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Open Loop Saddle Point on Linear Quadratic Stochastic Differential Games

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Abstract: In this paper, we deal with one kind of two-player zero-sum linear quadratic stochastic differential game problem. We give the existence of an open loop saddle point if and only if the lower and upper values exist.

Key words: stochastic differential game, saddle point, open loop strategy

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1 Introduction

In this paper, we consider the two-player zero-sum linear quadratic stochastic differential games on a finite horizon. The fundamental theory of differential games was given in 1965 by [1]. Pontryagin's Maximum Principle (see [2]) and Bellman's Dynamic Programming (see [3]) are applied to games. Bensoussan[4], Bensoussan and Friedman[5] studied stochastic differential games. It is well known that the existence of open loop saddle points guarantees the existence of the value of the differential games; the existence and equivalence of the lower and upper values guarantee the existence of the value of the differential games. These statements can be found, for instance, in [6–8].

Zhang[9] considered the two-person linear quadratic differential games and showed that the value of the game exists if and only if both the upper and lower values exist. The same outcomes were proved by Delfour[10] by using another way. Specially, Mou and Yong[11] discussed two-person zero-sum linear quadratic stochastic differential games in Hilbert spaces. The stochastic form of this problem is studied in this paper and we can achieve the same outcomes: No need of equivalence of the lower and upper values, we can prove the existence of the saddle point if and only if the lower and upper values exist. Due to stochastic op-

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timal control (see [4], [12]) is concerned, in the present paper we use the Peng's stochastic maximum principle (see [12]) to gain the adjoint equation of this stochastic state system.

This paper is organized as follows: Section 2 provides the basic framework. Some results of payoff function are discussed in Section 3. The main outcomes are characterized in Section 4, where we prove the existence of the saddle point by the existence of lower and upper values in this differential game.

2 Statement of the Problem

Let Ω be a bounded smooth domain in $\mathbf{R}^n$, $(\Omega,\ \mathcal{F},\ P)$ be a probability space with filtration $\mathcal{F}^t$, and $W(\ \cdot\ )$ be an $\mathbf{R}^n$-valued standard Wiener process. We assume that

$$\mathcal{F}^t = \sigma\{W(s);\ 0 \le s \le t\}.$$

Let $\boldsymbol{x}$ be a solution of the following stochastic differential equation:

$$\begin{cases} \mathrm{d}\boldsymbol{x}(t) = (\boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}_1(t)\boldsymbol{u}(t) + \boldsymbol{B}_2(t)\boldsymbol{v}(t))\mathrm{d}t \\ \qquad\qquad +(\boldsymbol{D}(t)\boldsymbol{x}(t) + \boldsymbol{C}_1(t)\boldsymbol{u}(t) + \boldsymbol{C}_2(t)\boldsymbol{v}(t))\mathrm{d}w_t, \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases} \tag{2.1}$$

where $\boldsymbol{x}_0$ is the initial state at time $t = 0$. We call that $\boldsymbol{u}(t) \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^m)$, $m \ge 1$, is the strategy of the first player if, $\boldsymbol{u}(\ \cdot\ )$ is an $\mathcal{F}^t$-adapted process with values in U (a nonempty subset of $\mathbf{R}^m$ (control domain)) such that

$$E\bigg(\int_0^T |\boldsymbol{u}(t)|^2 \mathrm{d}t\bigg) < \infty,$$

and $\boldsymbol{v}(t) \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^k)$, $k \ge 1$, is the strategy of the second player.

For any choice of controls $\boldsymbol{u}$, $\boldsymbol{v}$, we have the following payoff function:

$$C_{\boldsymbol{x}_0}(\boldsymbol{u},\ \boldsymbol{v}) = \frac{1}{2}E\Big(\boldsymbol{F}\boldsymbol{x}(T)\cdot\boldsymbol{x}(T) + \int_0^T \boldsymbol{Q}(t)\boldsymbol{x}(t)\cdot\boldsymbol{x}(t) + |\boldsymbol{u}(t)|^2 - |\boldsymbol{v}(t)|^2\mathrm{d}t\Big). \tag{2.2}$$

We assume that $\boldsymbol{F}$ is an $n \times n$ matrix, and $\boldsymbol{A}(t)$, $\boldsymbol{B}_1(t)$, $\boldsymbol{B}_2(t)$, $\boldsymbol{C}_1(t)$, $\boldsymbol{C}_2(t)$, $\boldsymbol{D}(t)$ and $\boldsymbol{Q}(t)$ are matrix functions of appropriate order that are measurable and bounded a.e. in $[0, T]$. Moreover, $\boldsymbol{F}$ and $\boldsymbol{Q}(t)$ are symmetrical. We write $\boldsymbol{A}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, $\boldsymbol{D}$ and $\boldsymbol{Q}$ instead of $\boldsymbol{A}(t)$, $\boldsymbol{B}_1(t)$, $\boldsymbol{B}_2(t)$, $\boldsymbol{C}_1(t)$, $\boldsymbol{C}_2(t)$, $\boldsymbol{D}(t)$ and $\boldsymbol{Q}(t)$ throughout this paper and use the above assumptions. $T > 0$ is a given final time. $|\boldsymbol{x}|$ and $\boldsymbol{x}\cdot\boldsymbol{y}$ are the usual norm and inner product, respectively.

The more general quadratic structure involving cross terms and different quadratic weights $\boldsymbol{N}_1\boldsymbol{u}\cdot\boldsymbol{u}$ and $\boldsymbol{N}_2\boldsymbol{v}\cdot\boldsymbol{v}$ on $\boldsymbol{u}$ and $\boldsymbol{v}$ can be simplified to our model (see [10]).

Definition 2.1 *The game is said to achieve its open loop lower value if*

$$v^-(\boldsymbol{x}_0) = \sup_{\boldsymbol{v}(t)\in\mathcal{L}^2(0,T;\mathbf{R}^k)} \inf_{\boldsymbol{u}(t)\in\mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v})$$

is finite and is said to achieve its open loop upper value if

$$v^+(\boldsymbol{x}_0) = \inf_{\boldsymbol{u}(t)\in\mathcal{L}^2(0,T;\mathbf{R}^m)} \sup_{\boldsymbol{v}(t)\in\mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v})$$

is finite.

Obviously, we always have

$$v^-(\boldsymbol{x}_0) \le v^+(\boldsymbol{x}_0).$$

Definition 2.2 *If both $v^-(\boldsymbol{x}_0)$ and $v^+(\boldsymbol{x}_0)$ exist and $v^-(\boldsymbol{x}_0) = v^+(\boldsymbol{x}_0)$, then we say that the open loop value of the game exists and is denoted by $v(\boldsymbol{x}_0)$.*

Definition 2.3 *A pair of controls $(\bar{\boldsymbol{u}},\ \bar{\boldsymbol{v}}) \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^m) \times \mathcal{L}^2(0,\ T;\ \mathbf{R}^k)$ is called an open loop saddle point of the stochastic differential game* (2.1) *with payoff* (2.2), *if for all $(t,x) \in (0,T) \times \Omega$, $\boldsymbol{u} \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^m)$ and $\boldsymbol{v} \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^k)$,*

$$C_{\boldsymbol{x}_0}(\bar{\boldsymbol{u}},\ \boldsymbol{v}) \le C_{\boldsymbol{x}_0}(\bar{\boldsymbol{u}},\ \bar{\boldsymbol{v}}) \le C_{\boldsymbol{x}_0}(\boldsymbol{u},\ \bar{\boldsymbol{v}}). \tag{2.3}$$

By Definition 2.3, (2.3) is equivalent to

$$\sup_{\boldsymbol{v}\in\mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\boldsymbol{x}_0}(\bar{\boldsymbol{u}},\ \boldsymbol{v}) = C_{\boldsymbol{x}_0}(\bar{\boldsymbol{u}},\ \bar{\boldsymbol{v}}) = \inf_{\boldsymbol{u}\in\mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\ \bar{\boldsymbol{v}}).$$

Definition 2.4 *For $\boldsymbol{x}_0 \in \mathbf{R}^n$, we define*

$$\begin{aligned}
\boldsymbol{V}(\boldsymbol{x}_0) &= \{\boldsymbol{v} \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^k);\ \inf_{\boldsymbol{u}\in\mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v}) > -\infty\},\\
\boldsymbol{U}(\boldsymbol{x}_0) &= \{\boldsymbol{u} \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^m);\ \sup_{\boldsymbol{v}\in\mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v}) < +\infty\},\\
J^-_{\boldsymbol{x}_0}(\boldsymbol{v}) &= \inf_{\boldsymbol{u}\in\mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v}),\\
J^+_{\boldsymbol{x}_0}(\boldsymbol{u}) &= \sup_{\boldsymbol{v}\in\mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v}).
\end{aligned}$$

3 Some Results of Payoff Function

Since the payoff function (2.2) is quadratic, it is infinitely differentiable. We can prove

$$\mathrm{d}C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v};\ \bar{\boldsymbol{u}},\bar{\boldsymbol{v}}) = E(\boldsymbol{F}\boldsymbol{x}(T)\cdot\bar{\boldsymbol{y}}(T) + (\boldsymbol{Q}\boldsymbol{x},\bar{\boldsymbol{y}}) + (\boldsymbol{u},\bar{\boldsymbol{u}}) - (\boldsymbol{v},\bar{\boldsymbol{v}})), \tag{3.1}$$

where $\boldsymbol{x}$ is the solution of (2.1) and $\bar{\boldsymbol{y}}$ is the solution of

$$\begin{cases} \mathrm{d}\bar{\boldsymbol{y}} = (\boldsymbol{A}\bar{\boldsymbol{y}} + \boldsymbol{B}_1\bar{\boldsymbol{u}} + \boldsymbol{B}_2\bar{\boldsymbol{v}})\mathrm{d}t + (\boldsymbol{D}\bar{\boldsymbol{y}} + \boldsymbol{C}_1\bar{\boldsymbol{u}} + \boldsymbol{C}_2\bar{\boldsymbol{v}})\mathrm{d}w_t,\\ \bar{\boldsymbol{y}}(0) = \boldsymbol{0}. \end{cases}$$

Definition 3.1 *Given a real function f defined on a Banach space B, the first directional semiderivative at $\boldsymbol{x}$ in the direction $\boldsymbol{v}$ (when it exists) is defined as*

$$\mathrm{d}f(\boldsymbol{x};\boldsymbol{v}) = \lim_{t\to0}\frac{f(\boldsymbol{x}+t\boldsymbol{v}) - f(\boldsymbol{x})}{t}.$$

The second order bidirectional derivative at $\boldsymbol{x}$ in the directions $(\boldsymbol{v},\boldsymbol{w})$ (when it exists) is defined as

$$\mathrm{d}^2f(\boldsymbol{x};\ \boldsymbol{v},\boldsymbol{w}) = \lim_{t\to0}\frac{\mathrm{d}f(\boldsymbol{x}+t\boldsymbol{w};\ \boldsymbol{v}) - \mathrm{d}f(\boldsymbol{x};\ \boldsymbol{v})}{t}.$$

According to the definition of directional derivative, we have

$$\mathrm{d}C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v};\ \bar{\boldsymbol{u}},\bar{\boldsymbol{v}}) = \lim_{l\to0}\frac{1}{l}[C_{\boldsymbol{x}_0}(\boldsymbol{u}+l\bar{\boldsymbol{u}},\ \boldsymbol{v}+l\bar{\boldsymbol{v}}) - C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v})].$$

According to adjoint equation of (2.1) and (2.2), we can rewrite expression (3.1) in another form. Therefore, we quote some remarks on the stochastic differential control.

We define the Hamiltonian by

$$H(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v},\boldsymbol{p},\boldsymbol{K}) = (\boldsymbol{p},\ \boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}_1\boldsymbol{u}+\boldsymbol{B}_2\boldsymbol{v}) + \sum_{j=1}^{d}(\boldsymbol{K}_j,\ \boldsymbol{\sigma}^j(t,\boldsymbol{x},\boldsymbol{u},\boldsymbol{v}))$$
$$+\frac{1}{2}(\boldsymbol{Q}\boldsymbol{x}\cdot\boldsymbol{x}+|\boldsymbol{u}|^2-|\boldsymbol{v}|^2).$$

The adjoint equation of (2.1) and (2.2) is

$$\begin{cases} \mathrm{d}\boldsymbol{p}+(\boldsymbol{A}^*\boldsymbol{p}+\boldsymbol{Q}\boldsymbol{x}+\langle\boldsymbol{\sigma}_{\boldsymbol{x}}'^*,\ \boldsymbol{K}\rangle)\mathrm{d}t-\boldsymbol{K}\mathrm{d}w_t=0, \\ \boldsymbol{p}(T)=\boldsymbol{F}\boldsymbol{x}(T), \end{cases} \tag{3.2}$$

where

$$\begin{aligned}\boldsymbol{\sigma} &= (\sigma^1,\sigma^2,\cdots,\sigma^d)\\ &= \boldsymbol{D}\boldsymbol{x}+\boldsymbol{C}_1\boldsymbol{u}+\boldsymbol{C}_2\boldsymbol{v}\in[0,T]\times\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}^k.\end{aligned}$$

Moreover, $(\boldsymbol{p}(\,\cdot\,),\ \boldsymbol{K}(\,\cdot\,))\in\mathcal{L}^2(0,\ T;\ \mathbf{R}^n)\times(\mathcal{L}^2(0,\ T;\ \mathbf{R}^n))^d$ and $\boldsymbol{K}=(\boldsymbol{K}_1,\boldsymbol{K}_2,\cdots,\boldsymbol{K}_d)$,

$$\boldsymbol{K}(t)=\boldsymbol{\sigma}_{\boldsymbol{x}}'^*(t,\boldsymbol{x},\boldsymbol{u},\boldsymbol{v})\boldsymbol{p}(t)-\boldsymbol{\Psi}^*(t)\boldsymbol{G}(t),$$

where $\boldsymbol{\Psi}(t)$ is defined by

$$\begin{cases} \mathrm{d}\boldsymbol{\Psi}=-\langle\boldsymbol{\Psi}\boldsymbol{\sigma}_{\boldsymbol{x}}',\ \mathrm{d}w_t\rangle+\langle\boldsymbol{\Psi}\boldsymbol{\sigma}_{\boldsymbol{x}}',\ \boldsymbol{\sigma}_{\boldsymbol{x}}'\rangle\mathrm{d}t-\boldsymbol{\Psi}\boldsymbol{A}\mathrm{d}t, \\ \boldsymbol{\Psi}(0)=\boldsymbol{I}, \end{cases}$$

and

$$\int_0^t \boldsymbol{G}(s)\mathrm{d}w(s)=E^{\mathcal{F}_t}\boldsymbol{X}-E\boldsymbol{X}$$

with

$$\boldsymbol{X}=\boldsymbol{\Phi}^*(T)\boldsymbol{F}\boldsymbol{x}(T)+\int_0^T\boldsymbol{\Phi}^*(s)\boldsymbol{Q}\boldsymbol{x}(s)\mathrm{d}s,$$

where $\boldsymbol{\Phi}(t)$ is defined by

$$\begin{cases} \mathrm{d}\boldsymbol{\Phi}=\boldsymbol{A}\boldsymbol{\Phi}\mathrm{d}t+\langle\boldsymbol{\sigma}_{\boldsymbol{x}}'\boldsymbol{\Phi},\ \mathrm{d}w_t\rangle, \\ \boldsymbol{\Phi}(0)=\boldsymbol{I}, \end{cases}$$

and the following property holds

$$\boldsymbol{\Psi}(t)\boldsymbol{\Phi}(t)=\boldsymbol{I}=\boldsymbol{\Phi}(t)\boldsymbol{\Psi}(t).$$

For the above assumptions and discussions about Hamiltonian and the adjoint equation, see [12] and [4].

Proposition 3.1 *According to adjoint equation* (3.2), *we can rewrite expression* (3.1) *in the following form*:

$$\mathrm{d}C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v};\ \bar{\boldsymbol{u}},\bar{\boldsymbol{v}})=E((\boldsymbol{B}_1^*\boldsymbol{p}+\boldsymbol{C}_1^*\boldsymbol{K}+\boldsymbol{u},\ \bar{\boldsymbol{u}})+(\boldsymbol{B}_2^*\boldsymbol{p}+\boldsymbol{C}_2^*\boldsymbol{K}-\boldsymbol{v},\ \bar{\boldsymbol{v}})).$$

Proof. By Itô formula,

$$\mathrm{d}(\boldsymbol{p}\bar{\boldsymbol{y}})=\bar{\boldsymbol{y}}\mathrm{d}\boldsymbol{p}+\boldsymbol{p}\mathrm{d}\bar{\boldsymbol{y}}+(\boldsymbol{C}_1\bar{\boldsymbol{u}}+\boldsymbol{C}_2\bar{\boldsymbol{v}})\cdot\boldsymbol{K}\mathrm{d}t.$$

Thus

$$\begin{aligned}\mathrm{d}C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v};\ \bar{\boldsymbol{u}},\bar{\boldsymbol{v}}) &= E\bigg(\boldsymbol{F}\boldsymbol{x}(T)\cdot\bar{\boldsymbol{y}}(T)+\int_0^T\boldsymbol{Q}(t)\boldsymbol{x}(t)\cdot\bar{\boldsymbol{y}}(t)\mathrm{d}t+(\boldsymbol{u},\bar{\boldsymbol{u}})-(\boldsymbol{v},\bar{\boldsymbol{v}})\bigg)\\ &= E\bigg(\boldsymbol{p}(T)\cdot\bar{\boldsymbol{y}}(T)+\int_0^T\bar{\boldsymbol{y}}\cdot\boldsymbol{K}\mathrm{d}w-\int_0^T\bar{\boldsymbol{y}}\cdot\boldsymbol{A}^*\boldsymbol{p}\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T \bar{\boldsymbol{y}}\mathrm{d}\boldsymbol{p}-\int_0^T \boldsymbol{D}^*\boldsymbol{K}\cdot\bar{\boldsymbol{y}}\mathrm{d}t+(\boldsymbol{u},\bar{\boldsymbol{u}})-(\boldsymbol{v},\bar{\boldsymbol{v}})\Big)\\
=&\,E\Big(\int_0^T \boldsymbol{p}\mathrm{d}\bar{\boldsymbol{y}}+\int_0^T(\boldsymbol{D}\bar{\boldsymbol{y}}+\boldsymbol{C}_1\bar{\boldsymbol{u}}+\boldsymbol{C}_2\bar{\boldsymbol{v}})\cdot\boldsymbol{K}\mathrm{d}t\\
&-\int_0^T \bar{\boldsymbol{y}}\cdot\boldsymbol{A}^*\boldsymbol{p}\mathrm{d}t-\int_0^T \boldsymbol{D}^*\boldsymbol{K}\cdot\bar{\boldsymbol{y}}\mathrm{d}t+(\boldsymbol{u},\ \bar{\boldsymbol{u}})-(\boldsymbol{v},\bar{\boldsymbol{v}})\Big)\\
=&\,E\Big(\int_0^T(\boldsymbol{B}_1\bar{\boldsymbol{u}}+\boldsymbol{B}_2\bar{\boldsymbol{v}})\cdot\boldsymbol{p}\mathrm{d}t+\int_0^T(\boldsymbol{D}\bar{\boldsymbol{y}}+\boldsymbol{C}_1\bar{\boldsymbol{u}}+\boldsymbol{C}_2\bar{\boldsymbol{v}})\cdot\boldsymbol{K}\mathrm{d}t\\
&-\int_0^T \boldsymbol{D}^*\boldsymbol{K}\cdot\bar{\boldsymbol{y}}\mathrm{d}t+(\boldsymbol{u},\ \bar{\boldsymbol{u}})-(\boldsymbol{v},\ \bar{\boldsymbol{v}})\Big)\\
=&\,E((\boldsymbol{B}_1^*\boldsymbol{p},\ \bar{\boldsymbol{u}})+(\boldsymbol{B}_2^*\boldsymbol{p},\ \bar{\boldsymbol{v}})+(\boldsymbol{C}_2^*\boldsymbol{K},\bar{\boldsymbol{u}})+(\boldsymbol{C}_2^*\boldsymbol{K},\ \bar{\boldsymbol{v}})+(\boldsymbol{u},\ \bar{\boldsymbol{u}})-(\boldsymbol{v},\ \bar{\boldsymbol{v}}))\\
=&\,E((\boldsymbol{B}_1^*\boldsymbol{p}+\boldsymbol{C}_1^*\boldsymbol{K}+\boldsymbol{u},\ \bar{\boldsymbol{u}})+(\boldsymbol{B}_2^*\boldsymbol{p}+\boldsymbol{C}_2^*\boldsymbol{K}-\boldsymbol{v},\ \bar{\boldsymbol{v}})).
\end{aligned}
$$

Similarly, the second order bidirectional derivative of payoff function is of the following form:

$$\mathrm{d}^2C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v};\ \bar{\boldsymbol{u}},\bar{\boldsymbol{v}};\ \tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}})=E(\boldsymbol{F}\bar{\boldsymbol{y}}(T)\cdot\tilde{\boldsymbol{y}}(T)+(\boldsymbol{Q}\bar{\boldsymbol{y}},\ \tilde{\boldsymbol{y}})+(\bar{\boldsymbol{u}},\ \tilde{\boldsymbol{u}})-(\bar{\boldsymbol{v}},\ \tilde{\boldsymbol{v}})),$$

where

$$\begin{cases}\mathrm{d}\tilde{\boldsymbol{y}}=(A\tilde{\boldsymbol{y}}+\boldsymbol{B}_1\tilde{\boldsymbol{u}}+\boldsymbol{B}_2\tilde{\boldsymbol{v}})\mathrm{d}t+(\boldsymbol{D}\tilde{\boldsymbol{y}}+\boldsymbol{C}_1\tilde{\boldsymbol{u}}+\boldsymbol{C}_2\tilde{\boldsymbol{u}})\mathrm{d}w_t,\\ \tilde{\boldsymbol{y}}(0)=\boldsymbol{0}.\end{cases}$$

In particular, for all $\boldsymbol{x}_0$, $\boldsymbol{u}$, $\boldsymbol{v}$, $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{v}}$,

$$\mathrm{d}^2C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v};\ \bar{\boldsymbol{u}},\bar{\boldsymbol{v}};\ \bar{\boldsymbol{u}},\bar{\boldsymbol{v}})=C_{\boldsymbol{0}}(\bar{\boldsymbol{u}},\ \bar{\boldsymbol{v}}).\tag{3.3}$$

Namely, the second order bidirectional derivative of payoff function is independent of $\boldsymbol{x}_0$ and $(\boldsymbol{u},\ \boldsymbol{v})$. So we have the following lemma.

Lemma 3.1 *The following statements are equivalent*:

(1) *The map* $\boldsymbol{u}\to C_{\boldsymbol{0}}(\boldsymbol{u},\boldsymbol{0}):\mathcal{L}^2(0,\ T;\ \mathbf{R}^m)\to\mathbf{R}$ *is convex*;

(2) *For all* $\boldsymbol{u}\in\mathcal{L}^2(0,\ T;\ \mathbf{R}^m)$, $C_{\boldsymbol{0}}(\boldsymbol{u},\boldsymbol{0})\geq 0$;

(3) $\inf\limits_{\boldsymbol{u}\in\mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\boldsymbol{0}}(\boldsymbol{u},\boldsymbol{0})=C_{\boldsymbol{0}}(\boldsymbol{0},\boldsymbol{0})$;

(4) *For all* $\boldsymbol{v}$ *and* $\boldsymbol{x}_0$, *the map* $\boldsymbol{u}\to C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v}):\mathcal{L}^2(0,\ T;\ \mathbf{R}^m)\to\mathbf{R}$ *is convex.*

Corollary 3.1 *The following statements are equivalent*:

(1) *The map* $\boldsymbol{v}\to C_{\boldsymbol{0}}(\boldsymbol{0},\boldsymbol{v}):\mathcal{L}^2(0,\ T;\ \mathbf{R}^k)\to\mathbf{R}$ *is concave*;

(2) *For all* $\boldsymbol{v}\in\mathcal{L}^2(0,\ T;\ \mathbf{R}^k)$, $C_{\boldsymbol{0}}(\boldsymbol{0},\boldsymbol{v})\leq 0$;

(3) $\sup\limits_{\boldsymbol{v}\in\mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\boldsymbol{0}}(\boldsymbol{0},\boldsymbol{v})=C_{\boldsymbol{0}}(\boldsymbol{0},\boldsymbol{0})$;

(4) *For all* $\boldsymbol{u}$ *and* $\boldsymbol{x}_0$, *the map* $\boldsymbol{v}\to C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v}):\mathcal{L}^2(0,\ T;\ \mathbf{R}^k)\to\mathbf{R}$ *is concave.*

Corollary 3.2 *The following statements are equivalent*:

(1) *The map* $(\boldsymbol{u},\boldsymbol{v})\to C_{\boldsymbol{0}}(\boldsymbol{u},\boldsymbol{v}):\mathcal{L}^2(0,\ T;\ \mathbf{R}^m)\times\mathcal{L}^2(0,\ T;\ \mathbf{R}^k)\to\mathbf{R}$ *is* $(\boldsymbol{u},\boldsymbol{v})$*-convex-concave. That is, for any* $\boldsymbol{v}\in\mathcal{L}^2(0,\ T;\ \mathbf{R}^k)$,

$$\boldsymbol{u}\to C_{\boldsymbol{0}}(\boldsymbol{u},\boldsymbol{v})$$

is convex, and for any $\boldsymbol{u} \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^m)$,

$$\boldsymbol{v} \to C_{\mathbf{0}}(\boldsymbol{u}, \boldsymbol{v})$$

is concave;

(2) *The pair* $(\mathbf{0}, \mathbf{0})$ *is a saddle point of* $C_{\mathbf{0}}(\boldsymbol{u}, \boldsymbol{v})$:

$$\inf_{\boldsymbol{u} \in \mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\mathbf{0}}(\boldsymbol{u}, \mathbf{0}) = C_{\mathbf{0}}(\mathbf{0}, \mathbf{0}) = \sup_{\boldsymbol{v} \in \mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\mathbf{0}}(\mathbf{0}, \boldsymbol{v});$$

(3) *For all* $\boldsymbol{x}_0$, *the map* $(\boldsymbol{u}, \boldsymbol{v}) \to C_{\boldsymbol{x}_0}(\boldsymbol{u}, \boldsymbol{v}) : \mathcal{L}^2(0,\ T;\ \mathbf{R}^m) \times \mathcal{L}^2(0,\ T;\ \mathbf{R}^k) \to \mathbf{R}$ *is* $(\boldsymbol{u}, \boldsymbol{v})$*-convex-concave. That is, for any* $\boldsymbol{v} \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^m)$,

$$\boldsymbol{u} \to C_{\boldsymbol{x}_0}(\boldsymbol{u}, \boldsymbol{v})$$

is convex, and for any $\boldsymbol{u} \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^m)$,

$$\boldsymbol{v} \to C_{\boldsymbol{x}_0}(\boldsymbol{u}, \boldsymbol{v})$$

is concave.

Theorem 3.1 *If* $\boldsymbol{V}(\boldsymbol{x}_0) \neq \emptyset$ *and* $\boldsymbol{U}(\boldsymbol{x}_0) \neq \emptyset$, *then the saddle point of payoff* $C_{\mathbf{0}}(\boldsymbol{u}, \boldsymbol{v})$ *exists and it is* $(\mathbf{0}, \mathbf{0})$.

Proof. By the assumption, there exists a $\boldsymbol{v} \in V(\boldsymbol{x}_0)$ such that $\inf\limits_{\boldsymbol{u} \in \mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\boldsymbol{x}_0}(\boldsymbol{u}, \boldsymbol{v})$ exists, i.e., there exists an $\bar{\boldsymbol{u}} \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^m)$ such that

$$\inf_{\boldsymbol{u} \in \mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\ \boldsymbol{v}) = C_{\boldsymbol{x}_0}(\bar{\boldsymbol{u}},\ \boldsymbol{v}).$$

Thus

$$\mathrm{d}C_{\boldsymbol{x}_0}(\bar{\boldsymbol{u}}, \boldsymbol{v};\ \mathbf{0}, \boldsymbol{w}) = E((\boldsymbol{B}_2^* \boldsymbol{p} + \boldsymbol{C}_2^* \boldsymbol{K} - \boldsymbol{v},\ \boldsymbol{w})), \qquad \boldsymbol{w} \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^k).$$

From the definition of directional derivative one has

$$\begin{aligned}
&\mathrm{d}^2 C_{\boldsymbol{x}_0}(\bar{\boldsymbol{u}}, \boldsymbol{v};\ \mathbf{0}, \boldsymbol{w};\ \mathbf{0}, \boldsymbol{w}) \\
=\ & \lim_{l \to 0} \frac{1}{l} E((\boldsymbol{B}_2^* \boldsymbol{p} + \boldsymbol{C}_2^* \boldsymbol{K} - (\boldsymbol{v} + l\boldsymbol{w}),\ \boldsymbol{w}) - (\boldsymbol{B}_2^* \boldsymbol{p} + \boldsymbol{C}_2^* \boldsymbol{K} - \boldsymbol{v},\ \boldsymbol{w})) \\
=\ & -E((\boldsymbol{w},\ \boldsymbol{w})) \\
\leq\ & 0.
\end{aligned}$$

By (3.3), it follows that

$$0 \geq \mathrm{d}^2 C_{\boldsymbol{x}_0}(\bar{\boldsymbol{u}}, \boldsymbol{v};\ \mathbf{0}, \boldsymbol{w};\ \mathbf{0}, \boldsymbol{w}) = C_{\mathbf{0}}(\mathbf{0}, \boldsymbol{w}), \qquad \boldsymbol{w} \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^k).$$

By Corollary 3.1 we have

$$\sup_{\boldsymbol{v} \in \mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\mathbf{0}}(\mathbf{0}, \boldsymbol{v}) = C_{\mathbf{0}}(\mathbf{0}, \mathbf{0}).$$

So

$$\inf_{\boldsymbol{u} \in \mathcal{L}^2(0,T;\mathbf{R}^m)} \sup_{\boldsymbol{v} \in \mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\mathbf{0}}(\boldsymbol{u}, \boldsymbol{v}) \leq \sup_{\boldsymbol{v} \in \mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\mathbf{0}}(\mathbf{0}, \boldsymbol{v}) = C_{\mathbf{0}}(\mathbf{0}, \mathbf{0}).$$

Similarly, since $\boldsymbol{U}(\boldsymbol{x}_0) \neq \emptyset$, by Lemma 3.1, we have

$$\sup_{\boldsymbol{v} \in \mathcal{L}^2(0,T;\mathbf{R}^k)} \inf_{\boldsymbol{u} \in \mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\mathbf{0}}(\boldsymbol{u}, \boldsymbol{v}) \geq \inf_{\boldsymbol{u} \in \mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\mathbf{0}}(\boldsymbol{u}, \mathbf{0}) = C_{\mathbf{0}}(\mathbf{0}, \mathbf{0}).$$

Hence

$$v^-(0) = \sup_{\boldsymbol{v} \in \mathcal{L}^2(0,T;\mathbf{R}^k)} \inf_{\boldsymbol{u} \in \mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\mathbf{0}}(\boldsymbol{u}, \boldsymbol{v})$$

$$
\begin{aligned}
&\geq \inf_{\boldsymbol{u}\in\mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\mathbf{0}}(\boldsymbol{u},\mathbf{0})\\
&= C_{\mathbf{0}}(\mathbf{0},\mathbf{0})\\
&= \sup_{\boldsymbol{v}\in\mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\mathbf{0}}(\mathbf{0},\boldsymbol{v})\\
&\geq \inf_{\boldsymbol{u}\in\mathcal{L}^2(0,T;\mathbf{R}^m)} \sup_{\boldsymbol{v}\in\mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\mathbf{0}}(\boldsymbol{u},\boldsymbol{v})\\
&= v^+(0),
\end{aligned}
$$

which shows that the saddle point of the payoff $C_{\mathbf{0}}(\boldsymbol{u},\boldsymbol{v})$ exists and it is $(\mathbf{0},\ \mathbf{0})$. The proof is completed.

Now we show the payoff $C_{\boldsymbol{x}_0}$ of the game when

$$\boldsymbol{u} = -\boldsymbol{B}_1^*\boldsymbol{p} - \boldsymbol{C}_1^*\boldsymbol{K}, \qquad \boldsymbol{v} = \boldsymbol{B}_2^*\boldsymbol{p} + \boldsymbol{C}_2^*\boldsymbol{K}$$

in (2.1).

Theorem 3.2 *There exists a solution $(\boldsymbol{x},\boldsymbol{p},\boldsymbol{K})$ of the adjoint system*

$$
\begin{cases}
\mathrm{d}\boldsymbol{x} = [\boldsymbol{A}\boldsymbol{x} + (-\boldsymbol{B}_1\boldsymbol{B}_1^* + \boldsymbol{B}_2\boldsymbol{B}_2^*)\boldsymbol{p} + (-\boldsymbol{B}_1\boldsymbol{C}_1^* + \boldsymbol{B}_2\boldsymbol{C}_2^*)\boldsymbol{K}]\mathrm{d}t\\
\qquad\quad +[\boldsymbol{D}\boldsymbol{x} + (-\boldsymbol{C}_1\boldsymbol{B}_1^* + \boldsymbol{C}_2\boldsymbol{B}_2^*)\boldsymbol{p} + (-\boldsymbol{C}_1\boldsymbol{C}_1^* + \boldsymbol{C}_2\boldsymbol{C}_2^*)\boldsymbol{K}]\mathrm{d}w_t,\\
\mathrm{d}\boldsymbol{p} + (\boldsymbol{A}^*\boldsymbol{p} + \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{D}^*\boldsymbol{K})\mathrm{d}t - K\mathrm{d}w_t = 0,\\
\boldsymbol{x}(0) = \boldsymbol{x}_0,\\
\boldsymbol{p}(T) = \boldsymbol{F}\boldsymbol{x}(T).
\end{cases}
\tag{3.4}
$$

If

$$\boldsymbol{u}^* = -\boldsymbol{B}_1^*\boldsymbol{p} - \boldsymbol{C}_1^*\boldsymbol{K}, \qquad \boldsymbol{v}^* = \boldsymbol{B}_2^*\boldsymbol{p} + \boldsymbol{C}_2^*\boldsymbol{K}, \tag{3.5}$$

then

$$C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v}^*) = \frac{1}{2}E(\boldsymbol{p}(0)\cdot\boldsymbol{x}_0). \tag{3.6}$$

Proof. By Itô formula one has

$$\mathrm{d}(\boldsymbol{p}\boldsymbol{x}) = \boldsymbol{x}\mathrm{d}\boldsymbol{p} + \boldsymbol{p}\mathrm{d}\boldsymbol{x} + (\boldsymbol{D}\boldsymbol{x} + \boldsymbol{C}_1\boldsymbol{u}^* + \boldsymbol{C}_2\boldsymbol{v}^*)\cdot\boldsymbol{K}\mathrm{d}t,$$

and

$$
\begin{aligned}
\int_0^T |\boldsymbol{u}^*|^2 - |\boldsymbol{v}^*|^2\mathrm{d}t &= (\boldsymbol{u}^*,\boldsymbol{u}^*) - (\boldsymbol{v}^*,\boldsymbol{v}^*)\\
&= (-\boldsymbol{B}_1^*\boldsymbol{p} - \boldsymbol{C}_1^*\boldsymbol{K},\ -\boldsymbol{B}_1^*\boldsymbol{p} - \boldsymbol{C}_1^*\boldsymbol{K}) - (\boldsymbol{B}_2^*\boldsymbol{p} + \boldsymbol{C}_2^*\boldsymbol{K},\ \boldsymbol{B}_2^*\boldsymbol{p} + \boldsymbol{C}_2^*\boldsymbol{K})\\
&= (\boldsymbol{B}_1\boldsymbol{B}_1^*\boldsymbol{p} + \boldsymbol{B}_1\boldsymbol{C}_1^*\boldsymbol{K},\ \boldsymbol{p}) - (\boldsymbol{B}_2\boldsymbol{B}_2^*\boldsymbol{p} + \boldsymbol{B}_2\boldsymbol{C}_2^*\boldsymbol{K},\ \boldsymbol{p})\\
&\quad + (\boldsymbol{C}_1\boldsymbol{B}_1^*\boldsymbol{p} + \boldsymbol{C}_1\boldsymbol{C}_1^*\boldsymbol{K},\ \boldsymbol{K}) - (\boldsymbol{C}_2\boldsymbol{B}_2^*\boldsymbol{p} + \boldsymbol{C}_2\boldsymbol{C}_2^*\boldsymbol{K},\ \boldsymbol{K})\\
&= (-\boldsymbol{B}_1\boldsymbol{u}^* - \boldsymbol{B}_2\boldsymbol{v}^*,\ \boldsymbol{p}) + (-\boldsymbol{C}_1\boldsymbol{u}^* - \boldsymbol{C}_2\boldsymbol{v}^*,\ \boldsymbol{K}).
\end{aligned}
$$

Then

$$
\begin{aligned}
C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\boldsymbol{v}^*) &= \frac{1}{2}E\biggl(\boldsymbol{F}\boldsymbol{x}(T)\cdot\boldsymbol{x}(T) + \int_0^T \boldsymbol{Q}(t)\boldsymbol{x}(t)\cdot\boldsymbol{x}(t) + |\boldsymbol{u}^*(t)|^2 - |\boldsymbol{v}^*(t)|^2\mathrm{d}t\biggr)\\
&= \frac{1}{2}E\biggl(\boldsymbol{p}(T)\cdot\boldsymbol{x}(T) - \int_0^T \boldsymbol{x}\cdot\boldsymbol{A}^*\boldsymbol{p}\mathrm{d}t - \int_0^T \boldsymbol{x}\mathrm{d}\boldsymbol{p}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \int_0^T \boldsymbol{x} \cdot \boldsymbol{D}^* \boldsymbol{K} \mathrm{d}t + \int_0^T |\boldsymbol{u}^*|^2 - |\boldsymbol{v}^*|^2 \mathrm{d}t \bigg) \\
&= \frac{1}{2} E\bigg(\boldsymbol{p}(0) \cdot \boldsymbol{x}_0 + \int_0^T \boldsymbol{p} \mathrm{d}\boldsymbol{x} + \int_0^T (\boldsymbol{D}\boldsymbol{x} + \boldsymbol{C}_1 \boldsymbol{u}^* + \boldsymbol{C}_2 \boldsymbol{v}^*) \cdot \boldsymbol{K} \mathrm{d}t \\
&\quad - \int_0^T \boldsymbol{x} \cdot \boldsymbol{D}^* \boldsymbol{K} \mathrm{d}t - \int_0^T \boldsymbol{x} \cdot \boldsymbol{A}^* \boldsymbol{p} \mathrm{d}t + (-\boldsymbol{B}_1 \boldsymbol{u}^* - \boldsymbol{B}_2 \boldsymbol{v}^*,\ \boldsymbol{p}) \\
&\quad + (-\boldsymbol{C}_1 \boldsymbol{u}^* - \boldsymbol{C}_2 \boldsymbol{v}^*,\ \boldsymbol{K}) \bigg) \\
&= \frac{1}{2} E\bigg(\boldsymbol{p}(0) \cdot \boldsymbol{x}_0 + \int_0^T (\boldsymbol{D}\boldsymbol{x} + \boldsymbol{C}_1 \boldsymbol{u}^* + \boldsymbol{C}_2 \boldsymbol{v}^*) \cdot \boldsymbol{K} \mathrm{d}t - \int_0^T \boldsymbol{x} \cdot \boldsymbol{D}^* \boldsymbol{K} \mathrm{d}t \\
&\quad + \int_0^T \boldsymbol{p} \cdot \boldsymbol{B}_1 (-\boldsymbol{B}_1^* \boldsymbol{p} - \boldsymbol{C}_1^* \boldsymbol{K}) \mathrm{d}t + \int_0^T \boldsymbol{p} \cdot \boldsymbol{B}_2 (\boldsymbol{B}_2^* \boldsymbol{p} + \boldsymbol{C}_2^* \boldsymbol{K}) \mathrm{d}t \\
&\quad + (-\boldsymbol{B}_1 \boldsymbol{u}^* - \boldsymbol{B}_2 \boldsymbol{v}^*, \boldsymbol{p}) + (-\boldsymbol{C}_1 \boldsymbol{u}^* - \boldsymbol{C}_2 \boldsymbol{v}^*, \boldsymbol{K}) \bigg) \\
&= \frac{1}{2} E(\boldsymbol{p}(0) \cdot \boldsymbol{x}_0).
\end{aligned}
$$

4 Main Results

In this section, we prove the existence of the saddle point of the system (2.1)-(2.3) if and only if the lower and upper values exist.

Definition 4.1 *We define*

$$\mathcal{A}(\boldsymbol{v},\ \boldsymbol{x}_0) = \{(\boldsymbol{p},\ \boldsymbol{K});\ (\boldsymbol{x},\ \boldsymbol{p},\ \boldsymbol{K}) \text{ is the solution of } (4.1)\},$$

and

$$
\begin{cases}
\mathrm{d}\boldsymbol{x} = [\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1(-\boldsymbol{B}_1^* \boldsymbol{p} - \boldsymbol{C}_1^* \boldsymbol{K}) + \boldsymbol{B}_2 \boldsymbol{v}]\mathrm{d}t \\
\qquad\quad + [\boldsymbol{D}\boldsymbol{x} + \boldsymbol{C}_1(-\boldsymbol{B}_1^* \boldsymbol{p} - \boldsymbol{C}_1^* \boldsymbol{K}) + \boldsymbol{C}_2 \boldsymbol{v}]\mathrm{d}w_t, \\
\mathrm{d}\boldsymbol{p} + (\boldsymbol{A}^* \boldsymbol{p} + \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{D}^* \boldsymbol{K})\mathrm{d}t - \boldsymbol{K}\mathrm{d}w = 0, \\
\boldsymbol{x}(0) = \boldsymbol{x}_0, \\
\boldsymbol{p}(T) = \boldsymbol{F}\boldsymbol{x}(T).
\end{cases}
\tag{4.1}
$$

The main result in this paper is the following theorem.

Theorem 4.1 *Consider the stochastic differential game* (2.1) *and* (2.3). *The following statements are equivalent*:

(1) *There exists an open loop saddle point of* $C_{\boldsymbol{x}_0}(\boldsymbol{u},\ \boldsymbol{v})$;

(2) *The value of the game exists*;

(3) *Both the lower value and the upper value of the game exist.*

The proof of Theorem 4.1 is discussed later. To prove it, some other theorems and discussions are needed. Firstly, we consider a part of Theorem 4.1: the open loop lower value of the game.

Theorem 4.2 *The following statements are equivalent*:

(1) *There exist* $\boldsymbol{u}^* \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^m)$ *and* $\boldsymbol{v}^* \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^k)$ *such that*

$$C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v}^*) = \inf_{\boldsymbol{u}\in\mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\ \boldsymbol{v}^*) = \sup_{\boldsymbol{v}\in\mathcal{L}^2(0,T;\mathbf{R}^k)} \inf_{\boldsymbol{u}\in\mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\ \boldsymbol{v}); \tag{4.2}$$

(2) *The open loop lower value* $v^-(\boldsymbol{x}_0)$ *of the game exists*;

(3) *There exists a solution* $(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{K})$ *of the adjoint system* (3.4) *such that* $\boldsymbol{B}_2^*\boldsymbol{p}+\boldsymbol{C}_2^*\boldsymbol{K} \in \boldsymbol{V}(\boldsymbol{x}_0)$, *the solution pairs* $(\boldsymbol{u}^*,\ \boldsymbol{v}^*)$ *is* (3.5), *and the open loop lower value are given by* (3.6).

Proof. To prove this theorem, we need four steps.

(a) We show that if lower value exists, then for any $\boldsymbol{v} \in \boldsymbol{V}(\boldsymbol{x}_0)$, one has

$$\begin{aligned} J^-_{\boldsymbol{x}_0} &= C_{\boldsymbol{x}_0}(-\boldsymbol{B}_1^*\boldsymbol{p} - \boldsymbol{C}_1^*\boldsymbol{K},\ \boldsymbol{v}) \\ &= \frac{1}{2}E\bigg(\boldsymbol{p}(0)\cdot\boldsymbol{x}_0 + \int_0^T (\boldsymbol{v}\cdot\boldsymbol{B}_2^*\boldsymbol{p} - |\boldsymbol{v}|^2)\mathrm{d}t + (\boldsymbol{C}_2\boldsymbol{v},\ \boldsymbol{K})\bigg), \end{aligned} \tag{4.3}$$

where $(\boldsymbol{p}, \boldsymbol{K}) \in \mathcal{A}(\boldsymbol{v},\ \boldsymbol{x}_0)$.

By the standard stochastic extremal principle (see [4]), (2.1) and (3.2), $\boldsymbol{u}^*$ is an optimizer if

$$\boldsymbol{u}^* = -\boldsymbol{B}_1^*\boldsymbol{p} - \boldsymbol{C}_1^*\boldsymbol{K}.$$

Similarly to Theorem 3.2, we can get (4.3).

(b) We show that if lower value exists, then the following statements hold:

(i)

$$\mathcal{A}(\boldsymbol{v},\ \boldsymbol{x}_0) = (\boldsymbol{p},\ \boldsymbol{K}) + \mathcal{A}(\boldsymbol{0},\ \boldsymbol{0}), \qquad (\boldsymbol{p},\ \boldsymbol{K}) \in \mathcal{A}(\boldsymbol{v},\ \boldsymbol{x}_0), \tag{4.4}$$

where

$$\mathcal{A}(\boldsymbol{0},\ \boldsymbol{0}) = \{(\bar{\boldsymbol{p}},\ \bar{\boldsymbol{K}});\ (\bar{\boldsymbol{x}},\ \bar{\boldsymbol{p}},\ \bar{\boldsymbol{K}}) \text{ is the solution of (4.5)}\},$$

and

$$\begin{cases} \mathrm{d}\bar{\boldsymbol{x}} = [\boldsymbol{A}\bar{\boldsymbol{x}} + \boldsymbol{B}_1(-\boldsymbol{B}_1^*\bar{\boldsymbol{p}} - \boldsymbol{C}_1^*\bar{\boldsymbol{K}})]\mathrm{d}t + [\boldsymbol{D}\bar{\boldsymbol{x}} + \boldsymbol{C}_1(-\boldsymbol{B}_1^*\bar{\boldsymbol{p}} - \boldsymbol{C}_1^*\bar{\boldsymbol{K}})]\mathrm{d}w_t, \\ \mathrm{d}\bar{\boldsymbol{p}} + (\boldsymbol{A}^*\bar{\boldsymbol{p}} + \boldsymbol{Q}\bar{\boldsymbol{x}} + \boldsymbol{D}^*\bar{\boldsymbol{K}})\mathrm{d}t - \bar{\boldsymbol{K}}\mathrm{d}w_t = 0, \\ \bar{\boldsymbol{x}}(0) = \boldsymbol{0}, \\ \bar{\boldsymbol{p}}(T) = \boldsymbol{F}\bar{\boldsymbol{x}}(T); \end{cases} \tag{4.5}$$

(ii) For all $\boldsymbol{v} \in \boldsymbol{V}(\boldsymbol{0})$,

$$E((\boldsymbol{v},\ \boldsymbol{B}_2^*\bar{\boldsymbol{p}} + \boldsymbol{C}_2^*\bar{\boldsymbol{K}})) = 0,$$

and we denote $\boldsymbol{V}(\boldsymbol{0}) = \mathcal{B}^\perp$ in the sense of expectation;

(iii)

$$\boldsymbol{V}(\boldsymbol{x}_0) = \boldsymbol{v} + \boldsymbol{V}(\boldsymbol{0}), \qquad \boldsymbol{v} \in \boldsymbol{V}(\boldsymbol{x}_0).$$

The differential game (2.1) can be written as

$$\begin{aligned} \boldsymbol{x}(t) &= S\boldsymbol{x}_0 + L[\boldsymbol{B}_1\boldsymbol{u} + \boldsymbol{B}_2\boldsymbol{v} - (\boldsymbol{C}_1\boldsymbol{u} + \boldsymbol{C}_2\boldsymbol{v})\boldsymbol{D}] + Y(\boldsymbol{C}_1\boldsymbol{u} + \boldsymbol{C}_2\boldsymbol{v}), \\ \boldsymbol{x}(T) &= \hat{S}\boldsymbol{x}_0 + \hat{L}[\boldsymbol{B}_1\boldsymbol{u} + \boldsymbol{B}_2\boldsymbol{v} - (\boldsymbol{C}_1\boldsymbol{u} + \boldsymbol{C}_2\boldsymbol{v})\boldsymbol{D}] + \hat{Y}(\boldsymbol{C}_1\boldsymbol{u} + \boldsymbol{C}_2\boldsymbol{v}), \end{aligned}$$

where

$$L : \mathcal{L}^2(0,\ T;\ \mathbf{R}^n) \to \mathcal{L}^2(0,\ T;\ \mathbf{R}^n), \qquad (L\boldsymbol{x})(t) = \int_0^t \boldsymbol{\Lambda}(s;\ t)\boldsymbol{x}(s)\mathrm{d}s,$$

$$\begin{aligned}
&\hat{L} : \mathcal{L}^2(0,\ T;\ \mathbf{R}^n) \to \mathbf{R}^n, && (\hat{L}\boldsymbol{x})(t) = \int_0^T \boldsymbol{\Lambda}(s;\ T)\boldsymbol{x}(s)\mathrm{d}s,\\
&S : \mathbf{R}^n \to \mathcal{L}^2(0,\ T;\ \mathbf{R}^n), && (S\boldsymbol{y})(t) = \boldsymbol{\Lambda}(0;\ t)\boldsymbol{y},\\
&\hat{S} : \mathbf{R}^n \to \mathbf{R}^n, && (\hat{S}\boldsymbol{y})(t) = \boldsymbol{\Lambda}(0;\ T)\boldsymbol{y},\\
&Y : \mathcal{L}^2(0,\ T;\ \mathbf{R}^n) \to \mathcal{L}^2(0,\ T;\ \mathbf{R}^n), && (Y\boldsymbol{x})(t) = \int_0^t \boldsymbol{\Lambda}(s;\ t)\boldsymbol{x}(s)\mathrm{d}w_s,\\
&\hat{Y} : \mathcal{L}^2(0,\ T;\ \mathbf{R}^n) \to \mathbf{R}^n, && (\hat{Y}\boldsymbol{x})(t) = \int_0^T \boldsymbol{\Lambda}(s;\ T)\boldsymbol{x}(s)\mathrm{d}w_s,\\
&G : \mathcal{L}^2(0,\ T;\ \mathbf{R}^n) \to \mathcal{L}^2(0,\ T;\ \mathbf{R}^n), && (G\boldsymbol{x})(t) = \int_t^T \mathrm{e}^{\boldsymbol{A}^*(s-t)}\boldsymbol{x}(s)\mathrm{d}s,\\
&H : \mathcal{L}^2(0,\ T;\ \mathbf{R}^n) \to \mathcal{L}^2(0,\ T;\ \mathbf{R}^n), && (H\boldsymbol{x})(t) = \int_t^T \mathrm{e}^{\boldsymbol{A}^*(s-t)}\boldsymbol{x}(s)\mathrm{d}w_s,\\
&M : \mathbf{R}^n \to \mathcal{L}^2(0,\ T;\ \mathbf{R}^n), && (M\boldsymbol{x})(t) = \mathrm{e}^{\boldsymbol{A}^*(T-t)}\boldsymbol{x},
\end{aligned}$$

and

$$\boldsymbol{\Lambda}(s;\ t) = \mathrm{e}^{\boldsymbol{A}(t-s)}.$$

We denote by R^* the adjoint operator of the operator R. Let

$$W = GQL + MF\hat{L}, \qquad \hat{W} = GQS + MF\hat{S}, \qquad J = MF\hat{Y} + GQY.$$

Then (4.1) can be written as

$$\begin{aligned}
&[I + (WB_1 - WC_1D + JC_1)B_1^*]\boldsymbol{p}\\
&= \hat{W}x_0 + (WB_2 - WC_2D + JC_2)\boldsymbol{v} + [GD^* - H - (WB_1 - WC_1D + JC_1)C_1^*]\boldsymbol{K},
\end{aligned} \tag{4.6}$$

and (4.5) can be written as

$$[I + (WB_1 - WC_1D + JC_1)B_1^*]\bar{\boldsymbol{p}} = [GD^* - H - (WB_1 - WC_1D + JC_1)C_1^*]\bar{\boldsymbol{K}}.$$

Obviously, $(\boldsymbol{p} + \bar{\boldsymbol{p}},\ \boldsymbol{K} + \bar{\boldsymbol{K}})$ is also the solution of (4.6). Then

$$\mathcal{A}(\boldsymbol{v},\ \boldsymbol{x}_0) = (\boldsymbol{p},\ \boldsymbol{K}) + \mathcal{A}(\mathbf{0},\ \mathbf{0}), \qquad (\boldsymbol{p},\ \boldsymbol{K}) \in \mathcal{A}(\boldsymbol{v},\ \boldsymbol{x}_0).$$

Given $\boldsymbol{v} \in \boldsymbol{V}(\boldsymbol{x}_0)$, for all $(\boldsymbol{p},\ \boldsymbol{K}) \in \mathcal{A}(\boldsymbol{v},\ \boldsymbol{x}_0)$ and $(\bar{\boldsymbol{p}},\ \bar{\boldsymbol{K}}) \in \mathcal{A}(\mathbf{0},\ \mathbf{0})$, we have

$$\begin{aligned}
J_{\boldsymbol{x}_0}^- &= \frac{1}{2}E\left((\boldsymbol{p}(0) + \bar{\boldsymbol{p}}(0)) \cdot \boldsymbol{x}_0 + \int_0^T (\boldsymbol{v} \cdot \boldsymbol{B}_2^*(\boldsymbol{p} + \bar{\boldsymbol{p}}) - |\boldsymbol{v}|^2)\mathrm{d}t + (\boldsymbol{C}_2\boldsymbol{v},\ (\boldsymbol{K} + \bar{\boldsymbol{K}}))\right)\\
&= \frac{1}{2}E\bigg(\boldsymbol{p}(0) \cdot \boldsymbol{x}_0 + \int_0^T (\boldsymbol{v} \cdot \boldsymbol{B}_2^*\boldsymbol{p} - |\boldsymbol{v}|^2)\mathrm{d}t\\
&\qquad + (\boldsymbol{C}_2\boldsymbol{v},\ \boldsymbol{K}) + \bar{\boldsymbol{p}}(0) \cdot \boldsymbol{x}_0 + (\boldsymbol{v},\ \boldsymbol{B}_2^*\bar{\boldsymbol{p}}) + (\boldsymbol{C}_2\boldsymbol{v},\ \bar{\boldsymbol{K}})\bigg)\\
&= J_{\boldsymbol{x}_0}^- + \frac{1}{2}E(\bar{\boldsymbol{p}}(0) \cdot \boldsymbol{x}_0 + (\boldsymbol{v},\ \boldsymbol{B}_2^*\bar{\boldsymbol{p}}) + (\boldsymbol{C}_2\boldsymbol{v},\ \bar{\boldsymbol{K}})).
\end{aligned}$$

So

$$E(\bar{\boldsymbol{p}}(0) \cdot \boldsymbol{x}_0 + (\boldsymbol{v},\ \boldsymbol{B}_2^*\bar{\boldsymbol{p}}) + (\boldsymbol{C}_2\boldsymbol{v},\ \bar{\boldsymbol{K}})) = 0. \tag{4.7}$$

For all $\boldsymbol{v} \in \boldsymbol{V}(\mathbf{0})$, we have

$$E((\boldsymbol{v},\ \boldsymbol{B}_2^*\bar{\boldsymbol{p}} + \boldsymbol{C}_2^*\bar{\boldsymbol{K}})) = 0.$$

Let

$$\mathcal{B} = \{\boldsymbol{B}_2^*\bar{\boldsymbol{p}} + \boldsymbol{C}_2^*\bar{\boldsymbol{K}};\ (\bar{\boldsymbol{p}},\ \bar{\boldsymbol{K}}) \in \mathcal{A}(\mathbf{0},\ \mathbf{0})\}.$$

We say that $\boldsymbol{V}(\mathbf{0})=\mathcal{B}^{\perp}$ in the sense of expectation. It is easy to prove that

$$\boldsymbol{V}(\boldsymbol{x}_0)=\boldsymbol{v}+\boldsymbol{V}(\mathbf{0}),\qquad \boldsymbol{v}\in\boldsymbol{V}(\boldsymbol{x}_0).$$

(c) We show that if the lower value exists, then there exist $\boldsymbol{v}\in\boldsymbol{V}(\boldsymbol{x}_0)$ and $(\boldsymbol{p}^*,\ \boldsymbol{K}^*)\in\mathcal{A}(\boldsymbol{v},\ \boldsymbol{x}_0)$ such that

$$\boldsymbol{v}=\boldsymbol{B}_2^*\boldsymbol{p}^*+\boldsymbol{C}_2^*\boldsymbol{K}^*,$$

where $(\boldsymbol{x}^*,\ \boldsymbol{p}^*,\ \boldsymbol{K}^*)$ is the solution of (4.1).

Since the lower value of the game exists, there exists a $\boldsymbol{v}_0\in\boldsymbol{V}(\boldsymbol{x}_0)$ such that for any $\boldsymbol{w}\in\boldsymbol{V}(\mathbf{0})$,

$$\begin{aligned}\mathrm{d}J^-_{\boldsymbol{x}_0}(\boldsymbol{v}_0;\ \boldsymbol{w})&=\mathrm{d}C_{\boldsymbol{x}_0}(-\boldsymbol{B}_1^*\boldsymbol{p}-\boldsymbol{C}_1^*\boldsymbol{K},\ \boldsymbol{v}_0;\ \mathbf{0},\ \boldsymbol{w})\\&=E((\boldsymbol{B}_2^*\boldsymbol{p}+\boldsymbol{C}_2^*\boldsymbol{K}-\boldsymbol{v}_0,\ \boldsymbol{w}))\\&=0.\end{aligned}\tag{4.8}$$

We have $\boldsymbol{B}_2^*\boldsymbol{p}+\boldsymbol{C}_2^*\boldsymbol{K}-\boldsymbol{v}_0\in\boldsymbol{V}(\mathbf{0})^{\perp}$ in the sense of expectation. Since $\boldsymbol{V}(\mathbf{0})=\mathcal{B}^{\perp}$, there exists $(\bar{\boldsymbol{p}},\ \bar{\boldsymbol{K}})\in\mathcal{A}(\mathbf{0},\ \mathbf{0})$ such that

$$\boldsymbol{B}_2^*\boldsymbol{p}+\boldsymbol{C}_2^*\boldsymbol{K}-\boldsymbol{v}_0=\boldsymbol{B}_2^*\bar{\boldsymbol{p}}+\boldsymbol{C}_2^*\bar{\boldsymbol{K}}.$$

By (4.4), there exists $(\boldsymbol{p}^*,\ \boldsymbol{K}^*)\in\mathcal{A}(\boldsymbol{v}_0,\ \boldsymbol{x}_0)$ such that $\boldsymbol{v}_0=\boldsymbol{B}_2^*\boldsymbol{p}^*+\boldsymbol{C}_2^*\boldsymbol{K}^*$, and $(\boldsymbol{x}^*,\ \boldsymbol{p}^*,\ \boldsymbol{K}^*)$ is the solution of (4.1). Therefore, $(\boldsymbol{x}^*,\ \boldsymbol{p}^*,\ \boldsymbol{K}^*)$ is the solution of (3.4). By Theorem 3.2, the open loop lower value is given by (3.6).

(d) We show that if $\boldsymbol{v}^*=\boldsymbol{B}_2^*\boldsymbol{p}+\boldsymbol{C}_2^*\boldsymbol{K}\in\boldsymbol{V}(\boldsymbol{x}_0)$, then

$$J^-_{\boldsymbol{x}_0}=\inf_{\boldsymbol{u}\in\mathcal{L}^2(0,T;\mathbf{R})}\boldsymbol{C}_{\boldsymbol{x}_0}(\boldsymbol{u},\ \boldsymbol{v})=\boldsymbol{C}_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v})$$

can achieve maximization at $\boldsymbol{v}^*$.

By assumption $\boldsymbol{v}^*=\boldsymbol{B}_2^*\boldsymbol{p}+\boldsymbol{C}_2^*\boldsymbol{K}\in\boldsymbol{V}(\boldsymbol{x}_0)$, there exists a solution of (4.1) with $\boldsymbol{u}^*=-\boldsymbol{B}_1^*\boldsymbol{p}-\boldsymbol{C}_1^*\boldsymbol{K}$ as a minimizer by (1). For any $\boldsymbol{v}\in\boldsymbol{V}(\boldsymbol{x}_0)$, we have

$$\begin{aligned}C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v})=&\,C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v}^*)+\mathrm{d}C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v}^*;\ \mathbf{0},\ \boldsymbol{v}-\boldsymbol{v}^*)\\&+\frac{1}{2}\mathrm{d}^2C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v}^*;\ \mathbf{0},\ \boldsymbol{v}-\boldsymbol{v}^*;\ \mathbf{0},\ \boldsymbol{v}-\boldsymbol{v}^*).\end{aligned}$$

By (4.8) and $\boldsymbol{v}-\boldsymbol{v}^*\in\boldsymbol{V}(\mathbf{0})$ we have

$$\mathrm{d}C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v}^*;\ \mathbf{0},\ \boldsymbol{v}-\boldsymbol{v}^*)=0.$$

According to the definition of directional derivative, we have

$$\begin{aligned}&\mathrm{d}^2C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v}^*;\ \mathbf{0},\ \boldsymbol{v}-\boldsymbol{v}^*;\ \mathbf{0},\ \boldsymbol{v}-\boldsymbol{v}^*)\\=&\lim_{l\to0}\frac{1}{l}E[(\boldsymbol{B}_2^*\boldsymbol{p}+\boldsymbol{C}_2^*\boldsymbol{K}-(\boldsymbol{v}^*+l(\boldsymbol{v}-\boldsymbol{v}^*)),\ \boldsymbol{v}-\boldsymbol{v}^*)-(\boldsymbol{B}_2^*\boldsymbol{p}+\boldsymbol{C}_2^*\boldsymbol{K}-\boldsymbol{v}^*,\ \boldsymbol{v}-\boldsymbol{v}^*)]\\=&\,(\boldsymbol{v}-\boldsymbol{v}^*,\ \boldsymbol{v}-\boldsymbol{v}^*)\\\le&\,0.\end{aligned}$$

Thus

$$C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\boldsymbol{v})\le C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\boldsymbol{v}^*).$$

Now we go back to the proof of Theorem 4.2.

It is obvious that (1)⇒(2).

According to the above (a)–(c), we have (2)⇒(3).

(3)⇒(1). By (d),

$$C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v}) \le C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v}^*).$$

So

$$C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v}^*) = \inf_{\boldsymbol{u}\in\mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\ \boldsymbol{v}^*) = \sup_{\boldsymbol{v}\in\mathcal{L}^2(0,T;\mathbf{R}^k)} \inf_{\boldsymbol{u}\in\mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\ \boldsymbol{v}).$$

The proof is completed.

Corresponding to the Theorem 4.2, we have

Theorem 4.3 *The following statements are equivalent*:

(1) *There exist* $\boldsymbol{u}^* \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^m)$ *and* $\boldsymbol{v}^* \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^k)$ *such that*

$$C_{\boldsymbol{x}_0}(\boldsymbol{u}^*, \boldsymbol{v}^*) = \sup_{\boldsymbol{v}\in\mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\boldsymbol{x}_0}(\boldsymbol{u}^*,\ \boldsymbol{v}) = \inf_{\boldsymbol{u}\in\mathcal{L}^2(0,T;\mathbf{R}^m)} \sup_{\boldsymbol{v}\in\mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\ \boldsymbol{v});$$

(2) *The open loop upper value* $v^+(\boldsymbol{x}_0)$ *of the game exists*;

(3) *There exists a solution* $(\boldsymbol{x},\ \boldsymbol{p},\ \boldsymbol{K})$ *of the adjoint system* (3.4) *such that* $-\boldsymbol{B}_1^*\boldsymbol{p} - \boldsymbol{C}_1^*\boldsymbol{K} \in \boldsymbol{U}(\boldsymbol{x}_0)$, *the solution pairs* $(\boldsymbol{u}^*,\ \boldsymbol{v}^*)$ *is* (3.5), *and the open loop lower value is given by* (3.6).

Now we give the proof of Theorem 4.1.

Proof of Theorem 4.1 (1)⇒(2)⇒(3) are obvious.

(3)⇒(1). By Theorems 4.2 and 4.3, there exists a solution $(\boldsymbol{x},\ \boldsymbol{p},\ \boldsymbol{K})$ of the system (3.10). Therefore, the game has a saddle point.

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