2+c\|T_t-r_hT_t\|_0^2.\end{aligned}$$

Combining the above estimates with (4.31), and noting (2.6a), (3.1), we derive

$$\begin{aligned}&\lambda^{-1}\frac{d}{dt}\|r_hT-T_h\|_1^2+\|r_hT_t-T_{ht}\|_0^2\\ \le& ch^2\|T_t\|_1^2+ch^{-2}\|u-u_h\|_1^2\|T-T_h\|_1^2+c(\|u-u_h\|_1^2+\|T-T_h\|_1^2).\end{aligned}\tag{4.32}$$

Multiplying (4.32) by $\tau(t)$ integrating it for t from 0 to t, using (2.6a), (3.1), (4.28a), Theorem 4.1, noting $T_h(0){=}r_hT(0)$, we obtain (4.28b). □

Lemma 4.4. *Under the Assumptions of* (A1)-(A3), *for* $t{\in}[0,t_n]$, (u_h,T_h,p_h) *satisfies*

$$\tau^2(t)\|u_t-u_{ht}\|_0^2+\int_0^t\tau^2(s)(\nu\|u_t-u_{ht}\|_1^2+G(\mu_{ht},\mu_{ht}))ds\le ch^2,\tag{4.33a}$$

$$\tau(t)\|p-p_h\|_0\le ch.\tag{4.33b}$$

Proof. Differentiating (4.12) with respect to time t gives that

$$\begin{aligned}
&(u_{tt}-u_{htt},v_h)+B_h((e_{ht},\mu_{ht});(v_h,q_h))+a_1(u_t-u_{ht},u,v_h)\\
&\quad+a_1(u,u_t-u_{ht},v_h)+a_1(u-u_h,u_t,v_h)+a_1(u_t,u-u_h,v_h)\\
&\quad-a_1(u_t-u_{ht},u-u_h,v_h)-a_1(u-u_h,u_t-u_{ht},v_h)\\
=&\lambda(j(T_t-T_{ht}),v_h),\quad \forall(v_h,q_h)\in(X_h,M_h).
\end{aligned}$$

By taking $(v_h,q_h)=(e_{ht},\mu_{ht})$ in the above relation, we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_t-u_{ht}\|_0^2-(u_{tt}-u_{htt},w_{ht})+\nu\|e_{ht}\|_1^2+G(\mu_{ht},\mu_{ht})\\
&+a_1(u_t-u_{ht},u,e_{ht})+a_1(u,u_t-u_{ht},e_{ht})+a_1(u-u_h,u_t,e_{ht})\\
&+a_1(u_t,u-u_h,e_{ht})-a_1(e_{ht}+w_{ht},u-u_h,e_{ht})\\
&-a_1(u-u_h,w_{ht},e_{ht})=\lambda(j(T_t-T_{ht}),e_{ht}).
\end{aligned}\tag{4.34}$$

Due to (A3), we find

$$\begin{aligned}
&|a_1(u_t-u_{ht},u,e_{ht})|+|a_1(u,u_t-u_{ht},e_{ht})|\\
\le& c\|u\|_2\|e_{ht}\|_1\|u_t-u_{ht}\|_0\le\frac{1}{8}\nu\|e_{ht}\|_1^2+c\|u\|_2^2\|u_t-u_{ht}\|_0^2,\\
&|a_1(u-u_h,u_t,e_{ht})|+|a_1(u_t,u-u_h,e_{ht})|\\
\le& c\|u_t\|_2\|u-u_h\|_0\|e_{ht}\|_1\le\frac{1}{8}\nu\|e_{ht}\|_1^2+c\|u_t\|_2^2\|u-u_h\|_0^2,\\
&|a_1(u-u_h,w_{ht},e_{ht})|+|a_1(w_{ht},u-u_h,e_{ht})|\\
\le& c\|u-u_h\|_1\|w_{ht}\|_1\|e_{ht}\|_1\le\frac{1}{8}\nu\|e_{ht}\|_1^2+c\|w_{ht}\|_1^2\|u-u_h\|_1^2,\\
&|a_1(e_{ht},u-u_h,e_{ht})|\le\|u-u_h\|_0^{\frac{1}{2}}\|u-u_h\|_1^{\frac{1}{2}}\|e_{ht}\|_0^{\frac{1}{2}}\|e_{ht}\|_1^{\frac{3}{2}}\\
\le&\frac{1}{8}\nu\|e_{ht}\|_1^2+c\|u-u_h\|_0^2\|u-u_h\|_1^2(\|u_t-u_{ht}\|_0^2+\|w_{ht}\|_0^2),\\
&|\lambda(j(T_t-T_{ht}),e_{ht})|\le\frac{1}{8}\nu\|e_{ht}\|_1^2+c\|T_t-T_{ht}\|_0^2,\\
&|(u_{tt}-u_{htt},w_{ht})|\le(\|u_{tt}\|_0+\|u_{htt}\|_0)\|w_{ht}\|_0.
\end{aligned}$$

Combining the above estimates with (4.34) yields

$$\begin{aligned}
&\frac{d}{dt}\|u_t-u_{ht}\|_0^2+\nu\|e_{ht}\|_1^2+G(\mu_{ht},\mu_{ht})\\
\le& c\|T_t-T_{ht}\|_0^2+(\|u_{tt}\|_0+\|u_{htt}\|_0)\|w_{ht}\|_0\\
&+c\|u\|_2^2\|u_t-u_{ht}\|_0^2+c\|u_t\|_2^2\|u-u_h\|_0^2+c\|w_{ht}\|_1^2\|u-u_h\|_1^2\\
&+c\|u-u_h\|_0^2\|u-u_h\|_1^2(\|u_t-u_{ht}\|_0^2+\|w_{ht}\|_0^2).
\end{aligned}\tag{4.35}$$

Multiplying (4.35) by $\tau^2(t)$ and using (4.1), it follows that

$$\begin{aligned}
&\frac{d}{dt}(\tau^2(t)\|u_t-u_{ht}\|_0^2)+\tau^2(t)(\nu\|e_{ht}\|_1^2+G(\mu_{ht},\mu_{ht}))\\
\le&2\tau(t)\|u_t-u_{ht}\|_0^2+ch^2\tau(t)(\|u_{tt}\|_0+\|u_{htt}\|_0)\tau(t)(\|u_t\|_2+\|p_t\|_1)\\
&+c\|T_t-T_{ht}\|_0^2+c\|u\|_2^2\tau(t)\|u_t-u_{ht}\|_0^2+c\tau(t)\|u_t\|_2^2\|u-u_h\|_0^2\\
&+ch^2(\|u\|_1^2+\|u_h\|_1^2)\tau(t)(\|u_t\|_2+\|p_t\|_1)\\
&+c\tau^2\|u-u_h\|_0^2\|u-u_h\|_1^2\|u_t-u_{ht}\|_0^2.
\end{aligned}\tag{4.36}$$

Integrating (4.36) respect to t, and applying (2.6a), (2.6b), (4.1), Theorem 4.1, Lemma 4.2, 4.3, we obtain (4.33a).

The inf-sup condition (3.9b) and (4.12) guarantee that

$$\begin{aligned}
\beta\|\mu_h(t)\|_0\le&\sup_{(v_h,q_h)\in(X_h,M_h)}\frac{B_h((e_h,\mu_h);(v_h,q_h))}{\|v_h\|_1+\|q_h\|_0}\\
\le&c\|u_t-u_{ht}\|_0+c(\|u\|_1+\|u_h\|_1)\|u-u_h\|_1+c\|T-T_h\|_0.
\end{aligned}\tag{4.37}$$

Combining the above estimates with (4.1) yields

$$\begin{aligned}
\tau(t)\|p-p_h\|_0\le&ch(\|u\|_2+\|p\|_1)+c\tau(t)\|u_t-u_{ht}\|_0\\
&+c(\|u\|_1+\|u_h\|_1)\tau^{\frac{1}{2}}(t)\|u-u_h\|_1+c\|T-T_h\|_0.
\end{aligned}\tag{4.38}$$

Therefore, we derive from (4.38), (2.6a), and Theorem 4.1, Lemma 4.3, 4.4 that (4.33b) holds. □

Theorem 4.2. *Under the Assumptions of* (A1)-(A3), *for* $t\in[0,t_n]$, (u_h,T_h,p_h) *satisfies the following error estimates*

$$\tau^{\frac{1}{2}}\|T-T_h\|_1+\tau^{\frac{1}{2}}(t)\|u-u_h\|_1+\tau(t)\|p-p_h\|_0\le ch.\tag{4.39}$$

Proof. The proof of Theorem 4.2 consists of Lemma 4.3 and Lemma 4.4. □

5 Numerical examples

This section presents the numerical results that complement the theoretical analysis of Theorem 4.2. Our goal is to illustrate the convergence theory of the new stabilized finite element method for the two-dimensional non-stationary conduction-convection equations approximated by the lowest equal-order finite element pair. In the experiment, $\Omega=[0,1]\times[0,1]$ is the unit square in $\mathbb{R}^2$. Let $T_0=0$ on left and lower boundary of the cavity, $\partial T_0/\partial y=0$ on upper boundary of the cavity, and $T_0=4y(1-y)$ on right boundary of the cavity (see Fig. 1). We take a time step increment as $\Delta t=0.0025$ and

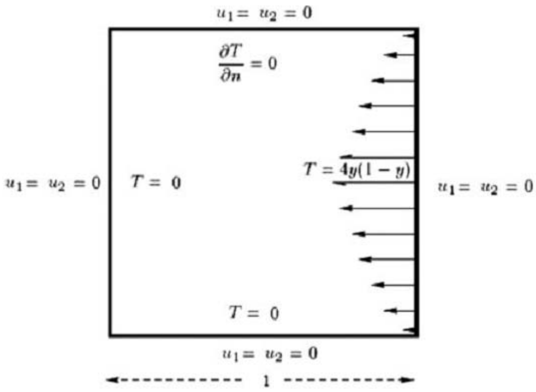


Figure 1: Physics model of the cavity flows: $t = 0$, i.e., $n = 0$ initial values on boundary.

Re=1, Pr=0.76. The pressure, temperature and velocity are approximated by piecewise linear finite elements defined with respect to the same uniform triangulation of Ω into triangles.

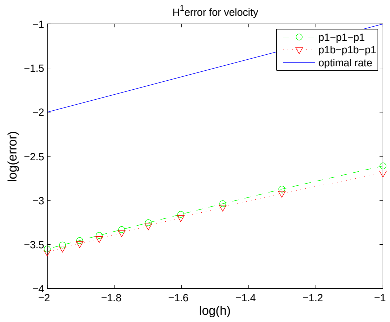
In general, we cannot obtain the exact solution of the non-stationary conduction-convection equations. Instead of the exact solution, we design the procedure as follows. Firstly, solving the non-stationary conduction-convection equations by using the $P_2 - P_2 - P_1$ finite element pair, which holds super-convergent results, on the finer mesh, we take the solution as the "exact solution". Secondly, the absolute error is

Table 1: Finite element errors after one implicit Euler step: $P_1b - P_1b - P_1$ element.

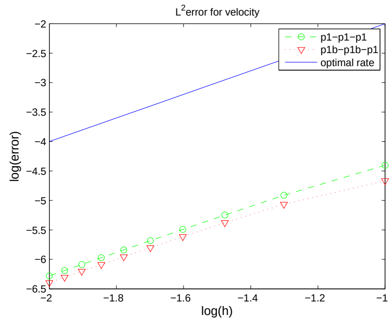
1/h	u_{L^2}error	u_{H^1} error	T_{L^2}error	T_{H^1} error	p_{L^2}error
10	2.161069052e-005	0.002041128593	0.01210821557	1.129714465	0.0006968399149
20	8.587757689e-006	0.001201400427	0.003649249181	0.5968362208	0.0003100472273
30	4.183078128e-006	0.0008356521855	0.001694778894	0.4022805405	0.0001662723572
40	2.429678457e-006	0.0006377358846	0.0009721978661	0.3029450738	0.0001045828393
50	1.577175939e-006	0.0005141213411	0.0006294078592	0.2427151473	7.276792312e-005
60	1.102619891e-006	0.0004295275926	0.0004403087771	0.2022589666	5.414809844e-005
70	8.154289092e-007	0.0003691859281	0.0003258934557	0.1733628531	4.219004553e-005
80	6.259809651e-007	0.0003234726459	0.0002504970938	0.1517675479	3.421466619e-005
90	4.953658083e-007	0.0002876186462	0.0001986174516	0.1348881567	2.823291358e-005
100	4.015756543e-007	0.0002597116994	0.0001613840114	0.1215504956	2.389275781e-005

Table 2: Finite element errors after one implicit Euler step: $P_1 - P_1 - P_1$ element.

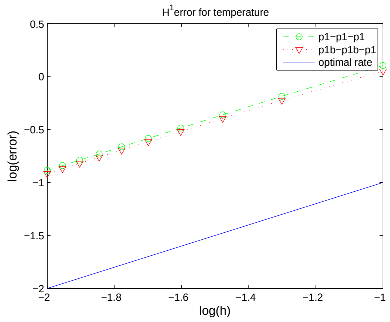
1/h	u_{L^2}error	u_{H^1} error	T_{L^2}error	T_{H^1} error	p_{L^2}error
10	3.962177025e-005	0.002458148978	0.01071806872	1.272346485	0.002844217132
20	1.22583322e-005	0.001341513608	0.002850501213	0.651449188	0.0009108718006
30	5.659030486e-006	0.0009171252869	0.001286168987	0.4353938578	0.0004276764799
40	3.225825006e-006	0.0006948664295	0.0007303147193	0.3264555611	0.0002461800831
50	2.077353113e-006	0.0005585922733	0.0004707322105	0.2609507555	0.0001594971937
60	1.447655849e-006	0.0004664863452	0.0003287808424	0.2172587166	0.000111327787
70	1.064912282e-006	0.0004005768075	0.0002426441947	0.1860945993	8.221164042e-005
80	8.171199932e-007	0.0003507953758	0.0001866233548	0.1627396491	6.328448505e-005
90	6.464128076e-007	0.0003120412498	0.0001479984785	0.1445587663	5.010566454e-005
100	5.238819897e-007	0.0002812166149	0.000120305271	0.130075494	4.110902564e-005



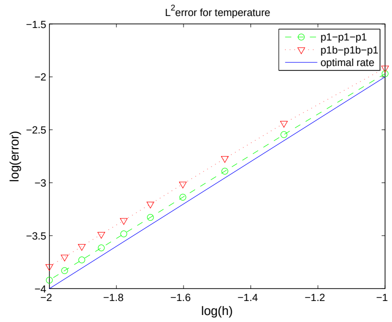
(a)



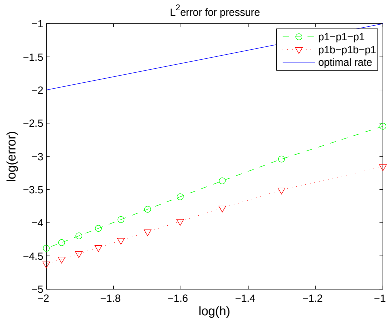
(b)



(c)



(d)



(e)

Figure 2: Comparison of rate analysis for the non-stationary conduction-convection equations by using different methods.

obtained by comparing the ”exact” solution and the finite element solutions with different methods. Finally, we can easily obtain ”errors” and ”convergence rates”.

The experimental ”convergence rates” with respect to the mesh size h are calculated by the formula $\log(E_i/E_{i+1})/\log(h_i/h_{i+1})$, where E_i and E_{i+1} are the ”errors” corresponding to the meshes of size h_i and h_{i+1} respectively. For comparison, we also present the result of standard Galerkin method with MINI element. From Tables 1-2, we see that there is not much difference in H_1-errors of the velocity and H_1-errors of the temperature between two methods. A large difference is observed between two methods for the pressure approximation in accuracy and convergence rates. Both the stabilized $P_1-P_1-P_1$ method and the $P_1b-P_1b-P_1$ method show a superconvergence behavior for the pressure approximation; the latter is better than the former in

Table 3: Finite element errors after one implicit Euler step: $P_1b - P_1b - P_1$ element.

1/h	u_{L^2}rate	u_{H^1} rate	T_{L^2}rate	T_{H^1} rate	p_{L^2}rate
10					
20	1.33139	0.76465	1.73031	0.920551	1.16834
30	1.77399	0.895344	1.89158	0.97294	1.53675
40	1.8885	0.939538	1.93181	0.985804	1.61167
50	1.93652	0.965589	1.94843	0.993365	1.62543
60	1.96327	0.986023	1.95974	1.0001	1.62105
70	1.95737	0.982062	1.95202	1.00007	1.6188
80	1.98002	0.989923	1.9705	0.996296	1.56914
90	1.9869	0.997419	1.97029	1.00103	1.63152
100	1.99221	0.968705	1.97032	0.98819	1.58421

Table 4: Finite element errors after one implicit Euler step: $P_1 - P_1 - P_1$ element.

1/h	u_{L^2}rate	u_{H^1} rate	T_{L^2}rate	T_{H^1} rate	p_{L^2}rate
10					
20	1.69253	0.873711	1.91076	0.965767	1.64271
30	1.90634	0.937959	1.96275	0.993793	1.86461
40	1.95377	0.964692	1.96727	1.00096	1.91984
50	1.97225	0.978293	1.96818	1.00367	1.9451
60	1.98084	0.988315	1.96849	1.00505	1.97205
70	1.9919	0.988145	1.97077	1.00443	1.96679
80	1.98352	0.993792	1.96586	1.00428	1.95951
90	1.98966	0.993927	1.96879	1.00579	1.98251
100	1.99479	0.987183	1.96629	1.002	1.87837

Table 5: CPU time by using two methods: one implicit Euler step.

1/h	10	20	30	40	50	60	70	80	90	100
$P_1 - P_1 - P_1$	0.484	1.593	3.735	6.844	10.579	15.563	21.89	28.406	36.844	46.688
$P_1b - P_1b - P_1$	0.672	2.453	5.5	9.969	15.984	23.203	31.89	42.907	56.672	70.844

terms of accuracy; however, the former is better than the latter in terms of convergence rate see Fig. 2(e).

The numerical results support our theoretical results and show that the new stabilized method is highly efficient for the non-stationary conduction-convection equations. It takes less CPU time than the $P_1b - P_1b - P_1$ method, see Table 5. Especially, the stabilized $P_1 - P_1 - P_1$ method has a better performance than the $P_1b - P_1b - P_1$ method in terms of convergence rate for the pressure approximation.

6 Conclusions

In this paper we have provided a theoretical analysis for a stabilized finite element method based on two local Gauss integrations. The analysis has extended the work in [21] for the transient Navier-Stokes equations to the non-stationary conduction-convection equations. The discretization uses a pair of spaces of finite elements $P_1 - P_1 - P_1$ over triangles or $Q_1 - Q_1 - Q_1$ over quadrilateral elements. This new method is computationally efficient. It does not require a selection of mesh-dependent stabilization parameters or a calculation of higher-order derivatives. Its another valuable feature is that the action of stabilization operators can be performed locally at the element level with minimal additional cost. The numerical tests performed are in a good agreement with the theoretical results established.

Acknowlegments

This work was supported by the NSF of China (No. 10971166) and the National High Technology Research and Development Program of China (863 Program, No. 2009AA01A135).

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