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MULTIPLE POSITIVE SOLUTIONS FOR A CLASS OF INTEGRAL BOUNDARY VALUE PROBLEM*†

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Abstract

In this paper, the existence and multiplicity of positive solutions for a class of non-resonant fourth-order integral boundary value problem

$$\begin{cases} u^{(4)}(t)+\beta u''(t)-\alpha u(t)=f\left(t,u(t),u''(t)\right), \quad t\in(0,1),\\ u''(0)=u''(1)=0,\\ u(0)=0, \quad u(1)=\left(\dfrac{1}{\lambda_2}-\dfrac{1}{\lambda_1}\right)\displaystyle\int_0^1 q(s)f(s,u(s),u''(s))\mathrm{d}s \end{cases}$$

with two parameters are established by using the Guo-Krasnoselskii's fixed-point theorem, where $f\in C\left([0,1]\times[0,+\infty)\times(-\infty,0],[0,+\infty)\right)$, $q(t)\in L^1[0,1]$ is nonnegative, $\alpha,\beta\in R$ and satisfy $\beta<2\pi^2$, $\alpha>0$, $\alpha/\pi^4+\beta/\pi^2<1$, $\lambda_{1,2}=(-\beta\mp\sqrt{\beta^2+4\alpha})/2$. The corresponding examples are raised to demonstrate the results we obtained.

Keywords positive solutions; fixed point; integral boundary conditions

2000 Mathematics Subject Classification 34B15

1 Introduction

By the fact of wide applications in a number of scientific fields of boundary value problems for ordinary differential equations, much attention and discussion has been attracted to many scholars [1–5]. Especially, an increasing interest in the existence and multiplicity of positive solutions to boundary value problems with integral boundary conditions has been evolved recent years, which arises in the fields of thermo-elasticity, heat conduction, plasma physics and underground water. Moreover, this kind of nonlocal boundary value problems include two-point and multi-point cases [3–6]. For more details about boundary value problems with integral boundary conditions, one can refer to [6–9].

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In 2011, Ma [8] considered the existence of positive solutions for the following boundary value problem

$$\begin{cases} u^{(4)}(t) = f\left(t, u(t), u''(t)\right), \quad t \in (0,1), \\ u(0) = \displaystyle\int_0^1 g(s)u(s)\mathrm{d}s, \quad u(1) = 0, \\ u''(0) = \displaystyle\int_0^1 h(s)u''(s)\mathrm{d}s, \quad u''(1) = 0 \end{cases}$$

with integral boundary conditions by using the Krein-Rutman theorem and the global bifurcation techniques, where $f \in C\left([0,1] \times [0,+\infty) \times (-\infty,0], [0,+\infty)\right)$ and $g, h \in L^1[0,1]$ are nonnegative. At the same year, by using the operator spectrum theorem together with the fixed point theorem on cone, Chai [9] established some results on the existence of positive solution for the following integral boundary value problem

$$\begin{cases} u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f\left(t, u(t)\right), \quad t \in (0,1), \\ u(0) = u(1) = 0, \\ u''(0) = \displaystyle\int_0^1 u(s)\phi_1(s)\mathrm{d}s, \quad u''(1) = \displaystyle\int_0^1 u(s)\phi_2(s)\mathrm{d}s \end{cases}$$

with two parameters, where $f \in C\left([0,1] \times [0,+\infty), (-\infty,+\infty)\right)$ is allowed to change sign and $\alpha, \beta \in R$, $\beta < 2\pi^2$, $\alpha \geq -\beta^2/4$, $\alpha/\pi^4 + \beta/\pi^2 < 1$, $\phi_1, \phi_2 \in C\left([0,1], (-\infty,0]\right)$.

The above results mentioned are mainly dealt with the existence of positive solutions. However, there are not few results on the existence of multiple positive solutions for integral boundary value problems. For completeness, the main purpose of this paper is to investigate the multiplicity of positive solutions for the following non-resonant fourth-order integral boundary value problem

$$\begin{cases} u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f\left(t, u(t), u''(t)\right), \quad t \in (0,1), \\ u''(0) = u''(1) = 0, \\ u(0) = 0, \quad u(1) = \left(\dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_1}\right) \displaystyle\int_0^1 q(s) f(s, u(s), u''(s))\mathrm{d}s \end{cases} \tag{1.1}$$

with two parameters.

Throughout this paper, we make the following assumptions:

(H_1) $\alpha, \beta \in R,\ \beta < 2\pi^2,\ \alpha > 0,\ \alpha/\pi^4 + \beta/\pi^2 < 1$;

(H_2) λ_1, λ_2 are the two roots of the polynomial $P(\lambda) = \lambda^2 + \beta\lambda - \alpha$ and $\lambda_{1,2} = (-\beta \mp \sqrt{\beta^2 + 4\alpha})/2$;

(H_3) $f \in C\left([0,1] \times [0,+\infty) \times (-\infty,0], [0,+\infty)\right)$ and $f(t,u,v) > 0$, for any $t \in [\frac{1}{4}, \frac{3}{4}]$ and $|u| + |v| > 0$; $q(t) \in L^1[0,1]$ are nonnegative satisfying that there exists a number $M > 0$ such that

$$0 \le q(t) \le \frac{M\sinh(\sqrt{|\lambda_2|}t)\sinh[\sqrt{|\lambda_2|}(1-t)]}{\sqrt{|\lambda_2|}\sinh\sqrt{|\lambda_2|}}, \quad \text{for any } t \in [0,1].$$

The assumption (H_1) implies an nonresonance condition by the fact that $\alpha > 0 \ge -\beta^2/4$. By ($H_1$) and ($H_2$), it is not difficult to check that $-\pi^2 < \lambda_1 < 0 < \lambda_2$.

2 Preliminaries

In this section, we first state some definitions and preliminary results which will be applied to prove the main results in this paper.

Denote

$$\overline{f_0} = \limsup_{|u|+|v|\to 0^+} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}, \quad \overline{f_\infty} = \limsup_{|u|+|v|\to +\infty} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}, \tag{2.1}$$

$$\underline{f_0} = \liminf_{|u|+|v|\to 0^+} \min_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}, \quad \underline{f_\infty} = \liminf_{|u|+|v|\to +\infty} \min_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}. \tag{2.2}$$

Let $G_i(t,s)\,(i=1,2)$ be the Green's function of the following boundary value problem

$$-u''(t) + \lambda_i u(t) = 0, \quad t \in (0,1), \quad u(0) = u(1) = 0, \quad i = 1,2.$$

From [5], $G_i(t,s)\,(i=1,2)$ is given by

$$G_i(t,s) = \begin{cases} F_i(\omega_i t)\cdot \dfrac{F_i[\omega_i(1-s)]}{\omega_i F_i(\omega_i)}, & 0 \le t \le s \le 1, \\ F_i(\omega_i s)\cdot \dfrac{F_i[\omega_i(1-t)]}{\omega_i F_i(\omega_i)}, & 0 \le s \le t \le 1, \end{cases}$$

where $F_1(t) = \sin t,\ F_2(t) = \sinh t,\ \omega_i = \sqrt{|\lambda_i|}\,(i=1,2)$ and $\omega_2 > 0,\ 0 < \omega_1 < \pi$ by the fact that $-\pi^2 < \lambda_1 < 0 < \lambda_2$.

Lemma 2.1[5] *$G_i(t,s)\,(i=1,2)$ has the following properties:*

(i) *$G_i(t,s) > 0$, for any $t,s \in (0,1)$.*

(ii) *$G_i(t,s) \le C_i G_i(s,s)$, for any $t,s \in [0,1]$, where $C_i > 0$ is a constant, and $C_1 = 1/\sin\omega_1,\ C_2 = 1$.*

(iii) *$G_i(t,s) \ge \delta_i G_i(t,t)\cdot G_i(s,s)$, for any $t,s \in [0,1]$, where $\delta_i > 0$ is a constant, and $\delta_1 = \omega_1 \sin\omega_1,\ \delta_2 = \omega_2/\sinh\omega_2$.*

Theorem 2.1[10] (Guo-Krasnoselskii's fixed-point theorem) *Let E be a Banach space, $K \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and $T : K \cap (\overline{\Omega}_2\backslash\Omega_1) \to K$ is a completely continuous operator such that either*

(i) *$\|Tu\| \ge \|u\|,\ u \in K \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|,\ u \in K \cap \partial\Omega_2$; or*

(ii) *$\|Tu\| \le \|u\|,\ u \in K \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|,\ u \in K \cap \partial\Omega_2$*

holds, then T has a fixed point in $K \cap (\overline{\Omega}_2\backslash\Omega_1)$.

Let $C[0,1]$ be equipped with the maximum norm $\|u\| = \max\limits_{t\in[0,1]} |u(t)|$, and $C^2[0,1]$ be equipped with the norm $\|u\|_2 = \|u\| + \|u''\| = \max\limits_{t\in[0,1]} |u(t)| + \max\limits_{t\in[0,1]} |u''(t)|$. Denote

$$\begin{aligned}
&M_i = \max_{s\in(0,1)} G_i(s,s), \quad m_i = \min_{s\in[\frac{1}{4},\frac{3}{4}]} G_i(s,s), \quad i=1,2,\\
&C_0 = \int_0^1 G_1(s,s)G_2(s,s)\mathrm{d}s, \quad A_1 = C_1C_2M_1 + M\Big(\frac{1}{\omega_1^2\sin\omega_1} + \frac{1}{\omega_2^2}\Big),\\
&A_2 = \omega_1^2 C_1C_2M_1 + C_2 + \frac{M}{\sin\omega_1}, \quad B_1 = \delta_1\delta_2 m_1 C_0, \quad B_2 = \delta_2 m_2,\\
&N_i = \frac{B_i}{A_i}, \quad i=1,2, \quad N = \min\{N_1, N_2\},
\end{aligned}$$

and they are all positive. Define a cone K in $C^2[0,1]$ by

$$K = \big\{u \in C^2[0,1] : u \geq 0,\ u'' \leq 0,\ \min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t) \geq N_1\|u\|,\ \min_{t\in[\frac{1}{4},\frac{3}{4}]}[-u''(t)] \geq N_2\|u''\|\big\}.$$

Define an integral operator $A : K \to C^2[0,1]$ by

$$Au(t) = \int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\mathrm{d}s + g(t)\int_0^1 q(s)f(s,u(s),u''(s))\mathrm{d}s, \tag{2.3}$$

where $g(t) = \frac{\sin\omega_1 t}{\omega_1^2\sin\omega_1} + \frac{\sinh\omega_2 t}{\omega_2^2\sinh\omega_2}$. Meanwhile, it is trivial to testify the following conclusion.

Lemma 2.2 *Assume that* (H_1)-(H_3) *hold. Then* $u(t)$ *is a solution for the BVP* (1.1) *if and only if* $u(t)$ *is a fixed point of the integral operator* A.

Lemma 2.3 *Assume that* (H_1)-(H_3) *hold. Then* $A : K \to K$ *is completely continuous.*

Proof By the definition of $g(t)$ and $\omega_2 > 0,\ 0 < \omega_1 < \pi$, it is easy to verify that $g(t) \geq 0$ for any $t \in [0,1]$. Combining $g(t) \geq 0$, (i) of Lemma 2.1, (H_3) and (2.3), we obtain $Au(t) \geq 0$.

Next, from (2.3), for any $u(t) \in K$ and $t \in [0,1]$, it follows that

$$\begin{aligned}
-(Au)''(t) = {}& \omega_1^2\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\mathrm{d}s\\
&+\int_0^1 G_2(t,s)f(s,u(s),u''(s))\mathrm{d}s + h(t)\int_0^1 q(s)f(s,u(s),u''(s))\mathrm{d}s,
\end{aligned} \tag{2.4}$$

where $h(t) = \sin\omega_1 t/\sin\omega_1 - \sinh\omega_2 t/\sinh\omega_2$, and thus $h(0) = h(1) = 0,\ h''(t) \leq 0$ for $t \in [0,1]$, which implies $h(t) \geq 0$ for $t \in [0,1]$. Therefore, we can conclude that $(Au)''(t) \leq 0$.

By (2.3), (H_3) and (ii) of Lemma 2.1, for any $u(t)\in K$ and $t \in [0,1]$, it follows that

$$\begin{aligned} Au(t) &\le C_1C_2M_1 \int_0^1 G_2(s,s)f(s,u(s),u''(s))\mathrm{d}s \\ &\quad + \left(\frac{1}{\omega_1^2 \sin\omega_1} + \frac{1}{\omega_2^2}\right)\int_0^1 q(s)f(s,u(s),u''(s))\mathrm{d}s \\ &\le \left[C_1C_2M_1 + M\left(\frac{1}{\omega_1^2\sin\omega_1} + \frac{1}{\omega_2^2}\right)\right]\int_0^1 G_2(s,s)f(s,u(s),u''(s))\mathrm{d}s, \\ &= A_1\int_0^1 G_2(s,s)f(s,u(s),u''(s))\mathrm{d}s. \end{aligned} \tag{2.5}$$

By (2.3), (2.5) and (iii) of Lemma 2.1, for any $u(t) \in K$ and $t \in [\frac{1}{4}, \frac{3}{4}]$, we have

$$\begin{aligned} Au(t) &\ge \delta_1\delta_2 m_1 C_0 \int_0^1 G_2(s,s)f(s,u(s),u''(s))\mathrm{d}s \\ &= B_1\int_0^1 G_2(s,s)f(s,u(s),u''(s))\mathrm{d}s \\ &\ge \frac{B_1}{A_1}\|Au\|, \end{aligned} \tag{2.6}$$

which implies that $\min\limits_{t\in[\frac{1}{4},\frac{3}{4}]} Au(t) \ge N_1\|Au\|$.

Similarly, by (2.4), (H_3) and (ii),(iii) of Lemma 2.1, for any $u(t) \in K$, we get

$$\begin{aligned} -(Au)''(t) &\le \omega_1^2 C_1C_2M_1 \int_0^1 G_2(s,s)f(s,u(s),u''(s))\mathrm{d}s \\ &\quad +C_2\int_0^1 G_2(s,s)f(s,u(s),u''(s))\mathrm{d}s \\ &\quad +\frac{M}{\sin\omega_1}\int_0^1 G_2(s,s)f(s,u(s),u''(s))\mathrm{d}s \\ &= A_2\int_0^1 G_2(s,s)f(s,u(s),u''(s))\mathrm{d}s, \quad \text{for any } t\in[0,1]; \end{aligned} \tag{2.7}$$

$$\begin{aligned} -(Au)''(t) &\ge \int_0^1 G_2(t,s)f(s,u(s),u''(s))\mathrm{d}s \\ &\ge B_2\int_0^1 G_2(s,s)f(s,u(s),u''(s))\mathrm{d}s \\ &\ge N_2\|(Au)''\|, \quad \text{for any } t\in\left[\frac{1}{4},\frac{3}{4}\right]. \end{aligned} \tag{2.8}$$

Therefore, $\min\limits_{t\in[\frac{1}{4},\frac{3}{4}]}[-(Au)''(t)] \ge N_2\|(Au)''\|$, and thus $A(K) \subset K$.

Furthermore, by Arzela-Ascoli theorem, the operator A is completely continuous. This completes the proof.

Corollary 2.1 $|u(t)|+|u''(t)|\ge N\|u\|_2$, *for any* $u(t)\in K$ *and* $t\in[\frac14,\frac34]$.

Lemma 2.4 *Suppose that* (H_1)-(H_3) *hold, then for the operator* A *and any* $u\in K$, *the following conclusions hold:*

(i) *If* $\overline{f_0}<\frac{1}{M_2(A_1+A_2)}$, $r>0$ *is small enough and* $\|u\|_2=r$, *then* $\|Au\|_2\le\|u\|_2$;

(ii) *If* $\underline{f_0}>\frac{2}{m_2B_2N}$, $r>0$ *is small enough and* $\|u\|_2=r$, *then* $\|Au\|_2\ge\|u\|_2$;

(iii) *If* $\overline{f_\infty}<\frac{1}{M_2(A_1+A_2)}$, $R>0$ *is large enough and* $\|u\|_2=R$, *then* $\|Au\|_2\le\|u\|_2$;

(iv) *If* $\underline{f_\infty}>\frac{2}{m_2B_2N}$, $R>0$ *is large enough and* $\|u\|_2=R$, *then* $\|Au\|_2\ge\|u\|_2$.

Proof In view of the proofs of (ii) and (iii) are similar to those of (i) and (iv) respectively, here we give only the proofs of (i) and (iv).

(i) By $\overline{f_0}<\frac{1}{M_2(A_1+A_2)}$, there exists a number $r_0>0$ such that

$$f(t,u,v)\le\frac{1}{M_2(A_1+A_2)}(|u|+|v|),\quad \text{for any } |u|+|v|<r_0.$$

Letting $0<r\ll r_0$, for any $t\in[0,1]$, $u\in K$ with $\|u\|_2=r$, it follows that

$$f(t,u,u'')\le\frac{1}{M_2(A_1+A_2)}(|u|+|u''|)\le\frac{r}{M_2(A_1+A_2)}.$$

From (2.5) and (2.7), we obtain

$$\begin{aligned}\|Au\|&\le A_1M_2\int_0^1 f(s,u(s),u''(s))\mathrm{d}s\\&\le A_1\cdot M_2\cdot\frac{r}{M_2(A_1+A_2)}=\frac{rA_1}{A_1+A_2},\\ \|(Au)''\|&\le A_2M_2\int_0^1 f(s,u(s),u''(s))\mathrm{d}s\\&\le A_2\cdot M_2\cdot\frac{r}{M_2(A_1+A_2)}=\frac{rA_2}{A_1+A_2}.\end{aligned}$$

Therefore, $\|Au\|_2=\|Au\|+\|(Au)''\|\le r=\|u\|_2$.

(iv) By the fact that $\underline{f_\infty}>\frac{2}{m_2B_2N}$, there exists a number $R_0>0$ such that

$$f(t,u,v)\ge\frac{2}{m_2B_2N}(|u|+|v|),\quad \text{for any } |u|+|v|>R_0.$$

Letting $R\gg R_0/N$, by Corollary 2.1, we get

$$|u(t)|+|u''(t)|\ge N\|u\|_2\ge R_0,\quad \text{for any } t\in\Big[\frac14,\frac34\Big]$$

and $u\in K$ with $\|u\|_2=R$. Therefore,

$$f(t,u,u'')\ge\frac{2}{m_2B_2N}(|u|+|u''|)\ge\frac{2}{m_2B_2N}NR,\quad \text{for any } t\in\Big[\frac14,\frac34\Big].$$

From (2.8), we have

$$\begin{aligned}\|Au\|_2 &\geq \|(Au)''\| = \max_{t\in[0,1]} |(Au)''(t)| \geq -(Au)''\Big(\frac{1}{2}\Big)\\ &\geq m_2 B_2 \int_{\frac{1}{4}}^{\frac{3}{4}} f(s,u(s),u''(s))\mathrm{d}s\\ &\geq m_2\cdot B_2\cdot \frac{2}{m_2B_2N}\cdot N\cdot R\cdot\frac{1}{2}\\ &= R = \|u\|_2, \end{aligned} \tag{2.9}$$

for any $t\in[0,1]$ and $u\in K$ with $\|u\|_2 = R$. This completes the proof.

3 Main Results

In this section, we state and prove the main results of this paper on the existence and multiplicity of positive solutions for (1.1).

Theorem 3.1 *Suppose* (H_1)-(H_3) *hold. If either*

(i) $\overline{f_0} < \frac{1}{M_2(A_1+A_2)}$, $\underline{f_\infty} > \frac{2}{m_2B_2N}$; *or*

(ii) $\underline{f_0} > \frac{2}{m_2B_2N}$, $\overline{f_\infty} < \frac{1}{M_2(A_1+A_2)}$,

holds, then the BVP (1.1) *has at least one positive solution.*

Proof Let $E = C^2[0,1]$, $\Omega_1 = \{u\in E : \|u\|_2 < r\}$, $\Omega_2 = \{u\in E : \|u\|_2 < R\}$, where $0<r<R$. Meanwhile, from Lemma 2.3, we know that $A : K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$ is completely continuous.

Suppose (i) holds. Then, according to Lemma 2.4, we can get $\|Au\|_2\leq\|u\|_2$ for any $u\in E$ with $\|u\|_2 = r$, and $\|Au\|_2\geq\|u\|_2$ for any $u\in E$ with $\|u\|_2 = R$, respectively, which imply that Theorem 2.1 holds. Therefore, it immediately follows that A has a fixed point $u_0(t)\in K\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Similarly, under the condition (ii), it also follows that A has a fixed point $u_0(t)\in K\cap(\overline{\Omega}_2\backslash\Omega_1)$ from Lemma 2.4 and Theorem 2.1. Moreover, by (2.6) and (H_3), we have

$$u_0(t) = Au_0(t) \geq B_1\int_{\frac{1}{4}}^{\frac{3}{4}} G_2(s,s)f(s,u(s),u''(s))\mathrm{d}s > 0.$$

Thus, by Lemma 2.2, u_0 is also the positive solution for the BVP (1.1). The proof is completed.

Corollary 3.1 *Suppose* (H_1)-(H_3) *hold. If either*

(i) $\overline{f_0} = 0$, $\underline{f_\infty} = \infty$ (*superlinear*); *or*

(ii) $\underline{f_0} = \infty$, $\overline{f_\infty} = 0$ (*sublinear*),

holds, then the BVP (1.1) *has at least one positive solution.*

Example 3.1 Consider the following boundary value problem

$$\begin{cases} u^{(4)}(t)+\pi u''(t)-2\pi^2u(t)=f\big(t,u(t),u''(t)\big), \quad t\in(0,1),\\ u''(0)=u''(1)=0,\\ u(0)=0, \quad u(1)=\Big(\dfrac{1}{\lambda_2}-\dfrac{1}{\lambda_1}\Big)\displaystyle\int_0^1 G_2(s,s)f\big(s,u(s),u''(s)\big)\mathrm{d}s, \end{cases} \tag{3.1}$$

where $q(t)=G_2(t,t)$, $M=1$, $f(t,u,v)=\frac{(|u|+|v|)^3}{1+|u|+|v|}+\frac{1}{3}(|u|+|v|)$, $\alpha=2\pi^2$ and $\beta=\pi$. Then, by some calculations, we get

$$\begin{aligned}
&\lambda_1=\frac{-\pi-\sqrt{\pi^2+8\pi^2}}{2}=-2\pi\approx-6.283, \quad \lambda_2=\frac{-\pi+\sqrt{\pi^2+8\pi^2}}{2}=\pi\approx 3.142,\\
&\omega_1=\sqrt{|\lambda_1|}=\sqrt{2\pi}\approx 2.507, \quad \omega_2=\sqrt{|\lambda_2|}=\sqrt{\pi}\approx 1.772,\\
&M_1=\max_{s\in(0,1)}\frac{\sin\omega_1 s\sin\omega_1(1-s)}{\omega_1\sin\omega_1}\approx 0.607,\\
&M_2=\max_{s\in(0,1)}\frac{\sinh\omega_2 s\sinh\omega_2(1-s)}{\omega_2\sinh\omega_2}\approx 0.200,\\
&C_1=\frac{1}{\sin\omega_1}\approx 1.686, \quad C_2=1,\\
&A_1=C_1C_2M_1+M\Big(\frac{1}{\omega_1^2\sin\omega_1}+\frac{1}{\omega_2^2}\Big)\approx 1.610,\\
&A_2=\omega_1^2C_1C_2M_1+C_2+\frac{M}{\sin\omega_1}\approx 9.116,\\
&\overline{f_0}=\frac{1}{3}<\frac{1}{M_2(A_1+A_2)}\approx 0.466, \quad \underline{f_\infty}=\infty,
\end{aligned}$$

which implies (i) of Theorem 3.1 holds. Therefore, by Theorem 3.1, the BVP (3.1) has at least one positive solution.

Theorem 3.2 *Suppose* (H_1)-(H_3) *hold. If one of the following two conditions holds:*

(i) $\overline{f_0},\overline{f_\infty}<\frac{1}{M_2(A_1+A_2)}$ *and there exists a number* $R_0>0$ *satisfying* $r\ll R_0\ll R$, *such that* $m(R_0)\geq\frac{2R_0}{m_2B_2}$, *where* $m(R)=\min\{f(t,u,v): NR\leq|u|+|v|\leq R, t\in[\frac{1}{4},\frac{3}{4}]\}$, $r>0$ *is small enough and* $R>0$ *is large enough;*

(ii) $\underline{f_0},\underline{f_\infty}>\frac{2}{m_2B_2N}$ *and there exists a number* $\tilde{R}_0>0$ *satisfying* $\tilde{r}\ll\tilde{R}_0\ll\tilde{R}$, *such that* $M(\tilde{R}_0)\leq\frac{\tilde{R}_0}{M_2(A_1+A_2)}$, *where* $M(\tilde{R})=\max\{f(t,u,v): |u|+|v|\leq\tilde{R}, t\in[0,1]\}$, $\tilde{r}>0$ *is small enough and* $\tilde{R}>0$ *is large enough,*

then the BVP (1.1) *has at least two positive solutions.*

Proof Since the proof of (ii) is similar to that of (i), here we only give the proof of (i).

Let $E = C^2[0,1]$, $\Omega_1 = \{u \in E : \|u\|_2 < r\}$, $\Omega_2 = \{u \in E : \|u\|_2 < R\}$. From the condition (i) or (iii) of Lemma 2.4, it follows that $\|Au\|_2 \le \|u\|_2$ when $u \in K \cap \partial\Omega_1$ or $u \in K \cap \partial\Omega_2$, respectively. Choose $\Omega_3 = \{u \in E : \|u\|_2 < R_0\}$ such that $\overline{\Omega}_1 \subset \Omega_3, \overline{\Omega}_3 \subset \Omega_2$. By Corollary 2.1, we know that $NR_0 \le |u(t)| + |u''(t)| \le R_0$ for any $t \in [\frac{1}{4}, \frac{3}{4}]$ and $u \in K \cap \partial\Omega_3$. Then by (2.9), we have

$$\|Au\|_2 \ge m_2 B_2 \int_{\frac{1}{4}}^{\frac{3}{4}} f(s, u(s), u''(s))\mathrm{d}s \ge m_2 \cdot B_2 \cdot m(R_0) \cdot \frac{1}{2} \ge R_0 = \|u\|_2.$$

Therefore, $\|Au\|_2 \ge \|u\|_2$ for any $u \in K \cap \partial\Omega_3$.

Applying Theorem 2.1, the BVP (1.1) has two positive solutions $u_1(t) \in K \cap (\overline{\Omega}_3 \backslash \Omega_1)$, $u_2(t) \in K \cap (\overline{\Omega}_2 \backslash \Omega_3)$. Together with Theorem 3.1, it follows that the BVP (1.1) has two distinct positive solutions $u_1(t)$ and $u_2(t)$. The proof is completed.

Corollary 3.2 *Suppose* (H$_1$)-(H$_3$) *hold, and f satisfies either*

(i) $\overline{f_0} = \overline{f_\infty} = 0$ *and* $m(1) \ge \frac{2}{m_2 B_2}$*; or*

(ii) $\underline{f_0} = \underline{f_\infty} = \infty$ *and* $M(1) \le \frac{1}{M_2(A_1+A_2)}$,

then the BVP (1.1) *has at least two positive solutions.*

Example 3.2 Consider the following boundary value problem

$$\begin{cases} u^{(4)}(t) + 3u''(t) - 4u(t) = f\left(t, u(t), u''(t)\right), \quad t \in (0,1), \\ u''(0) = u''(1) = 0, \\ u(0) = 0, \quad u(1) = \left(\dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_2}\right) \displaystyle\int_0^1 G_2(s,s) f\left(s, u(s), u''(s)\right) \mathrm{d}s, \end{cases} \tag{3.2}$$

where $q(t) = G_2(t,t)$, $M = 1$, $f(t,u,v) = \frac{1}{2}(|u| + |v|)^{\frac{1}{2}} + \frac{1}{5}(|u| + |v|)^2$, $\alpha = 4$ and $\beta = 3$. Then, by some calculations, we obtain

$$\begin{aligned} &\lambda_1 = -4, \quad \lambda_2 = 1, \quad \omega_1 = 2, \quad \omega_2 = 1, \quad M_1 \approx 0.389, \quad M_2 \approx 0.231, \\ &C_1 \approx 1.100, \quad C_2 = 1, \quad A_1 \approx 1.703, \quad A_2 \approx 3.812, \quad \overline{f_0} = \infty, \quad \underline{f_\infty} = \infty, \\ &M(1) = \max\{f : |u| + |v| \le 1, t \in [0,1]\} = \frac{1}{2} + \frac{1}{5} = 0.7 \le \frac{1}{M_2(A_1 + A_2)} \approx 0.785, \end{aligned}$$

which implies (ii) of Theorem 3.2 holds. Therefore, by Theorem 3.2, the BVP (3.2) has at least two positive solutions.

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