Commun. Math. Anal. Appl. Vol. **1**, No. 1, pp. 1-71
doi: 10.4208/cmaa.2021-0001 February 2022

# The Dirichlet Problem for Stochastic Degenerate Parabolic-Hyperbolic Equations

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Received 14 September 2021; Accepted 17 October 2021

**Abstract.** We consider the Dirichlet problem for a quasilinear degenerate parabolic stochastic partial differential equation with multiplicative noise and nonhomogeneous Dirichlet boundary condition. We introduce the definition of kinetic solution for this problem and prove existence and uniqueness of solutions. For the uniqueness of kinetic solutions we prove a new version of the doubling of variables method and use it to deduce a comparison principle between solutions. The proof requires a delicate analysis of the boundary values of the solutions for which we develop some techniques that enable the usage of the existence of normal weak traces for divergence measure fields in this stochastic setting. The existence of solutions, as usual, is obtained through a two-level approximation scheme consisting of nondegenerate regularizations of the equations which we show to be consistent with our definition of solutions. In particular, the regularity conditions that give meaning to the boundary values of the solutions are shown to be inherited by limits of nondegenerate parabolic approximations provided by the vanishing viscosity method.

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**AMS subject classifications**: Primary: 26B20, 28C05, 35L65, 35B35; Secondary: 26B35, 26B12, 35L67

**Key words**: Stochastic partial differential equations, quasilinear degenerate parabolic equations, divergence measure fields, normal traces.

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# 1 Introduction

Let $\mathcal{O}$ be a bounded smooth open subset of $\mathbb{R}^d$. We consider the Dirichlet problem for a quasilinear degenerate parabolic stochastic partial differential equation

$$du+\operatorname{div}\big(\mathbf{A}(u)\big)\,dt=D^2:\mathbf{B}(u)\,dt+\Phi(u)\,dW, \quad x\in\mathcal{O}, \quad t\in(0,T), \tag{1.1}$$

$$u(0)=u_0, \tag{1.2}$$

$$u(t)\big|_{\partial\mathcal{O}}=u_b(t), \tag{1.3}$$

where, $\mathbf{A}:\mathbb{R}\to\mathbb{R}^d$ and $\mathbf{B}:\mathbb{R}\to\mathbb{M}^d$ are smooth maps, $\mathbb{M}^d$ denote the space of $d\times d$ matrices. For $\mathbf{R}=(R_{ij}),\mathbf{S}=(S_{ij})\in\mathbb{M}^d$ we denote $\mathbf{R}:\mathbf{S}:=\sum_{i,j}R_{ij}S_{ij}$ and, by extrapolation, $D^2:\mathbf{B}:=\sum_{i,j}\partial^2_{x_ix_j}B_{ij}$ . The matrix $\mathbf{B}(u)$ is symmetric and its derivative $\mathbf{b}(u)=\mathbf{B}'(u)$ is a symmetric nonnegative $d\times d$ matrix. $W$ is a cylindrical Wiener process.

## 1.1 Hypotheses

The flux function $\mathbf{A}=(A_1,\cdots,A_d):\mathbb{R}\to\mathbb{R}^d$ is assumed to be of class $C^2$ and we denote its derivative by $\mathbf{a}=(a_1,\cdots,a_d)$. The diffusion matrix $\mathbf{b}=(b_{ij})_{i,j=1}^d:\mathbb{R}\to\mathbb{M}^d$ is symmetric and positive semidefinite. Its square-root matrix, also symmetric and positive semidefinite, is denoted by $\sigma$, which is assumed to be bounded and locally $\gamma$-Hölder continuous for some $\gamma>1/2$, that is,

$$|\sigma(\xi)-\sigma(\zeta)|\le C(R)|\xi-\zeta|^\gamma \quad \text{for all} \quad \xi,\zeta\in\mathbb{R}, \quad |\xi-\zeta|<R. \tag{1.4}$$

Moreover, we assume that, for some $b\in C^1(\mathbb{R})$ with $b'(u)\ge 0$ for all $u\in\mathbb{R}$, and a constant $\Lambda>1$ we have

$$b'(u)^2|\xi|^2\le\sum_{i,j}b_{ij}(u)\xi_i\xi_j\le\Lambda b'(u)^2|\xi|^2. \tag{1.5}$$

As it was observed in [19], (1.5) implies that the $B_{ij}$ are locally Lipschitz functions of $b(u)$, that is, there exists a locally Lipschitz continuous functions $\tilde{B}_{ij}$ such that $B_{ij}(u) = \tilde{B}_{ij}(b(u))$ for all $i,j = 1,\dots,d$. Relation (1.5) immediately implies

$$b'(u)|\xi|^2 \le \sum_{i,j} \sigma_{ij}(u)\xi_i\xi_j \le \Lambda^{\frac{1}{2}} b'(u)|\xi|^2, \tag{1.6}$$

and similarly to **B**, we deduce that

$$\Sigma(u) = \int_0^u \sigma(\zeta)\,d\zeta \tag{1.7}$$

is a locally Lipschitz function of $b(u)$, i.e. there is a locally Lipschitz function $\tilde{\Sigma}$ such that $\Sigma(u) = \tilde{\Sigma}(b(u))$.

Further, we require a nondegeneracy condition for the symbol $\mathcal{L}$ associated to the kinetic form of (1.1). In order to have spatial regularity of kinetic solutions we localise the $\chi$-function associated to such solution and so, for $\ell > 0$ sufficiently large, we may view our localised $\chi$-functions as periodic with period $\ell$. The symbol is given by

$$\mathcal{L}(i\tau, in, \xi) := i\big(u + \mathbf{a}(\xi)\cdot n\big) + n^*\mathbf{b}(\xi)n,$$

where $n \in \ell\mathbb{Z}^d$. For a certain $L_0 > 0$ and any $J, \delta > 0$ let

$$\begin{aligned}
&\Omega_{\mathcal{L}}(u,n;\delta) := \big\{\xi \in (-L_0, L_0) : |\mathcal{L}(i\tau, in, \xi)| \le \delta\big\},\\
&\omega_{\mathcal{L}}(J;\delta) := \sup_{\substack{\tau\in\mathbb{R}, n\in\ell\mathbb{Z}^d\\ |n|\sim J}} |\Omega_{\mathcal{L}}(\tau, in;\delta)|
\end{aligned}$$

and $\mathcal{L}_\xi := \partial_\xi \mathcal{L}$. We suppose that there exist $\alpha \in (0,1)$ and $\beta > 0$ such that

$$\begin{aligned}
&\omega_{\mathcal{L}}(J;\delta) \lesssim \left(\frac{\delta}{J^\beta}\right)^\alpha,\\
&\sup_{\substack{\tau\in\mathbb{R}, n\in\mathbb{Z}^d\\ |n|\sim J}} \sup_{\xi\in(-L_0,L_0)} |\mathcal{L}_\xi(i\tau, in;\xi)| \lesssim J^\beta, \quad \forall \delta > 0, \quad J \gtrsim 1,
\end{aligned} \tag{1.8}$$

where we employ the usual notation $x \lesssim y$, if $x \le Cy$, for some absolute constant $C > 0$, and $x \sim y$, if $x \lesssim y$ and $y \lesssim x$.

The following example in the case where $d = 2$, corresponds to the one in [42, Corollary 4.5] where conditions (1.8) are verified

$$du + \partial_{x_1}\left(\frac{1}{l+1}u^{l+1}\right)dt = \partial_{x_2}^2\left(\frac{1}{n+1}|u|^n u\right)dt + \Phi(u)\,dW,$$

where $l,n\in\mathbb{N}$ satisfy $n\geq 2l$. The same argument as in [42, Corollary 4.5] applies to the corresponding equation in any space dimension $d$, replacing $\partial_{x_2}^2$ in the above equation by $\partial_{x_2}^2+\cdots+\partial_{x_d}^2$. Clearly, many other similar examples may be given.

As to the stochastic term, we adopt a framework similar to that in [14, 15, 21]. Let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\geq 0})$ be a stochastic basis, where $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space and $(\mathcal{F}_t)_{t\geq 0}$ is a complete right-continuous filtration. We assume that the initial datum $u_0$ is $\mathcal{F}_0$-measurable and that $W$ is a cylindrical Wiener process $W=\sum_{k\geq 1}\beta_k e_k$, where $\beta_k$ are independent Brownian processes and $(e_k)_{k\geq 1}$ is a complete orthonormal basis in a Hilbert space $\mathfrak{U}$. For each $u\in L^2(\mathcal{O})$, $\Phi(u):\mathfrak{U}\to L^2(\mathcal{O})$ is defined by $\Phi(u)e_k=g_k(\cdot,u(\cdot))$, where $g_k(\cdot,u(\cdot))$ is a regular function on $\mathcal{O}$. In this setting, we may assume, without loss of generality, that $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by the Wiener process and the initial condition.

We assume that $g_k$ are smooth functions defined on $(\bar{\mathcal{O}}\times\mathbb{R})$ with the bounds

$$|g_k(x,0)|+\sum_{|\alpha|+j\leq 2}\left|\partial_x^\alpha\partial_\xi^j g_k(x,\xi)\right|\leq \alpha_k,\quad \forall x\in\mathcal{O},\quad \xi\in\mathbb{R}, \tag{1.9}$$

where $(\alpha_k)_{k\geq 1}$ is a sequence of positive numbers satisfying $D:=4\sum_{k\geq 1}\alpha_k^2<\infty$. Observe that (1.9) implies

$$G^2(x,u)=\sum_{k\geq 1}|g_k(x,u)|^2\leq D\big(1+|u|^2\big), \tag{1.10}$$

$$\sum_{k\geq 1}|g_k(x,u)-g_k(y,v)|^2\leq D\big(|x-y|^2+|u-v|^2\big) \tag{1.11}$$

for all $x,y\in\mathcal{O}$, $u,v\in\mathbb{R}$.

In addition, we assume that, for some $\delta_0>0$,

$$\nabla_x g_k(x,u)=0,\quad \text{if}\quad \operatorname{dist}(x,\partial\mathcal{O})<\delta_0\quad \text{for all}\quad k\geq 1. \tag{1.12}$$

The conditions on $\Phi$ imply that $\Phi: L^2(\mathcal{O})\to L_2(\mathfrak{U};L^2(\mathcal{O}))$, where the latter denotes the space of Hilbert-Schmidt operators from $\mathfrak{U}$ to $L^2(\mathcal{O})$. In particular, given a predictable process $u\in L^2(\Omega\times[0,T];L^2(\mathcal{O}))$, the stochastic integral is a well defined process taking values in $L^2(\mathcal{O})$. Indeed, for $u\in L^2(\mathcal{O})$, from (1.9), it follows

$$\sum_{k\geq 1}\|g_k(\cdot,u(\cdot))\|_{L^2(\mathcal{O})}^2\leq D\big(1+\|u\|_{L^2(\mathcal{O})}^2\big).$$

Since, clearly, the series defining $W$ does not converge in $\mathfrak{U}$, in order to have $W$ properly defined as a Hilbert space valued Wiener process, one usually introduces an auxiliary space $\mathfrak{U}_0\supset\mathfrak{U}$ such as

$$\mathfrak{U}_0:=\left\{v=\sum_{k\geq 1}a_k e_k:\sum_{k\geq 1}\frac{a_k^2}{k^2}<\infty\right\},$$

endowed with the norm

$$\|v\|_{\mathfrak{U}_0}^2=\sum_{k\geq 1}\frac{a_k^2}{k^2},\quad v=\sum_{k\geq 1}a_k e_k.$$

In this way, one may check that the trajectories of $W$ are $\mathbb{P}$-a.s. in $C([0,T],\mathfrak{U}_0)$ (see [13]).

We look for bounded solutions of the initial-boundary value problem (1.1)-(1.3) which assume values in an interval, say $[u_{\min},u_{\max}]$. Accordingly we assume that $[u_{\min},u_{\max}]\subset(-L_0,L_0)$, where $L_0$ is as in the nondegeneracy condition (1.8). Moreover, we assume that

$$g_k(x,u_{\min})=g_k(x,u_{\max})=0,\quad x\in\mathcal{O},\quad k=1,2,\dots. \tag{1.13}$$

We also assume that $u_0\in L^\infty(\mathcal{O})$ is deterministic (for simplicity) and satisfies $u_{\min}\leq u_0\leq u_{\max}$. Let us point out that $u_0$ may be random, in which case it should be assumed that it is $\mathcal{F}_0$-measurable. The extension to this more general setting is straightforward and follows the arguments below line by line just by adding an expectation where integration in the random parameter takes place. The conditions on $u_b$ are given below (see (1.18)).

## 1.2 Definitions and main result

We will work with the notion of kinetic solutions of Eq. (1.1). This notion of solution is consistent with the one introduced by Debussche and Vovelle [15], in the context of stochastic conservation laws, where they extended to the stochastic setting the concept of kinetic solutions originally introduced by Lions *et al.* [34] for deterministic conservation laws. The methods in [15] were later extended to degenerated parabolic problems by Debussche *et al.* [14] and by Gess and Hofmanová [21], both of them in the periodic case. Essentially, a function $u$ is a kinetic solution of Eq. (1.1) if it satisfies certain regularity requirements and if there exists a kinetic measure $m$ such that the pair $(f,m)$ satisfies a certain kinetic equation related to (1.1), where $f(t,x,\xi)=1_{u(t,x)>\xi}$ (see Definition 1.1 and parts (i) and (ii) of Definition 1.2 below). Note that if we ignore the boundary condition (1.3) we may easily construct kinetic solutions by applying the theory from [14, 21]. Indeed it suffices to put the domain $\mathcal{O}$ inside a $d$-cell $\mathcal{U}=(-R,R)^d$ large enough and extend the initial data to $\mathbb{R}^d$ in any way so that we may assume that it is periodic (for instance, we may extend $u_0$ to a function with compact support in $\mathcal{U}$ and extend it periodically to $\mathbb{R}^d$). In this case, the results from [14, 21] guarantee the existence of solutions, which we may then restrict to $\mathcal{O}$. Here, we are going to introduce a set of conditions that will allow us to give meaning to the boundary condition and will characterize uniquely, among all possible solutions, a particular kinetic solution of (1.1), (1.2) satisfying (1.3).

**Definition 1.1** (Kinetic measure). *A mapping $m$ from $\Omega$ to $\mathcal{M}_b^+([0,T]\times\mathcal{O}\times\mathbb{R})$, the set of nonnegative bounded measures over $[0,T]\times\mathcal{O}\times\mathbb{R}$, is said to be a kinetic measure if the following holds:*

*(i) $m$ is measurable, in the sense that for each $\psi\in C_0([0,T]\times\mathcal{O}\times\mathbb{R})$ the mapping $m(\psi):\Omega\to\mathbb{R}$ is measurable, where by $C_0$ we denote the space of continuous functions vanishing at the boundary or when the norm of the argument goes to infinity;*

*(ii) $m$ vanishes for large $\xi$: if $B_R^c=\{\xi\in\mathbb{R}:|\xi|\geq R\}$, then*

$$\lim_{R\to\infty}\mathbb{E}m\big([0,T]\times\mathcal{O}\times B_R^c\big)=0;$$

*(iii) for any $\psi\in C_0(\mathcal{O}\times\mathbb{R})$*

$$\int_{[0,t]\times\mathcal{O}\times\mathbb{R}}\psi(x,\xi)\,dm(s,x,\xi)\in L^2(\Omega\times[0,T])$$

*admits a predictable representative.*

Concerning the Dirichlet condition in the next definition we make the following comments and further assumptions. First, we assume that $\mathbf{B}(u)$ is diagonal, i.e.

$$B_{ij}(u)\equiv 0\quad\text{for}\quad i\neq j. \tag{1.14}$$

Second, we introduce the functions

$$\begin{aligned}
&F(u,v):=\operatorname{sgn}(u-v)\big(\mathbf{A}(u)-\mathbf{A}(v)\big),\\
&\mathbb{B}(u,v)=\big(\operatorname{sgn}(u-v)(B_{ij}(u)-B_{ij}(v))\big)_{i,j=1}^d,\\
&K_x(u,v):=\nabla_x\cdot\mathbb{B}(u,v)-F(u,v),\\
&H_x(u,v,w):=K_x(u,v)+K_x(u,w)-K_x(w,v),\\
&\mathbb{G}(u,v):=\operatorname{sgn}(u-v)\big(\Phi(u)-\Phi(v)\big),
\end{aligned} \tag{1.15}$$

where $\nabla_x\cdot\mathbb{B}(u,v)$ is the $d$-vector with components

$$\big(\nabla_x\cdot\mathbb{B}(u,v)\big)_j=\sum_{i=d}^d\partial_{x_i}\big(\operatorname{sgn}(u-v)(B_{ij}(u)-B_{ij}(v))\big),$$

and $\mathbb{G}:L^2(\mathcal{O})^2\to L_2(\mathfrak{U};L^2(\mathcal{O}^2))$ is given by

$$\mathbb{G}(u,v)e_k = \mathrm{sgn}(u-v)\big(g_k(x,u)-g_k(x,v)\big), \quad k\geq 1.$$

Henceforth, whenever dealing with the Dirichlet boundary conditions, we will omit the dependence of $g_k$ on $x$, because of (1.12).

Similarly, we also define $F_+$, $\mathbb{B}_+$ and $\mathbb{G}_+$ as their counterparts in (1.15) with $\mathrm{sgn}(\cdot)_+$ instead of $\mathrm{sgn}(\cdot)$. Furthermore, let

$$\mathcal{A}(u,v,w) = |u-v|+|u-w|-|w-v|. \tag{1.16}$$

Third, in order to take advantage of the fact that $\partial\mathcal{O}$ is locally the graph of a $C^2$ function, we introduce a system of balls $\mathcal{B}$ with the following property. For each $B=B(x_0,r)\in\mathcal{B}$, a ball with center at an arbitrary $x_0\in\partial\mathcal{O}$ with radius $0<r<\delta_0$ (cf. (1.12)), we have that for some $\gamma\in C^2(\mathbb{R}^{d-1})$,

$$B\cap\mathcal{O} = \Big\{(\bar{y},y_d)\in B : y_d<\gamma(\bar{y}), \bar{y}=(y_1,\cdots,y_{d-1})\in\mathbb{R}^{d-1}\Big\}, \tag{1.17}$$

where the coordinate system $(y_1,\cdots,y_d)$ is obtained from the original $(x_1,\cdots,x_d)$ by relabelling, reorienting and performing a translation. By relabelling we mean a permutation of the coordinates, by reorienting we mean changing the orientation of one of the coordinate axes, and the translation changes the origin of the coordinate system if necessary.

Fourth, we assume that

$$\begin{aligned} u_b \in L^2\big(\Omega\times[0,T];H^4(\partial\mathcal{O})\big)\cap L^2\big(\Omega;H^1(0,T;L^2(\partial\mathcal{O}))\big) \\ \cap L^4\big(\Omega\times[0,T];W^{1,4}(\mathcal{O})\big) \end{aligned} \tag{1.18}$$

is predictable and satisfies $u_{\min}\leq u_b(t,x)\leq u_{\max}$, $(t,x)\in(0,T)\times\partial\mathcal{O}$, and given $B\in\mathcal{B}$ satisfying (1.17) we consider an extension $u_B\in L^2(\Omega\times[0,T];H^4(\mathcal{O}))$ of $u_b$ satisfying the following problem:

$$du_B = -\Delta^2 u_B dt + \Phi(u_B)dW(t), \quad x\in\mathcal{O}, \quad t\in(0,T), \tag{1.19}$$

$$u_B(0) = u_{B0}, \tag{1.20}$$

$$u_B(t)\big|_{\partial\mathcal{O}} = u_b(t), \tag{1.21}$$

$$\frac{\partial u_B}{\partial x_d}(t)\big|_{\partial\mathcal{O}\cap B} = 0, \tag{1.22}$$

where $\Delta^2$ denotes the bi-Laplacian operator and $u_{B0}$ is a smooth extension of $u_b(0,\cdot)$ to $B$ such that $\frac{\partial u_{B0}}{\partial x_d}|_{\partial\mathcal{O}\cap B}=0$.

**Remark 1.1.** Actually, we only need to assume that, for each $B\in\mathcal{B}$, $u_b|\partial\mathcal{O}\cap B$ is the restriction to $\partial\mathcal{O}\cap B$ of a strong solution of a stochastic equation whose noise term is given by $\Phi(u_B)dW$, also satisfying (1.22). It is possible to show that under the hypotheses (1.9)-(1.12) and assuming (1.18), then there are strong solutions to (1.19)-(1.22), in particular (see Appendix A). In the deterministic setting, that is, when $\Phi=0$, $u_B$ may simply be obtained using (1.17) by setting $u_B(\bar{x},x_d)=u_b(\bar{x})$, for $x=(\bar{x},x_d)\in B\cap\overline{\mathcal{O}}$ (cf. [19, 36]). We will comment further on the extension of the boundary data satisfying (1.19)-(1.22) in Subsection 1.4 below.

**Definition 1.2.** *A predictable function $u\in L^\infty(\Omega\times[0,T]\times\mathcal{O})$ is a kinetic solution of* (1.1), (1.2) *if the following conditions are satisfied:*

*(i) Regularity:*

$$\nabla b(u)\in L^2(\Omega\times[0,T]\times\mathcal{O}). \tag{1.23}$$

*In particular,* $\operatorname{div}\int_0^u\sigma(\xi)d\xi\in L^2(\Omega\times[0,T]\times\mathcal{O})$.

*(ii) Kinetic equation: There exists a kinetic measure $m\geq n_1$, $\mathbb{P}$-a.s., such that the pair $(f=\mathbf{1}_{u>\xi},m)$ satisfies, for all $\varphi\in C_c^\infty([0,T)\times\mathcal{O}\times\mathbb{R})$, $\mathbb{P}$-a.s.,*

$$\begin{aligned}
&\int_0^T\big\langle f(t),\partial_t\varphi(t)\big\rangle dt+\big\langle f_0,\varphi(0)\big\rangle+\int_0^T\big\langle f(t),\mathbf{a}\cdot\nabla\varphi(t)\big\rangle dt\\
&\qquad+\int_0^T\big\langle f(t),\mathbf{b}:D^2\varphi(t)\big\rangle dt\\
&=-\sum_{k\geq1}\int_0^T\int_{\mathcal{O}}g_k\big(x,u(t,x)\big)\varphi\big(t,x,u(t,x)\big)dxd\beta_k(t)\\
&\qquad-\frac{1}{2}\int_0^T\int_{\mathcal{O}}G^2\big(x,u(t,x)\big)\partial_\xi\varphi\big(t,x,u(t,x)\big)dxdt+m(\partial_\xi\varphi),
\end{aligned}\tag{1.24}$$

*where $n_1:\Omega\to\mathcal{M}^+([0,T]\times\mathcal{O})$, called the parabolic dissipation measure, is defined by*

$$n_1(\varphi)=\int_0^T\int_{\mathcal{O}}\int_{\mathbb{R}}\varphi(t,x,\xi)\left|\operatorname{div}\int_0^u\sigma(\zeta)d\zeta\right|^2d\delta_{u(s,x)}(\xi)dxdt.$$

*Additionally, among all possible functions that satisfy (i) and (ii), item (iii) below will characterize uniquely the one that satisfies the boundary condition* (1.3)*.*

*(iii) Dirichlet condition on $\partial\mathcal{O}$: We say that $u$ satisfies* (1.3) *if the following conditions hold. For each $B\in\mathcal{B}$, and some random constant $C_*>0$ with finite expectation,*

*depending only on* $\mathbf{A},\mathbf{B}$ *and* $u_b$, *we have a.s. and for all* $0\le\tilde{\varphi}\in C_c^\infty((0,T)\times B)$ *that*

$$\begin{aligned}&\int_0^T\int_{\mathcal{O}}\{|u(t,x)-u_B(t,x)|\partial_t\tilde{\varphi}-K_x(u(t,x),u_B(t,x))\cdot\nabla\tilde{\varphi}\}dxdt\\&\quad+\sum_{k\ge1}\int_0^T\int_{\mathcal{O}}\mathbb{G}_k\big(u(t,x),u_B(t,x)\big)\tilde{\varphi}dxd\beta_k(t)\ge -C_*\|\tilde{\varphi}\|_{L^2(\mathcal{O}\times[0,T])}.\end{aligned}\tag{1.25}$$

*Also, if* $v$ *is any other kinetic solution of Eq.* (1.1) *(possibly with different initial data* $v_0$*) and* $\zeta_\delta$ *is any boundary layer sequence (for whose precise definition we refer to Section* 2*), then for all* $0\le\tilde{\phi}\in C_c^\infty(B\times\mathcal{O})$ *and* $0\le t\le T$, *we have that*

$$\liminf_{\delta\to0}\mathbb{E}\int_0^t\int_{\mathcal{O}^2}H_x\big(u(s,x),v(s,y),u_B(s,x)\big)\cdot\nabla_x\zeta_\delta(x)\tilde{\phi}(x,y)dxdyds\ge0.\tag{1.26}$$

*Moreover, for all* $B\in\mathcal{B}$ *and for all* $\phi\in C_c^1(B)$, *a.s. we have*

$$b\big(u(t,x)\big)=b\big(u_b(t,x)\big)\ \text{on}\ \partial\mathcal{O}\times(0,T)\tag{1.27}$$

*in the sense of traces in* $L^2(0,T;H^1(B\cap\mathcal{O}))$.

**Remark 1.2.** Note that in the deterministic setting, that is, when $g_k\equiv0$ for all $k\ge1$, the constants are solutions of Eq. (1.1) and the condition (1.26) is only necessary to hold for constant solutions $v(t,y)=k$ (cf. [19,36]). In the present setting, this is no longer the case.

**Remark 1.3** (Chain rule)**.** Since $\Sigma(u)$ in (1.7) is a locally Lipschitz function of $b(u)$ (see (1.5)), condition (1.23) implies that $\nabla\Sigma(u)\in L^2(\Omega\times[0,T]\times\mathcal{O})$. Consequently, for any $0\le\vartheta\in C_b(\mathbb{R})$ the following chain rule formula holds in $L^2(\Omega\times[0,T]\times\mathcal{O})$ (see the appendix in [12]):

$$\operatorname{div}\int_0^u\vartheta(\xi)\sigma(\xi)d\xi=\vartheta(u)\operatorname{div}\int_0^u\sigma(\xi)d\xi\quad\text{in }\mathcal{D}'(\mathcal{O}),\text{ a.e. }(\omega,t).\tag{1.28}$$

We now state the main theorem of this paper.

**Theorem 1.1.** *Let* $u_0\in L^\infty(\Omega\times\mathcal{O})$ *with* $u_{\min}\le u_0\le u_{\max}$ *and assume that* (1.4)-(1.14) *hold. Then, there is a unique kinetic solution* $u$ *of* (1.1)-(1.3)*. Moreover,* $u$ *has almost surely continuous trajectories in* $L^p(\mathcal{O})$*, for all* $p\in[1,\infty)$ *and satisfies* $u_{\min}\le u\le u_{\max}$*. Furthermore, if* $u$ *and* $v$ *are two kinetic solutions with initial data* $u_0,v_0$, *we have*

$$\mathbb{E}\int_{\mathcal{O}}|u(t,x)-v(t,x)|dx\le\int_{\mathcal{O}}|u_0(x)-v_0(x)|dx.\tag{1.29}$$

It is important to have also at hand the notion of entropy solution.

**Definition 1.3.** *A bounded measurable function* $u \in L^\infty(\Omega\times[0,T]\times\mathcal{O})$ *is a weak entropy solution of* (1.1)-(1.3) *if* $u \in L^p(\Omega\times[0,T],\mathcal{P},d\mathbb{P}\otimes dt;L^p(\mathcal{O}))\cap L^p(\Omega;L^\infty(0,T;L^p(\mathcal{O})))$ *for all* $p\geq 1$ *and it satisfies conditions (i) and (iii) of Definition* 1.2 *and*

*(ii') For all* $\eta\in C^2(\mathbb{R})$ *with* $\mathbf{A}_\eta,\mathbf{B}_\eta$ *such that* $\mathbf{A}'_\eta=\eta'\mathbf{A}'$, $\mathbf{B}'_\eta=\eta'\mathbf{B}'$ *and for all* $0\leq\varphi\in C_c^\infty([0,T)\times\mathcal{O})$

$$\begin{aligned}&\int_0^T\langle\eta(u(t)),\partial_t\varphi\rangle dt+\langle\eta(u_0),\varphi(0)\rangle+\int_0^T\langle\mathbf{A}_\eta(u(t)),\nabla\varphi\rangle dt\\&\quad+\int_0^T\langle\operatorname{div}(\mathbf{B}_\eta(u(t))),\nabla\varphi\rangle dt\\&\geq\int_0^T\int_{\mathcal{O}}\eta''(u)\sum_{k=d'+1}^{d}\left(\sum_{i=d'+1}^{d}\partial_{x_i}\sigma_{ik}(u)\right)^2\varphi dxdt\\&\quad-\sum_{k\geq1}\int_0^T\langle g_k(x,u(t))\eta'(u(t)),\varphi\rangle d\beta_k(t)\\&\quad-\frac12\int_0^T\langle G^2(x,u(t))\eta''(u(t)),\varphi\rangle dt,\end{aligned}\tag{1.30}$$

*a.s. where* $\langle\cdot,\cdot\rangle$ *represents the inner product of* $L^2(\mathcal{O})$ *or* $L^2(\mathcal{O};\mathbb{R}^d)$.

The following proposition establishes the equivalence between the notions of kinetic and weak entropy solutions, in the context of $L^\infty$ solutions.

**Proposition 1.1.** *Let* $u_b\in L^2(\Omega\times[0,T];H^2(\partial\mathcal{O}))\cap L^2(\Omega;H^1((0,T);L^2(\partial\mathcal{O})))$ *and* $u_0\in L^\infty(\Omega\times\mathcal{O})$. *For a measurable function* $u:\Omega\times[0,T]\times\mathcal{O}\to\mathbb{R}$ *it is equivalent to be a kinetic solution of* (1.1)-(1.3) *and a weak entropy solution of* (1.1)-(1.3).

## 1.3 Earlier works

In the deterministic case, i.e. in the absence of the stochastic term $\Phi(u)dW$, the problem (1.1)-(1.3) in the hyperbolic case $\mathbf{B}\equiv0$ has a well known history beginning with Bardos *et al.* [1], where BV solutions are addressed, then Otto [38] (see also [35]) where $L^\infty$ solutions are considered. Still in the deterministic case, for degenerate parabolic-hyperbolic equations, Mascia *et al.* [36] proved the uniqueness and consistency for their notion of solution, and Michel and Vovelle [37] proved the existence of such solutions, both works in the isotropic case. For the anisotropic case we mention the papers by Kobayasi and Owha [30] and Frid and Li [19].

On the other hand, stochastic conservation laws have a recent yet intense history. For the sake of examples, we mention Kim [28] for the first result of existence and uniqueness of entropy solutions of the Cauchy problem for a one-dimensional stochastic conservation law, in the additive case, that is, $\Phi$ does not depend on $u$. Feng and Nualart [17], where a notion of strong entropy solution is introduced, which is more restrictive than that of entropy solution, and for which the uniqueness is established in the class of entropy solutions in any space dimension, in the multiplicative case, i.e., $\Phi$ depending on $u$; existence of such strong entropy solutions is proven only in the one-dimensional case. Chen *et al.* [7], where the result in [17] was improved and existence in any dimension was proven in the context of the functions of bounded variation. Debussche and Vovelle [15], where a major step in the development of this theory was made with the extension of the concept of kinetic solution, originally introduced by Lions *et al.* [34], for deterministic conservation laws, to the context of stochastic conservation laws, for which the well-posedness of the Cauchy problem was established in the periodic setting in any space dimension. Bauzet *et al.* [2], where the existence and uniqueness of entropy solutions for the general Cauchy problem was proved in any space dimension (see also [27]). Concerning boundary value problems, in the hyperbolic case, Vallet and Wittbold [44], in the additive case, and Bauzet *et al.* [3], in the multiplicative case, obtain existence and uniqueness of entropy solutions to the homogeneous Dirichlet problem, i.e., null boundary condition. Kobayasi and Noboriguchi [29], address the well-posedness of the non-homogeneous Dirichlet problem. As for stochastic degenerate parabolic-hyperbolic equations, we recall that the concept of kinetic solution introduced in [15] together with its well-posedness was achieved also for such equations (in the general non-isotropic case) by Debussche *et al.* [14] and Gess and Hofmanová [21] (see also [2], for entropy solutions of such equations, in the isotropic case, and, more recently, [11] in the non-isotropic case).

## 1.4 Outline of the content

The Dirichlet problem for stochastic degenerated parabolic-hyperbolic equations is addressed here for the first time. In this paper, we introduce the definition of kinetic solution for the Dirichlet problem (1.1)-(1.3) and prove existence and uniqueness of solutions. For the uniqueness of kinetic solutions we prove a new version of the doubling of variables method and use it to deduce a comparison principle between solutions. The proof of uniqueness requires a delicate analysis of the boundary values of the solutions for which we develop some techniques that enable the usage of the existence of normal weak traces for divergence measure fields in the present stochastic setting.

The existence of kinetic solutions for (1.1)-(1.3) will be obtained through a two-level approximation scheme consisting of solutions to adequate regularizations of the equations by the introduction of higher order terms. The notion of kinetic solutions of Definition 1.2 is shown to be consistent with the limit of non-degenerate parabolic approximations provided by the vanishing viscosity method. In particular, the regularity conditions that give meaning to the boundary values of the solutions are inherited from such approximations.

In order to deal with the boundary condition we rely on the theory of divergence measure fields and the normal trace formula (see [8,36]). A key observation is to write a stochastic term of the form $\Psi(v(t))dW(t)$ as $\partial_t(\int_0^t \Psi(v(s))dW(s))$, a.s. in the sense of distributions, so that it can be incorporated to the partial derivative with respect to $t$ of the first coordinate of the corresponding field. This kind of reasoning will allow us to apply the normal trace formula from [8,36] in our present stochastic setting. This is particularly important in the proof of uniqueness, as any comparison principle between kinetic solutions requires a delicate analysis of the boundary values.

The extension of the boundary values to the interior of the spatial domain that we choose plays a key role in the analysis at the boundary developed in the proof of uniqueness. On the one hand, condition (1.20), satisfied by the extension $u_B$ on $\partial\mathcal{O}\cap B$ is essential to control the values of the solution near the boundary. On the other hand, in order to deduce the consistency of the definition of kinetic solutions to (1.1)-(1.3) with limits obtained from the vanishing viscosity method, we are bound to impose that $u_B$ be a strong solution to a stochastic equation whose noise term is given by $\Phi(u_B)dW$, in order to avoid infinite quadratic variation in the limit when comparing a solution with the boundary data near the boundary. This imposition precludes us from using the trivial extension given by $\tilde{u}_B(\bar{x},x_d)=u_b(\bar{x})$ considered in the deterministic case treated in [19,36].

Let us point out that the non-degeneracy condition (1.8) plays a crucial role in the proof of the compactness of the sequence of approximate solutions. Indeed, it guarantees certain uniform regularity of the sequence of solutions, provided by the averaging lemma by Gess and Hofmanová [21], which guarantees the tightness of the sequence of laws. After this, having uniqueness of solutions of the limit problem as well as the consistency of the definition of kinetic solutions with the pointwise limit of non-degenerate parabolic regularizations, the usual argument involving Skorohod's representation together with Gyöngy-Krylov's criterion yields the existence of solutions to the original problem.

The rest of the paper is organized as follows. In Section 2 we recall some results on the theory of divergence measure fields and the existence of normal weak traces. We also show an extension of this theory that allows us to apply the existence of normal traces to certain divergence measure random fields in connection with inequality (1.25) that concern us. In Section 3 we prove a version of the doubling of variables for kinetic solutions of (1.1)-(1.3) and use it to prove a comparison principle, which, in turn, yields uniqueness. In Section 4 we prove the consistency of Definition 1.2 with limits of solutions to regularized non-degenerate parabolic approximations of (1.1)-(1.3). More precisely, we show that a pointwise limit of solutions to a convenient viscous approximation of (1.1)-(1.3) inherits the conditions from item (iii) of Definition 1.2 that give meaning to the boundary conditions (1.3). Finally, in Section 5 we prove the existence part of Theorem 1.1 through a two-level approximation scheme. We also include an Appendix containing some results that we use throughout the text.

# 2 Divergence measure fields and normal traces for kinetic solutions

Let us first recall some definitions and results from the theory of divergence measure fields on domains with deformable boundary.

**Definition 2.1.** *Let $\mathcal{U}\subset\mathbb{R}^N$ be open. For $F\in L^p(\mathcal{U};\mathbb{R}^N)$, $1\le p\le\infty$ set*

$$|\mathrm{div}F|(\mathcal{U}):=\sup\left\{\int_{\mathcal{U}}\nabla\varphi\cdot F\,dx:\varphi\in C_0^1(\mathcal{U}),|\varphi(x)|\le 1,x\in\mathcal{U}\right\}.$$

*For $1\le p\le\infty$ we say that $F$ is an $L^p$-divergence-measure field over $\mathcal{U}$, i.e., $F\in\mathcal{DM}^p(\mathcal{U})$ if $F\in L^p(\mathcal{U};\mathbb{R}^N)$ and*

$$\|F\|_{\mathcal{DM}^p(\mathcal{U})}:=\|F\|_{L^p(\mathcal{U})}+|\mathrm{div}F|(\mathcal{U})<\infty.$$

The following result, first proved in [9, 10] and later extended by Silhavý in [41] establishes the Gauss-Green formula and, in particular, the existence of normal weak traces for general divergence-measure fields. Let us point out that we will only apply these results on bounded domains and fields belonging to $L^p$. Thus, it is enough to consider only $\mathcal{DM}^1$-fields.

**Theorem 2.1** (Chen & Frid [9,10], Silhavý [41])**.** *Let $\mathcal{U}\subset\mathbb{R}^N$ be an open set and let $F\in\mathcal{DM}^1(\mathcal{U})$. Then, there exists a linear functional $F\cdot\nu:\mathrm{Lip}(\partial\mathcal{U})\to\mathbb{R}$ such that*

$$F\cdot\nu(g|_{\partial\mathcal{U}})=\int_{\mathcal{U}}\nabla g\cdot F+\int_{\mathcal{U}}g\,\mathrm{div}F \tag{2.1}$$

*for every* $g \in \mathrm{Lip}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Moreover,*

$$|F\cdot\nu(h)| \le |F|_{\mathcal{DM}(\mathcal{U})} |h|_{\mathrm{Lip}(\partial\mathcal{U})} \tag{2.2}$$

*for all* $h \in \mathrm{Lip}(\partial\mathcal{U})$, *where we use the notation*

$$|g|_{\mathrm{Lip}(C)} := \sup_{x\in C} |g(x)| + Lip_C(g).$$

*Furthermore, let* $m:\mathbb{R}^N \to \mathbb{R}$ *be a nonnegative Lipschitz function with* $\operatorname{supp} m \subset \overline{U}$, *which is strictly positive on* $U$, *and for each* $\varepsilon > 0$ *let* $L_\varepsilon = \{x \in \mathcal{U} : 0 < m(x) < \varepsilon\}$. *Then*

*(i) (cf.* [9,10] *and* [41]*) If* $g \in \mathrm{Lip}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *we have*

$$F\cdot\nu(g|_{\partial\mathcal{U}}) = -\lim_{\varepsilon\to 0} \varepsilon^{-1} \int_{L_\varepsilon} g \nabla m \cdot F dx; \tag{2.3}$$

*(ii) (cf.* [41]*) If*

$$\liminf_{\varepsilon\to 0} \varepsilon^{-1} \int_{L_\varepsilon} |\nabla m \cdot F| dx < \infty, \tag{2.4}$$

*then* $F\cdot\nu$ *is a measure over* $\mathcal{U}$.

A particular example of a function $m$ for which (2.3) holds is given by a level set boundary layer sequence, provided that the domain has a Lipschitz deformable boundary; concepts whose definitions we recall subsequently.

**Definition 2.2.** *Let* $\mathcal{U} \subset \mathbb{R}^N$ *be an open set. We say that* $\partial\mathcal{U}$ *is a Lipschitz deformable boundary if the following conditions hold:*

*(i) For each* $x \in \partial\mathcal{U}$, *there exist* $r > 0$ *and a Lipschitz mapping* $\gamma:\mathbb{R}^{N-1} \to \mathbb{R}$ *such that, upon relabelling, reorienting and translation,*

$$\mathcal{U} \cap Q(x,r) = \Big\{ y \in \mathbb{R}^{N-1} : \gamma(y_1,\cdots,y_{N-1}) < y_N \Big\} \cap Q(x,r),$$

*where*

$$Q(x,r) = \Big\{ y \in \mathbb{R}^N : |y_i - x_i| \le r,\ i = 1,\dots,N \Big\}.$$

*We denote by* $\hat{\gamma}$ *the map* $\hat{y} \mapsto (\hat{y}, \gamma(\hat{y}))$, $\hat{y} = (y_1,\cdots,y_{N-1})$.

*(ii) There exists a map* $\Psi:[0,1]\times\partial\mathcal{U} \to \bar{\mathcal{U}}$ *such that* $\Psi$ *is a bi-Lipschitz homeomorphism over its image and* $\Psi(0,x) = x$ *for all* $x \in \partial\mathcal{U}$. *For* $s \in [0,1]$, *we denote by* $\Psi_s$ *the mapping from* $\partial\mathcal{U}$ *to* $\bar{\mathcal{U}}$ *given by* $\Psi_s(x) = \Psi(s,x)$, *and set* $\partial\mathcal{U}_s := \Psi_s(\partial\mathcal{U})$. *We call such map a Lipschitz deformation for* $\partial\mathcal{U}$.

**Definition 2.3.** *Let $\mathcal{U}\subset\mathbb{R}^N$ be an open set with a Lipschitz deformable boundary and $\Psi:[0,1]\times\partial\mathcal{U}\to\bar{\mathcal{U}}$ a Lipschitz deformation for $\partial\mathcal{U}$. The Lipschitz deformation is said to be regular over $\Gamma\subset\partial\mathcal{U}$, if $D\Psi_s\to\mathrm{Id}$, as $s\to0$, in $L^1(\Gamma,\mathcal{H}^{N-1})$. It is simply said to be regular if it is regular over $\partial\mathcal{U}$. The Lipschitz deformation is said to be strongly regular over $\Gamma\subset\partial\mathcal{U}$, if it is regular over $\Gamma$ and the Jacobian determinants $J[\Psi_s]$, $0\le s\le1$, defined through a convenient parametrization for $\Gamma$, belong to Lip $(\Gamma)$ and $J[\Psi_s]\to1$ in Lip $(\Gamma)$ as $s\to0$.*

The following theorem from [18], provides a formula for the normal weak trace of $L^p$-divergence measure fields, similar to the one for $L^\infty$-divergence measure fields established in [8], whose advantage is that it renders clear that the referred trace depends only on the part of the boundary where it is being evaluated on which the deformation is strongly regular and not on the whole boundary (see also [19]).

**Theorem 2.2** (cf. [18]). *Let $F\in\mathcal{DM}^1(U)$, where $U\subset\mathbb{R}^{N+1}$ is a bounded open set with a Lipschitz deformable boundary and Lipschitz deformation $\Psi:[0,1]\times\partial U\to\bar{U}$. Denoting by $F\cdot\nu|_{\partial U}$ the continuous linear functional Lip $(\partial U)\to\mathbb{R}$ given by the normal trace of $F$ at $\partial U$, we have the formula*

$$F\cdot\nu|_{\partial U}=\operatorname*{esslim}_{s\to0}F\circ\Psi_s(\cdot)\cdot\nu_s\big(\Psi_s(\cdot)\big)J[\Psi_s] \tag{2.5}$$

*with equality in the sense of $(Lip\,(\partial U))^*$, where on the right-hand side the functionals are given by ordinary functions in $L^1(\partial U)$. In particular, if $\Psi$ is strongly regular over $\Gamma\subset\partial U$ then, for all $\varphi\in Lip\,(\partial U)$ with $\operatorname{supp}\varphi\subset\Gamma$, we have*

$$\langle F\cdot\nu|_{\partial U},\varphi\rangle=\operatorname*{esslim}_{s\to0}\int_\Gamma F\circ\Psi_s(\omega)\cdot\nu_s\big(\Psi_s(\omega)\big)\varphi(\omega)d\mathcal{H}^N(\omega). \tag{2.6}$$

We also recall the following definition of boundary layer sequence (cf. [19,36]).

**Definition 2.4.** *Let $\mathcal{U}\subset\mathbb{R}^d$ be a smooth open set. We say that $\zeta_\delta$ is a boundary layer sequence if for each $\delta>0$, $\zeta_\delta\in Lip\,(\mathcal{U})$, $0\le\zeta_\delta\le1$, $\zeta_\delta(x)\to1$ for every $x\in\mathcal{U}$, as $\delta\to0$ and $\zeta_\delta=0$ on $\partial\mathcal{U}$.*

Let us also recall that if $\mathcal{O}$ has a Lipschitz deformable boundary and given a Lipschitz deformation for $\partial\mathcal{O}$, $\Psi:[0,1]\times\partial\mathcal{O}\to\mathcal{O}$, the associated level set function $h:\mathcal{O}\to[0,1]$ is given by $h(x)=s$ for $x\in\Psi(s,\partial\mathcal{O})$ and $h(x)=1$ for $x\in\mathcal{O}\backslash\Psi([0,1]\times\partial\mathcal{O})$. Then we can also define an associated boundary layer sequence by

$$\zeta_\delta(x)=\frac{1}{\delta}\min\{\delta,h(x)\},\quad 0<\delta<1,$$

which we call the level set boundary layer sequence associated with the deformation $\Psi$. In this case, as in [19], we note that if $\Psi$ is of class $C^{1,1}$, we have that

$$\begin{aligned}&\nabla\zeta_\delta(x)=-\frac{1}{\delta}\chi_{\{0<\zeta_\delta(x)<1\}}(x)N(x),\\&D^2\zeta_\delta(x)=-\frac{N(x)\otimes\nu(x)}{\delta}d\mathcal{H}^{d-1}(x)\big|\Psi(\delta,\partial\mathcal{O})-\frac{1}{\delta}\chi_{0<\zeta_\delta(x)<1}\nabla N(x),\end{aligned}\tag{2.7}$$

where $N(x)=\lambda(x)\nu(x)$, $\nu$ is the outward unit normal vector to $\Psi(\delta,\partial\mathcal{O})$, $\lambda(x)$ is a positive Lipschitz function and $\mathcal{H}^{d-1}(x)|\Psi(\delta,\partial\mathcal{O})$ denotes the $(d-1)$-dimensional Hausdorff measure restricted to the hyper-surface $\Psi(\delta,\partial\mathcal{O})$.

**Remark 2.1.** Let $U$ be endowed with a Lipschitz boundary and $\Gamma\subset\partial U$ be an open subset of $\partial U$. Let $o\in\Gamma$ and $\mathcal{R}_o:\mathbb{R}^{N+1}\to\mathbb{R}^{N+1}$ be a rigid motion in $\mathbb{R}^{N+1}$ such that, for some Lipschitz function $\gamma:\mathbb{R}^N\to\mathbb{R}$, denoting $y=\mathcal{R}_o x$, $\hat{y}=(y_1,\cdots,y_N)$, and defining $\hat{\gamma}(\hat{y}):=(\gamma(\hat{y}),\hat{y})$, we have that $\hat{\gamma}(D)=\Gamma$, for some open set $D\subset\mathbb{R}^N$. Let also $\tilde{U}=\{y\in\mathbb{R}^{N+1}:\gamma(\hat{y})<y_0\}$ and suppose $U\cap\tilde{U}\neq\emptyset$. If $F\in\mathcal{DM}^1(U)\cap\mathcal{DM}^1(\tilde{U})$, it is immediate to check, using the Gauss-Green formula, that $F\cdot\nu|_{\partial U}$ and $F\cdot\nu|_{\partial\tilde{U}}$ coincide over $\Gamma$, that is,

$$\langle F\cdot\nu|_{\partial U},\varphi\rangle=\langle F\cdot\nu|_{\partial\tilde{U}},\varphi\rangle$$

for all $\varphi\in\mathrm{Lip}_c(\Gamma)$, where $\mathrm{Lip}_c(\Gamma)$ denotes the subspace of functions in $\mathrm{Lip}(\Gamma)$ with compact support in $\Gamma$. Also recall that, to define $F\cdot\nu|_{\partial U}$, it is not necessary that $\partial U$ be Lipschitz deformable. In such cases, restricted to functions $\varphi\in\mathrm{Lip}_c(\Gamma)$ we will always view $\langle F\cdot\nu|_{\partial U},\varphi\rangle$ as obtained, after possible translation, relabelling and reorienting coordinates, through (2.6) by using the canonical deformation of $\partial\tilde{U}$, defined as $(s,(\gamma(\hat{y}),\hat{y}))\mapsto(\gamma(\hat{y})+s,\hat{y})$, evidently strongly regular over $\Gamma$, which is legitimate for $\tilde{U}$; we call that a local canonical deformation of $\partial U$. In agreement with this terminology, we will call the corresponding level set boundary layer sequence a local canonical boundary layer sequence.

Coming back to the main subject of the paper, we will use Theorem 2.1 to deal with the boundary values of kinetic solutions of Eqs. (1.1)-(1.3), which is essential to guarantee their uniqueness. In particular, we will now apply Theorem 2.1 in combination with (1.25) in order to guarantee the existence of the normal traces on $\partial\mathcal{O}$ for the field $K_x(u(t,x),u_B(t,x))$, where $u$ is a kinetic solution of Eqs. (1.1)-(1.3). More precisely, we have the following result.

**Proposition 2.1.** *Let $u\in L^\infty(\Omega\times[0,T]\times\mathcal{O})$ satisfy conditions* (1.23) *and* (1.25) *from Definition* 1.2 *and let $B\in\mathcal{B}$ (that is, $B$ as in* (1.17)*). Given $0\le\psi\in C_c^\infty(B)$, consider the field $(\psi F_0,\psi F_1)$, given by*

$$
\begin{aligned}
&F_0 := |u(t,x) - u_B(t,x)| - \int_0^t \mathbb{G}\big(u(s,x), u_B(s,x)\big)\, dW(s),\\
&F_1 := -K_x\big(u(t,x), u_B(t,x)\big),
\end{aligned}
$$

*where $u_B$ is any extension of $u_b$ to $\mathcal{O} \cap B$ satisfying* (1.19)-(1.22).

*Then, we have a.s. that $\partial_t(\psi F_0) + \nabla \cdot (\psi F_1)$ is a Radon measure with finite total variation in $(0,T) \times \mathcal{O}$. Moreover,*

$$
\mathbb{E}\, |\mathrm{div}_{t,x}(\psi F_0, \psi F_1)| \big((0,T) \times \mathcal{O}\big) \le \tilde{C},
$$

*where $\tilde{C}$ is a finite constant that depends only on $T$, $\Omega$, $\|u\|_{L^\infty}$, $u_b$, $\mathbf{A}$, $\|\nabla b(u)\|_{L^2}$ and $\|\psi, \nabla \psi\|_{L^\infty}$. In particular, $K_x(u(t,x), u_B(t,x))\psi(x)$ has a.s. a normal weak trace $\mathcal{K}(u(t), u_B(t)) \cdot \nu : \mathrm{Lip}([0,T] \times \partial \mathcal{O}) \to \mathbb{R}$ on $\partial \mathcal{O}$ (in the sense of Theorem* 2.1*) such that:*

*(i) for every $g \in \mathrm{Lip}(\mathbb{R}^{d+1})$, the map $\omega \mapsto \mathcal{K}(u, u_B) \cdot \nu(g | [0,T] \times \partial \mathcal{O})$ is measurable and*

$$
\mathbb{E}\, |\mathcal{K}(u, u_B) \cdot \nu(g | [0,T] \times \partial \mathcal{O})|\, dt \le C |g|_{\mathrm{Lip}([0,T] \times \partial \mathcal{O})} \tag{2.8}
$$

*for some positive constant $C$ which depends on $\tilde{C}$; and*

*(ii) for any $g \in \mathrm{Lip}(\mathbb{R}^{d+1})$ with* $\mathrm{supp}\, g \subset (0,T) \times \mathbb{R}^d$ *and any non-negative Lipschitz function $m : \mathbb{R}^d \to \mathbb{R}$ with* $\mathrm{supp}\, m \subset [0,T] \times \overline{\mathcal{O}}$*, which is strictly positive on $\mathcal{O}$, we have a.s. that*

$$
\begin{aligned}
&\mathcal{K}(u, u_B) \cdot \nu (g | [0,T] \times \partial \mathcal{O})\\
&= -\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^T \int_{L_\varepsilon} g \nabla m \cdot K_x\big(u(t,x), u_B(t,x)\big)\, dx ds,
\end{aligned} \tag{2.9}
$$

*where $L_\varepsilon = \{x \in \mathcal{O} : 0 < m(t,x) < \varepsilon\}$.*

*Proof.* Note that $(\psi F_0, \psi F_1) \in L^2((0,T) \times \mathcal{O}; \mathbb{R}^{d+1})$ a.s.. We also observe that, in order to show that $\mathrm{div}_{(t,x)}(\psi F_0, \psi F_1)$ is a Radon measure, it is enough to show that $\ell_1 := \psi(\partial_t F_0 + \nabla \cdot F_1)$ is a radon measure with a.s. finite total variation in $(0,T) \times \mathcal{O}$. For this, we first note that we may write the term $\mathbb{G}(u(t,x), u_B(t,x)) dW(t)$ as $\partial_t \int_0^t \mathbb{G}(u(s,x), u_B(s,x)) dW(s)$ in the sense of distributions (where $\mathbb{G}$ is given by (1.15)). Then, we would like to replace $\tilde{\varphi}(t,x) = \psi(x)(\|\varphi\|_{L^\infty} \pm \varphi(t,x))$ in (1.25), where $\varphi \in C^\infty((0,T) \times B)$. However, we must first extend inequality (1.25) to functions that do not vanish at $t=0$ and $t=T$.

Let $\xi_h(t) \in \mathbb{C}_c^\infty(0,T)$ with $0 \le \xi(t) \le 1$, such that $\xi_h(t) \to 1$ as $h \to 0$, everywhere in $(0,T)$, and let $0 \le \varphi \in C_c^\infty([0,T] \times B)$. Then, we may take $\tilde{\varphi}(t,x) = \xi_h(t) \varphi(t,x)$ in

(1.25) and let $h\to 0$ to obtain

$$\begin{aligned}
&\int_0^T\int_{\mathcal{O}}\big\{|u(t,x)-u_B(t,x)|\partial_t\varphi-K_x\big(u(t,x),u_B(t,x)\big)\cdot\nabla\varphi\big\}\,dxdt\\
&\quad+\sum_{k\geq 1}\int_0^T\int_{\mathcal{O}}\mathbb{G}_k\big(u(t,x),u_B(t,x)\big)\varphi\,dxd\beta_k(t)\\
&\geq -C(C_*,\|u\|_{L^\infty},\|u_b\|_{L^\infty})\left(\|\varphi\|_{L^\infty}+\int_{\mathcal{O}}\big(\varphi(0,x)+\varphi(T,x)\big)dx\right).
\end{aligned}\tag{2.10}$$

Then, we take any $\bar\varphi\in C_c^\infty(0,T)\times\mathcal{O}$ and substitute $\varphi=\psi(\|\bar\varphi\|_{L^\infty}\pm\bar\varphi)$ in (2.10). After some straightforward manipulation we have a.s. that

$$\begin{aligned}
|\langle\ell_1,\varphi\rangle|\leq{}& C\big(T,\Omega,\|u\|_{L^\infty},\|u_b\|_{L^\infty},\|\nabla b(u)\|_{L^2},\|\psi,\nabla\psi\|_{L^\infty}\big)\|\bar\varphi\|_{L^\infty}\\
&+\|\bar\varphi\|_{L^\infty}\left(1+\left(\sum_{k\geq 1}\int_0^T\int_{\mathcal{O}}\mathbb{G}_k\big(u(t,x),u_B(t,x)\big)\psi\,dxd\beta_k(t)\right)^2\right).
\end{aligned}$$

Concerning the term multiplying $\|\bar\varphi\|_{L^\infty}$ in the last line, we note that, by the Itô isometry, its expectation is finite and, therefore, it is finite almost surely. Thus, we conclude that $|\langle\ell,\bar\varphi\rangle|\leq C_1\|\bar\varphi\|_{L^\infty}$ a.s. for some finite random constant $C_1$ whose expected value is finite.

At last, we note that we may apply Theorem 2.1 to deduce the existence of the normal weak traces of $K_x(u(t,x),u_b(t,x))$, denoted by $\mathcal{K}(u(t),u_b(t))\cdot\nu$, on $\partial\mathcal{O}$ satisfying (i) and (ii) above due to (2.2) and (2.3). $\square$

# 3 Doubling of variables and uniqueness

Let us move on to the proof of the uniqueness part of Theorem 1.1. We start by recalling the result establishing the existence of left- and right-continuous representatives of a kinetic solution proved in [14, 15]. The same property holds here also and the proof is exactly the same as in [14, 15] to which we refer.

**Proposition 3.1** (Left- and right-continuous representatives)**.** *Let $u$ be a kinetic solution to* (1.1)-(1.3). *Then $f=1_{u>\xi}$ admits representatives $f^-$ and $f^+$ which are almost surely left- and right-continuous, respectively, at all points $t^*\in[0,T]$ in the sense of distributions over $\mathcal{O}\times\mathbb{R}$. More precisely, for all $t^*\in[0,T]$ there exist kinetic functions $f^{*,\pm}$ on $\Omega\times\mathcal{O}\times\mathbb{R}$ such that setting $f^\pm(t^*)=f^{*,\pm}$ yields $f^\pm=f$ almost everywhere and*

$$\langle f^\pm(t^*\pm\varepsilon),\psi\rangle\to\langle f^\pm(t^*),\psi\rangle,\quad \varepsilon\downarrow 0,\quad \forall\psi\in C_c^2(\mathcal{O}\times\mathbb{R}),\quad \mathbb{P}\text{-a.s.}$$

*Moreover, $f^+=f^-$ for all $t^*\in[0,T]$ except for some at most countable set.*

**Remark 3.1.** Let $0<\tau\le t<T$. Note that taking a test function of the form $\varphi(s,x,\xi)= \theta(s)\phi(x,\xi)$ in (1.24), with

$$\theta(s)=\begin{cases} 0, & s\le \tau-\varepsilon,\\ \dfrac{s+\varepsilon-\tau}{\varepsilon}, & \tau-\varepsilon\le s\le \tau,\\ 1, & \tau\le s\le t,\\ 1-\dfrac{s-t}{\varepsilon}, & t\le s\le t+\varepsilon,\\ 1, & t+\varepsilon\le s,\end{cases}$$

and letting $\varepsilon\to 0$, we obtain that

$$\begin{aligned} &-\langle f^+(t),\phi\rangle dt+\langle f^-(\tau),\phi\rangle+\int_\tau^t \langle f(s),\mathbf{a}(\xi)\cdot\nabla\phi\rangle\,ds+\int_\tau^t \langle f(s),\mathbf{b}:D^2\phi\rangle\,ds\\ =&-\sum_{k\ge 1}\int_\tau^t\int_{\mathcal{O}} g_k\big(x,u(s,x)\big)\phi\big(x,u(s,x)\big)\,dxd\beta_k(s)\\ &-\frac{1}{2}\int_\tau^t\int_{\mathcal{O}} G^2\big(x,u(s,x)\big)\partial_\xi\phi\big(x,u(t,x)\big)\,dxdt+m(\partial_\xi\phi)([\tau,t]). \end{aligned} \tag{3.1}$$

In order to prove the uniqueness of the kinetic solution, in view of the Dirichlet condition and the fact that the noise coefficients may not vanish close to the boundary, we need to split our comparison analysis in two different situations:

(i) one in which the test function has support far from the boundary, say, at a distance $\ge \delta_0/2$, in which case the analysis is standard and essentially repeats that of [14] with slight adaptations;

(ii) another where the support of test function lies at a distance less than $\delta_0$ from the boundary (cf. (1.12)). This splitting of cases is necessary, since in case (ii) we need a doubling of variables result in the macroscopic variables in order to deal with the boundary values of the solutions in the proof of the comparison principle (cf. Theorem 3.3 below), unlike the doubling of variables from [14] (cf. [15, 21]) which takes place in the microscopic variables (i.e. kinetic functions associated to the corresponding solutions).

For the situation (i) mentioned above, we have the following result whose proof is similar to that of the corresponding result in [14] with slight adaptations and so we omit it here.

**Theorem 3.1.** *Let $u_1,u_2$ be kinetic solutions of* (1.1)-(1.3). *Then, for $0\le t\le T$ and $0\le\phi\in C_c^\infty(\mathcal{O})$, we have*

$$\begin{aligned}&\mathbb{E}\int_{\mathcal{O}}\left|u_1^{\pm}(t,x)-u_2^{\pm}(t,x)\right|\phi(x)dx\\ \le&-\mathbb{E}\int_0^t\int_{\mathcal{O}}\mathrm{sgn}(u_1-u_2)\big(\mathbf{A}(u_1)-\mathbf{A}(u_2)\big)\cdot\nabla\phi(x)dxdt\\ &-\mathbb{E}\int_0^t\int_{\mathcal{O}}\mathbb{B}(u_1,u_2):D_x^2\phi(x)dxdt+\int_{\mathcal{O}}|u_{10}(x)-u_{20}(x)|\phi(x)dx. \end{aligned}\tag{3.2}$$

For the situation (ii) mentioned above, we introduce the following version of the doubling of variables for kinetic solutions of Eq. (1.1), which generalizes to the stochastic case a corresponding comparison inequality from the deterministic setting considered first by Carrillo [6] (see also [19, 36]). Such an extension will allow us to extend the comparison inequality (3.2) to admit test functions that do not necessarily vanish on the boundary (see Theorem 3.3 below).

**Theorem 3.2** (Doubling of variables 1)**.** *Let $u$ and $v$ be kinetic solutions of* (1.1)-(1.3) *with initial data $u_0$ and $v_0$, respectively. Denote $\nabla_{x+y}=\nabla_x+\nabla_y$. Then, for $0\le\tau\le t\le T$ and any non-negative test function $\phi\in C_c^\infty(\mathcal{O}^2)$, with support at a distance less than $\delta_0$ from the diagonal of $\partial\mathcal{O}\times\partial\mathcal{O}$, we have a.s. that*

$$\begin{aligned}&\int_{\mathcal{O}^2}\big(u^{\pm}(t,x)-v^{\pm}(t,y)\big)_+\phi(x,y)dxdy \qquad (3.3)\\ \le&\int_{\mathcal{O}^2}\big(u^{\pm}(\tau,x)-v^{\pm}(\tau,y)\big)_+\phi(x,y)dxdy\\ &+\int_\tau^t\int_{\mathcal{O}^2}F_+\big(u(s,x),v(s,y)\big)\cdot\nabla_{x+y}\phi(x,y)dxdyds\\ &-\int_\tau^t\int_{\mathcal{O}^2}\nabla_{x+y}\cdot\mathbb{B}_+\big(u(s,x),v(s,y)\big)\nabla_{x+y}\phi(x,y)dxdyds\\ &+\sum_{k\ge1}\int_\tau^t\int_{\mathcal{O}^2}\mathrm{sgn}\big(u(s,x)-v(s,x)\big)_+\big(g_k(u(s,x))-g_k(v(s,y))\big)\phi(x,y)dxdyd\beta_k(s).\end{aligned}$$

*Proof.* The proof draws on that of the comparison principle in [14] (cf. [15,21,25]). Let $f_1(t,x,\xi)=1_{u(t,x)>\xi}$ and $f_2(t,y,\zeta)=1_{v(t,y)>\zeta}$ and set $\bar{f}_2=1-f_2$. Let us denote $\langle\langle\cdot,\cdot\rangle\rangle_{L^2}$, the scalar product in $L^2(\mathcal{O}_x\times\mathcal{O}_y\times\mathbb{R}_\xi\times\mathbb{R}_\zeta)$. Set $G_1^2(x,\xi)=\sum_{k\ge1}|g_k(\xi)|^2$ and $G_2^2(x,\zeta)=\sum_{k\ge1}|g_k(\zeta)|^2$.

Proceeding as in [14,15,21,25] (but without taking the expectation), for $0\le\tau\le t\le T$ and any $\alpha\in C_c^\infty(\mathcal{O}^2\times\mathbb{R}^2)$, we have a.s. that

$$\langle\langle f_1^+(t)\overline{f}_2^+(t),\alpha\rangle\rangle_{L^2}=\sum_{j=1}^{10}I_j,\tag{3.4}$$

where

$$
\begin{aligned}
I_1 &= \langle\langle f_1^-(\tau)\overline{f}_2^-(\tau),\alpha\rangle\rangle_{L^2},\\
I_2 &= \int_\tau^t \int_{\mathcal{O}^2}\int_{\mathbb{R}^2} f_1\overline{f}_2\big(\mathbf{a}(\xi)\cdot\nabla_x+\mathbf{a}(\zeta)\cdot\nabla_y\big)\alpha d\xi d\zeta dxdyds,\\
I_3 &= \int_\tau^t \int_{\mathcal{O}^2}\int_{\mathbb{R}^2} f_1\overline{f}_2\Big(\mathbf{b}(\xi):D_x^2+\mathbf{b}(\zeta):D_y^2\Big)\alpha d\xi d\zeta dxdyds,\\
I_4 &= \frac{1}{2}\int_\tau^t \int_{\mathcal{O}^2}\int_{\mathbb{R}^2} \partial_\xi\alpha\overline{f}_2(s)G_1^2 d\nu^1_{(x,s)}(\xi)d\zeta dxdyds,\\
I_5 &= -\frac{1}{2}\int_\tau^t \int_{\mathcal{O}^2}\int_{\mathbb{R}^2} \partial_\zeta\alpha f_1(s)G_2^2 d\nu^2_{(y,s)}(\zeta)d\xi dydxds,\\
I_6 &= -\int_\tau^t \int_{\mathcal{O}^2}\int_{\mathbb{R}^2} G_{1,2}\alpha d\nu^1_{x,s}(\xi)d\nu^2_{y,s}(\zeta)dxdyds,\\
I_7 &= -\int_\tau^t \int_{\mathcal{O}^2}\int_{\mathbb{R}^2} \overline{f}_2^+(s)\partial_\xi\alpha dm_1(x,s,\xi)d\zeta dy,\\
I_8 &= \int_\tau^t \int_{\mathcal{O}^2}\int_{\mathbb{R}^2} f_1^-(s)\partial_\zeta\alpha dm_2(y,s,\zeta)d\xi dx,\\
I_9 &= -\sum_{k\geq 1}\int_\tau^t \int_{\mathcal{O}^2}\int_{\mathbb{R}^2} f_1 g_k(\zeta)\alpha d\nu^2_{y,s}(\zeta)d\xi dxdyd\beta_k(s),\\
I_{10} &= \sum_{k\geq 1}\int_\tau^t \int_{\mathcal{O}^2}\int_{\mathbb{R}^2} \overline{f}_2 g_k(\xi)\alpha d\nu^1_{x,s}(\xi)d\zeta dxdyd\beta_k(s),
\end{aligned}
$$

and also $\nu^1_{x,s}=\delta_{u(s,x)}$, $\nu^2_{y,s}=\delta_{v(s,y)}$ and $G_{1,2}(\xi,\zeta):=\sum_{k\geq 1}g_k(\xi)g_k(\zeta)$. Let $\psi_\delta$ be a standard mollifier in $\mathbb{R}$, i.e.

$$\psi_\delta(\xi)=\frac{1}{\delta}\psi\left(\frac{\xi}{\delta}\right)$$

for some $0\leq\psi\in C_c^\infty(\mathbb{R})$ such that

$$\int_{\mathbb{R}}\psi(\xi)d\xi=1,$$

and let us take $\alpha(x,y,\xi,\zeta)=\phi(x,y)\psi_\delta(\xi-\zeta)$ in (3.4). In particular,

$$\partial_\xi\alpha=-\partial_\zeta\alpha. \tag{3.5}$$

As in [15], we note that

$$\partial_\xi f_1=-\nu^1_{x,s},\quad \partial_\zeta\overline{f}_2=\nu^2_{y,s}, \tag{3.6}$$

which together with (3.5) can be used to deduce that

$$I_4+I_5+I_6=\frac{1}{2}\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}^2}\alpha\sum_{k\geq 1}|g_k(\xi)-g_k(\zeta)|^2 d\nu^1_{x,s}\otimes\delta\nu^2_{y,s}(\xi,\zeta)\,dx\,dy\,ds,$$

which, by (1.11), yields

$$|I_4+I_5+I_6|\leq Ct\delta. \tag{3.7}$$

Now, we write $I_3$ as

$$I_3=I_3+J-J, \tag{3.8}$$

where, denoting $[(\nabla_x\otimes\nabla_y)\alpha]_{i,j}=\partial_{x_i}\partial_{y_j}\alpha$ for $i,j=1,\ldots,d$, we define

$$J:=2\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}^2}f_1\overline{f}_2\sigma(\xi)\sigma(\zeta):(\nabla_x\otimes\nabla_y)\alpha\,d\xi\,d\zeta\,dx\,dy\,ds. \tag{3.9}$$

We claim that

$$I_7+I_8-J\leq 0. \tag{3.10}$$

Indeed, proceeding similarly as in [14], using (3.5) once again, and recalling the definition of the parabolic dissipation measure for $u$ and $v$, we have that

$$\begin{aligned}
I_7+I_8\leq&-\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}^2}\alpha\,d\nu^2_{y,s}(\zeta)\,dn^u_1(s,x,\xi)-\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}^2}\alpha\,d\nu^2_{x,s}(\xi)\,dn^v_2(s,y,\zeta)\\
=&-\int_\tau^t\int_{\mathcal{O}^2}\phi(x,y)\psi_\delta(u-v)\left|\mathrm{div}_x\int_0^u\sigma(\zeta)d\zeta\right|^2dx\,dy\,ds\\
&-\int_\tau^t\int_{\mathcal{O}^2}\phi(x,y)\psi_\delta(u-v)\left|\mathrm{div}_y\int_0^v\sigma(\zeta)d\zeta\right|^2dx\,dy\,ds.
\end{aligned}$$

On the other hand, integrating by parts in (3.9) and using the chain rule formula (1.28), we have that

$$\begin{aligned}
J=&2\int_\tau^t\int_{\mathcal{O}^2}\phi(x,y)\mathrm{div}_y\int_0^v\sigma(\zeta)\,\mathrm{div}_x\int_0^u\sigma(\xi)\psi_\delta(\xi-\zeta)\,d\xi\,d\zeta\,dx\,dy\,ds\\
=&2\int_\tau^t\int_{\mathcal{O}^2}\phi(x,y)\psi_\delta(u-v)\mathrm{div}_x\int_0^u\sigma(\xi)d\xi\,\mathrm{div}_y\int_0^v\sigma(\zeta)d\zeta\,dx\,dy\,ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
&I_7+I_8-J\\
\leq&-\int_\tau^t\int_{\mathcal{O}^2}\phi(x,y)\psi_\delta(u-v)\left|\mathrm{div}_x\int_0^u\sigma(\xi)d\xi-\mathrm{div}_y\int_0^v\sigma(\zeta)d\zeta\right|^2dx\,dy\,ds\leq 0,
\end{aligned}$$

as claimed.

In order to bound $I_9$ and $I_{10}$, we first observe that (3.5) and (3.6) yield

$$\begin{aligned}
&-\int_{\mathbb{R}^2} f_1 g_k(\zeta)\alpha d\nu^2_{y,s}(\zeta)d\xi \\
&=\int_{\mathbb{R}^2} f_1\overline{f}_2 g_k'(\zeta)\alpha d\xi d\zeta+\int_{\mathbb{R}^2} f^1\overline{f}_2 g_k(\zeta)\partial_\zeta\alpha d\xi d\zeta \\
&=\int_{\mathbb{R}^2} f_1\overline{f}_2 g_k'(\zeta)\alpha d\xi d\zeta-\int_{\mathbb{R}^2} f^1\overline{f}_2 g_k(\zeta)\partial_\xi\alpha d\xi d\zeta \\
&=\int_{\mathbb{R}^2} f_1\overline{f}_2 g_k'(\zeta)\alpha d\xi d\zeta-\int_{\mathbb{R}^2} \overline{f}_2 g_k(\zeta)\alpha d\nu^1_{x,s}(\xi)d\zeta,
\end{aligned}$$

so that

$$I_9+I_{10}=\sum_{k\geq 1}\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}^2} f_1\overline{f}_2 g_k'(\zeta)\alpha d\xi d\zeta dxdyd\beta_k(s)+I_{11}, \tag{3.11}$$

where

$$\begin{aligned}
I_{11}&=\sum_{k\geq 1}\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}^2}\overline{f}_2\big(g_k(\xi)-g_k(\zeta)\big)\alpha d\nu^1_{x,s}(\xi)d\zeta dxdyd\beta_k(s) \\
&=\sum_{k\geq 1}\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}^2}\overline{f}_2\big(g_k(\xi)-g_k(\zeta)\big)\phi(x,y)\psi_\delta(\xi-\zeta)d\nu^1_{x,s}(\xi)d\zeta dxdyd\beta_k(s).
\end{aligned}$$

We claim that $\mathbb{E}(\sup_{\tau\leq t\leq T}(I_{11})^2)\to 0$ as $\delta\to 0$. Indeed, using Doob's maximal inequality and Itô isometry we have that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\tau\leq t\leq T}(I_{11})^2\Big) \\
&\leq C\int_0^T\int_{\mathcal{O}^2}\phi(x,y)\sum_{k\geq 1}\left|\int_{\mathbb{R}^2}|g_k(\xi)-g_k(\zeta)|\psi_\delta(\xi-\zeta)d\nu_{x,s}(\xi)d\zeta\right|^2 dxdyds.
\end{aligned}$$

Noting that $\psi_\delta(\xi-\zeta)d\nu_{x,s}(\xi)d\zeta$ is a probability measure on $\mathbb{R}^2$, we can apply Jensen's inequality in order to obtain that

$$\begin{aligned}
&\sum_{k\geq 1}\left|\int_{\mathbb{R}^2}|g_k(\xi)-g_k(\zeta)|\psi_\delta(\xi-\zeta)d\nu_{x,s}(\xi)d\zeta\right|^2 \\
&\leq\int_{\mathbb{R}^2}\sum_{k\geq 1}|g_k(\xi)-g_k(\zeta)|^2\psi_\delta(\xi-\zeta)d\nu_{x,s}(\xi)d\zeta.
\end{aligned}$$

Thus, we can use (1.11) to conclude that

$$\mathbb{E}\Big(\sup_{\tau\leq t\leq T}(I_{11})^2\Big)\leq CT\delta^2,$$

which converges to zero as $\delta\to 0$, proving the claim.

Since mean square convergence implies convergence in probability, which in turn implies almost sure convergence along a subsequence, we may take a sequence $\delta_k \to 0$ along, which $I_{11} \to 0$ a.s. and for a.e. $t$. Consequently, we may take $\delta = \delta_k \to 0$ in (3.4) and use (3.7), (3.10) and (3.11) to obtain that a.s. we have

$$\begin{aligned}
&\int_{\mathcal{O}^2}\int_{\mathbb{R}} f_1^+(t,x,\xi)\overline{f}_2^+(t,x,\xi)\phi(x,y)d\xi,dxdy \\
\leq &\int_{\mathcal{O}^2}\int_{\mathbb{R}} f_1^-(\tau,x,\xi)\overline{f}_2^-(\tau,x,\xi)\phi(x,y)d\xi,dxdy \\
&+\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}} f_1\overline{f}_2\mathbf{a}(\xi)\cdot(\nabla_x+\nabla_y)\phi(x,y)d\xi dxdyds \\
&+\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}} f_1\overline{f}_2\mathbf{b}(\xi):\left(D_x^2+2\nabla_x\otimes\nabla_y+D_y^2\right)\phi(x,y)d\xi dxdyds \\
&+\sum_{k\geq 1}\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}} f_1\overline{f}_2 g_k'(\xi)\phi(x,y)d\xi dxdyd\beta_k(s).
\end{aligned} \tag{3.12}$$

Substituting $\tau$ by $\tau+1/n$ in (3.12) and letting $n\to\infty$, we conclude

$$\begin{aligned}
&\int_{\mathcal{O}^2}\int_{\mathbb{R}} f_1^+(t,x,\xi)\overline{f}_2^+(t,x,\xi)\phi(x,y)d\xi dxdy \\
\leq &\int_{\mathcal{O}^2}\int_{\mathbb{R}} f_1^+(\tau,x,\xi)\overline{f}_2^+(\tau,x,\xi)\phi(x,y)d\xi dxdy \\
&+\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}} f_1\overline{f}_2\mathbf{a}(\xi)\cdot(\nabla_x+\nabla_y)\phi(x,y)d\xi dxdyds \\
&+\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}} f_1\overline{f}_2\mathbf{b}(\xi):\left(D_x^2+2\nabla_x\otimes\nabla_y+D_y^2\right)\phi(x,y)d\xi dxdyds \\
&+\sum_{k\geq 1}\int_\tau^t\int_{\mathcal{O}^2}\int_{\mathbb{R}} f_1\overline{f}_2 g_k'(\xi)\phi(x,y)d\xi dxdyd\beta_k(s).
\end{aligned} \tag{3.13}$$

Finally, we see that (3.13) is exactly (3.3), upon performing the integration in $d\xi$ and integrating by parts in the resulting integral on the fourth line.

For the case of $f_1^-$, $f_2^-$, we take $\tau_n\uparrow\tau$ and $t_n\uparrow t$, write (3.3) for $f_i^+(t_n)$ and let $n\to\infty$. □

**Remark 3.2.** With the same hypotheses as in Theorem 3.2, the following inequality also holds: if $u$ and $v$ are kinetic solutions of (1.1)-(1.3), then for $0\leq\tau\leq t\leq T$ and any non-negative test function $\phi\in C_c^\infty(\mathcal{O}^2)$, we have a.s. that

$$\int_{\mathcal{O}^2}\left(u^\pm(t,x)-v^\pm(t,y)\right)_+\phi(x,y)dxdy \tag{3.14}$$

$$
\begin{aligned}
&\le \int_{\mathcal{O}^2} \big(u^\pm(\tau,x)-v^\pm(\tau,y)\big)_+ \phi(x,y)\,dxdy \\
&\quad + \int_\tau^t \int_{\mathcal{O}^2} F_+\big(u(s,x),v(s,y)\big)\cdot \nabla_{x+y}\phi(x,y)\,dxdyds \\
&\quad + \int_\tau^t \int_{\mathcal{O}^2} \mathbb{B}_+\big(u(s,x),v(s,y)\big) : \big(D_x^2+D_y^2\big)\phi(x,y)\,dxdyds \\
&\quad + \sum_{k\ge 1}\int_\tau^t \int_{\mathcal{O}^2} \operatorname{sgn}\big(u(s,x)-v(s,x)\big)_+ \big(g_k(u(s,x))-g_k(v(s,y))\big)\phi(x,y)\,dxdyd\beta_k(s).
\end{aligned}
$$

The difference between (3.3) and (3.14) is that the latter does not contain the mixed terms $\nabla_x\cdot\mathbb{B}_+(u(s,x),v(s,y))\nabla_y\phi(x,y)$ and $\nabla_y\cdot\mathbb{B}_+(u(s,x),v(s,y))\nabla_x\phi(x,y)$ in the integral on the third line. Indeed, the mixed terms, which are needed for the proof of uniqueness, are added artificially in the proof of Theorem 3.2 above. More precisely, they arise from the addition and subtraction of the term $J$ in formula (3.8). Note that if we do not add and subtract this term, we still have that $I_7\le 0$ and $I_8\le 0$ and the rest of the proof can be carried out line by line to obtain (3.14).

**Corollary 3.1** (Doubling of variables 2)**.** *Let $u$ and $v$ be kinetic solutions of* (1.1)-(1.3) *with initial data $u_0$ and $v_0$, respectively. Denote $\nabla_{x+y}=\nabla_x+\nabla_y$. Then, for $0\le t\le T$ and nonnegative test functions $\theta\in C_c^\infty([0,T))$ and $\phi\in C_c^\infty(\mathcal{O}^2)$ with support at a distance less than $\delta_0$ from the diagonal of $\partial\mathcal{O}\times\partial\mathcal{O}$, we have a.s. that*

$$
\begin{aligned}
&-\mathbb{E}\int_0^T \int_{\mathcal{O}^2} \big(u^\pm(s,x)-v^\pm(s,y)\big)_+ \theta'(s)\phi(x,y)\,dxdydt \\
&\le \mathbb{E}\int_{\mathcal{O}^2} \big(u_0(x)-v_0(y)\big)_+ \theta(0)\phi(x,y)\,dxdy \\
&\quad +\mathbb{E}\int_0^T \int_{\mathcal{O}^2} F_+\big(u(s,x),v(s,y)\big)\cdot\theta(s)\nabla_{x+y}\phi(x,y)\,dxdyds \\
&\quad -\mathbb{E}\int_0^T \int_{\mathcal{O}^2} \nabla_{x+y}\cdot\mathbb{B}_+\big(u(s,x),v(s,y)\big)\theta(s)\nabla_{x+y}\phi(x,y)\,dxdyds.
\end{aligned}
\tag{3.15}
$$

*Proof.* The method to obtain (3.15) from (3.3) is somewhat standard and goes as follows. From (3.3) we have, for $0\le\tau\le t\le T$, that

$$
\mathbb{E}h(t)\le \mathbb{E}h(\tau)+\mathbb{E}\int_\tau^t k(t)dt, \tag{3.16}
$$

where

$$
h(t)=\int_{\mathcal{O}^2} \big(u^\pm(t,x)-v^\pm(t,y)\big)_+ \phi(x,y)\,dxdy,
$$

and

$$\begin{aligned}k(t)=&\int_{\mathcal{O}^2} F_+\big(u(t,x),v(t,y)\big)\cdot\nabla_{x+y}\phi(x,y)\,dxdy\\&+\int_{\mathcal{O}^2}\mathbb{B}_+\big(u(t,x),v(t,y)\big):\big(D_x^2+D_y^2\big)\phi(x,y)\,dxdyds.\end{aligned}$$

Let $\theta\in C_c^\infty([0,T))$ and take $0=t_0<t_1<\cdots<t_{n+1}=T$. Then, from (3.16), for each $i=1,\dots,n+1$ we have that

$$\mathbb{E}h(t_i)\theta(t_{i-1})\le\mathbb{E}h(t_{i-1})\theta(t_{i-1})+\mathbb{E}\int_{t_{i-1}}^{t_i}k(t)\theta(t_{i-1})dt.$$

Taking the sum over $i=1,\dots,n+1$ we have that

$$\begin{aligned}&\mathbb{E}\sum_{i=1}^n h(t_i)\big(-\theta(t_i)+\theta(t_{i-1})\big)+h(T)\theta(t_n)\\&\le\mathbb{E}h(0)\theta(0)+\mathbb{E}\int_0^T k(t)\sum_{i=1}^{n+1}\theta(t_{i-1})1_{[t_{i-1},t_i]}(t)dt.\end{aligned}\tag{3.17}$$

Noting that $-\theta(t_i)+\theta(t_{i-1})=-\theta'(t_i^*)(t_i-t_{i-1})$ for some $t_i^*\in[t_{i-1},t_i]$, we may take the limit as $n\to\infty$ in (3.17) to conclude that

$$-\mathbb{E}\int_0^T h(t)\theta'(t)dt\le\mathbb{E}h(0)\theta(0)+\mathbb{E}\int_0^T k(t)\theta(t)dt,$$

which proves the result. □

**Theorem 3.3** (Uniqueness). *Let $u$ and $v$ be kinetic solutions to* (1.1)-(1.3) *with initial data $u_0$ and $v_0$, respectively. Then for all $t\in[0,T]$ we have*

$$\mathbb{E}\int_{\mathcal{O}}\big|u^\pm(t,x)-v^\pm(t,x)\big|dx\le\mathbb{E}\int_{\mathcal{O}}|u_0(x)-v_0(x)|dx\tag{3.18}$$

*for some $C>0$ depending only on the data of the problem.*

*Proof.* Our argument is inspired by the proof of uniqueness of the entropy solution given in [19] which in turn follows the lines of the proof of the corresponding result in [36]. Both the latter and [19] deal with the deterministic case. The analysis, as usual, is split into two cases, corresponding to situations (i) and (ii) above. In case (i), we can use directly Theorem 3.1. So, it only remains to analyse case (ii). With Theorem 3.2 at hand, combined with the existence of the normal weak traces for $K_x(u(t,x),u_B(t,x))$ and $K_y(v(t,y),u_B(t,y))$, established in Section 2, as well as (1.26), the proof of Theorem 3.3 is performed by adapting the arguments of the corresponding one in [19], with the following important exception. Since we are bound to use as extension of the boundary data a solution of a stochastic equation this precludes us to take advantage of the trivial extension used in [19] and forces us to make a careful analysis of the behaviour of our specific extension near the boundary. Let us indicate how to proceed.

Let $u$ and $v$ be two kinetic solutions to (1.1)-(1.3). Then, from Corollary 3.1, for any non-negative test functions $\theta\in C_c^\infty((0,T))$ and $\phi\in C_c^\infty(\mathcal{O}^2)$, we have that

$$\begin{aligned}
&-\mathbb{E}\int_0^T\int_{\mathcal{O}^2}\big|u^\pm(t,x)-v^\pm(t,y)\big|\theta'(s)\phi(x,y)\,dx\,dy\,ds\\
\le&\,\mathbb{E}\int_0^T\int_{\mathcal{O}^2}F\big(u(s,x),v(s,y)\big)\cdot\theta(s)\nabla_{x+y}\phi(x,y)\,dx\,dy\,ds\\
&-\mathbb{E}\int_0^T\int_{\mathcal{O}^2}\nabla_{x+y}\cdot\mathbb{B}\big(u(s,x),v(s,y)\big)\theta(s)\nabla_{x+y}\phi(x,y)\,dx\,dy\,ds.
\end{aligned}\tag{3.19}$$

The strategy is to take a test function $\varphi$ which is a product of a function that converges to the characteristic function of $\mathcal{O}^2$ times a mollifier, which will force $y$ to be equal to $x$ in the limit. Evidently, the difficulty lies in the analysis of the convergence near the boundary. Now, using a partition of unity we see that we may assume that the test function is supported in a ball $B\in\mathcal{B}$ satisfying (1.17).

With this in mind, we fix some ball $B\in\mathcal{B}$ centered at some point of the boundary $\partial\mathcal{O}$ satisfying (1.17) and take smooth functions $\psi_1,\psi_2\in C_c^\infty(B)$ with $0\le\psi_i\le 1$, $i=1,2$, such that $\psi_2(x)=1$ for $x\in\operatorname{supp}\psi_1$. Then, we define $\psi(x,y)=\psi_1(x)\psi_2(y)$. Next, as in [19, 36] we consider our coordinates $x=(\bar{x},x_d)\in\mathbb{R}^{d-1}\times\mathbb{R}$ already relabelled so that $\partial\mathcal{O}\cap B=\{x_d=\lambda(\bar{x})\}$ and we take, by approximation, the test function

$$\varphi(x,y)=\zeta_\delta(x)\zeta_\eta(y)\rho(x-y)\psi(x,y),$$

where $\zeta_\delta$ and $\zeta_\eta$ are canonical boundary layer sequences, $\rho=\rho_{m,n}$ is given by $\rho_{m,n}=\rho_m(\bar{x}-\bar{y})\rho_n(x_d-y_d)$ and $\rho_m$ and $\rho_n$ are sequences of symmetric mollifiers in $\mathbb{R}^{d-1}$ and in $\mathbb{R}$, respectively. Accordingly, by adding and subtracting a few terms, from (3.19) we obtain

$$\begin{aligned}
&-\mathbb{E}\int_0^T\int_{\mathcal{O}^2}\big|u^\pm-v^\pm\big|\zeta_\delta\zeta_\eta\rho\theta'(s)\psi\,dx\,dy\,ds\\
\le&\,\mathbb{E}\int_0^T\int_{\mathcal{O}^2}F(u,v)\cdot\zeta_\delta\zeta_\eta\rho\theta\nabla_{x+y}\psi\,dx\,dy\,ds
\end{aligned}$$

$$
\begin{aligned}
&-\mathbb{E}\int_0^T\int_{\mathcal{O}^2}\nabla_x\cdot\mathbb{B}(u,v)\zeta_\delta\zeta_\eta\rho\theta\nabla_{x+y}\psi dxdyds\\
&-\mathbb{E}\int_0^T\int_{\mathcal{O}^2}\nabla_y\cdot\mathbb{B}(u,v)\zeta_\delta\zeta_\eta\rho\theta\nabla_{x+y}\psi dxdyds\\
&-\mathbb{E}\int_0^T\int_{\mathcal{O}^2}H_x\big(u,v,u_B(s,x)\big)\nabla_x\zeta_\delta\zeta_\eta\rho\theta\psi dxdyds\\
&-\mathbb{E}\int_0^T\int_{\mathcal{O}^2}H_y\big(v,u,u_B(s,y)\big)\zeta_\delta\nabla_y\zeta_\eta\rho\theta\psi dxdyds+\sum_{j=1}^5 I_j,
\end{aligned}
\tag{3.20}
$$

where $u_B$ is any extension of $u_b$ to $\mathcal{O}\cap B$ satisfying (1.19)-(1.22) and

$$
\begin{aligned}
\sum_{j=1}^5 I_j=&\mathbb{E}\int_0^T\int_{\mathcal{O}^2}K_x\big(u,u_B(s,x)\big)\cdot\nabla_x\zeta_\delta\zeta_\eta\rho\theta\psi dxdyds\\
&+\mathbb{E}\int_0^T\int_{\mathcal{O}^2}K_y\big(v,u_B(s,y)\big)\cdot\zeta_\delta\nabla_y\zeta_\eta\rho\theta\psi dxdyds\\
&+\mathbb{E}\int_0^T\int_{\mathcal{O}^2}\Big\{F\big(u_B(s,x),v\big)\cdot\nabla_x\zeta_\delta\zeta_\eta+F\big(u_B(s,y),u\big)\cdot\zeta_\delta\nabla_y\zeta_\eta\Big\}\rho\theta\psi dxdyds\\
&-\mathbb{E}\int_0^T\int_{\mathcal{O}^2}\Big\{\nabla_x\cdot\mathbb{B}(u,v)\zeta_\delta\nabla_y\zeta_\eta+\nabla_y\cdot\mathbb{B}\big(u_B(s,y),u\big)\zeta_\delta\nabla_y\zeta_\eta\Big\}\rho\theta\psi dxdyds\\
&-\mathbb{E}\int_0^T\int_{\mathcal{O}^2}\Big\{\nabla_y\cdot\mathbb{B}(u,v)\nabla_x\zeta_\delta\zeta_\eta+\nabla_x\cdot\mathbb{B}\big(u_B(s,x),v\big)\nabla_x\zeta_\delta\zeta_\eta\Big\}\rho\theta\psi dxdyds.
\end{aligned}
$$

Note that, by virtue of (1.26) the $\liminf$ as $\delta\to 0$ of the fifth integral on the right-hand-side of (3.20) is non-positive. Similarly, the $\liminf$ as $\eta\to 0$ of the sixth integral on the right hand side is non-positive. Also, using Proposition 2.1 we see that

$$
\lim_{\eta\to 0}\lim_{\delta\to 0}I_1=-\mathbb{E}\left\langle\mathcal{K}(u,u_B)\cdot\nu,\int_0^T\int_{\mathcal{O}}\rho(x-y)\theta(s)\psi_2(y)dyds\right\rangle,
\tag{3.21}
$$

where $\mathcal{K}_1(u,u_B)\cdot\nu$ denotes the normal trace of $K_x(u,u_b)\psi_1$. Likewise,

$$
\lim_{\eta\to 0}\lim_{\delta\to 0}I_2=-\mathbb{E}\left\langle\mathcal{K}_2(v,u_B)\cdot\nu,\int_0^T\int_{\mathcal{O}}\rho(x-y)\theta(s)\psi_1(x)dxds\right\rangle,
\tag{3.22}
$$

where $\mathcal{K}_2(v,u_B)\cdot\nu$ denotes the normal trace of $K_y(v,u_B)\psi_2$.

Now, for $I_3$, since in each term the boundary layer sequence is in the integral of a smooth function, we immediately get

$$
\lim_{\eta\to 0}\lim_{\delta\to 0}I_3=-\mathbb{E}\int_0^T\iint\limits_{(\partial\mathcal{O})_x\times\mathcal{O}_y}F\big(u_b(s,x),v(s,y)\big)\cdot\nu(x)\rho\theta\psi\, ds
$$

$$-\mathbb{E}\int_0^T \iint\limits_{\mathcal{O}_x\times(\partial\mathcal{O})_y} F\big(u_b(s,y),u(s,x)\big)\cdot\nu(y)\rho\theta\psi\,ds. \tag{3.23}$$

Regarding $I_4$, following the reasoning in [19,36], we have

$$\begin{aligned}I_4=&\mathbb{E}\int_0^T\iint\limits_{\mathcal{O}\times\mathcal{O}}\mathbb{B}(u,v)\nabla_x\zeta_\delta\nabla_y\zeta_\eta\rho\theta\psi\,ds\\&+\mathbb{E}\int_0^T\iint\limits_{\mathcal{O}\times\mathcal{O}}\mathbb{B}(u,v)\zeta_\delta\nabla_y\zeta_\eta\rho\theta\nabla_x\psi\,ds\\&+\mathbb{E}\int_0^T\iint\limits_{\mathcal{O}\times\mathcal{O}}\mathbb{B}(u,v)\zeta_\delta\nabla_y\zeta_\eta\nabla_x\rho\theta\psi\,ds\\&-\mathbb{E}\int_0^T\iint\limits_{\mathcal{O}\times\mathcal{O}}\nabla_y\cdot\mathbb{B}\big(u_B(s,y),u\big)\zeta_\delta\nabla_y\zeta_\eta\rho\theta\psi\,ds.\end{aligned}\tag{3.24}$$

We observe that, from (1.27), as $\delta,\eta\to 0$ the first two integrals in the right-hand side of (3.24) converge to

$$\begin{aligned}&\mathbb{E}\int_0^T\iint\limits_{(\partial\mathcal{O})_x\times(\partial\mathcal{O})_y}\mathbb{B}\big(u_b(s,x),u_b(s,y)\big)\rho\theta\psi\nu_x\nu_y\,ds\\&\quad-\mathbb{E}\int_0^T\iint\limits_{\mathcal{O}_x\times(\partial\mathcal{O})_y}\mathbb{B}\big(u,u_b(s,y)\big)\rho\theta\nabla_x\psi\nu_y\,ds,\end{aligned}$$

both of which clearly converge to 0 when we make $m\to\infty$ and then $n\to\infty$.

Then, we only have to worry about the remaining two terms. Let us denote them by

$$\begin{aligned}\tilde{I}_4=\tilde{I}_{4,1}+\tilde{I}_{4,2}:=&\mathbb{E}\int_0^T\iint\limits_{\mathcal{O}\times\mathcal{O}}\mathbb{B}(u,v)\zeta_\delta\nabla_y\zeta_\eta\nabla_x\rho\theta\psi\,ds\\&-\mathbb{E}\int_0^T\iint\limits_{\mathcal{O}\times\mathcal{O}}\nabla_y\cdot\mathbb{B}(u_B(s,y),u)\zeta_\delta\nabla_y\zeta_\eta\rho\theta\psi\,ds.\end{aligned}$$

Since $\frac{\partial u_B}{\partial y_d}=0$ on $\partial\mathcal{O}\cap B$ and since $\operatorname{supp}\psi\subset B$ we see that

$$\lim_{\eta\to 0}\lim_{\delta\to 0}\tilde{I}_{4,2}=\mathbb{E}\int_0^T\iint\limits_{\mathcal{O}_x\times(\partial\mathcal{O})_y}\tilde{\nabla}_{\tilde{y}}\cdot\mathbb{B}\big(u_b(s,y),u\big)\nu\rho\theta\psi\,ds,$$

where we employ the notation $y=(\bar{y},y_d)$ and $\tilde{\nabla}_{\bar{y}}=\binom{\nabla_{\bar{y}}}{0}$. Let $U_y\subset\mathbb{R}^{d-1}$ such that $\partial\mathcal{O}\cap B$ is the graph of $\gamma(\bar{y})$ over $U_y$. Then, we may rewrite the last integral as

$$\lim_{\eta\to 0}\lim_{\delta\to 0}\tilde{I}_{4,2}=-\mathbb{E}\int_0^T\iint_{\mathcal{O}_x\times U_y}\tilde{\nabla}_{\bar{y}}\cdot\mathbb{B}\big(u_b(s,\bar{y},\gamma(\bar{y}),u)\big)N\rho\big(\bar{x}-\bar{y},x_d-\gamma(\bar{y})\big)\theta\psi\,ds$$

with $N=(-\nabla_{\bar{y}}\gamma,1)$ and we used the fact that the unit normal vector to $\partial\mathcal{O}$ is $\nu=\frac{1}{\sqrt{1+|\nabla_{\bar{y}}\gamma|^2}}N$ and the Jacobian is $\sqrt{1+|\nabla_{\bar{y}}\gamma|^2}$.

Now, taking the limit as $\delta\to 0$ and as $\eta\to 0$ in the term $\tilde{I}_{4,1}$, changing coordinates and noting that $\tilde{\nabla}_{\bar{y}}\rho=-\tilde{\nabla}_{\bar{x}}\rho$ we have that

$$\begin{aligned}\lim_{\eta\to 0}\lim_{\delta\to 0}\tilde{I}_{4,1}=&-\mathbb{E}\int_0^T\iint_{\mathcal{O}_x\times U_y}\mathbb{B}\big(u,u_b(s,\bar{y},\gamma(\bar{y}))\big)N\tilde{\nabla}_{\bar{y}}\rho\big(\bar{x}-\bar{y},x_d-\gamma(\bar{y})\big)\theta\psi\,ds\\&-\mathbb{E}\int_0^T\iint_{\mathcal{O}_x\times U_y}\mathbb{B}\big(u,u_b(s,\bar{y},\gamma(\bar{y}))\big)N\otimes N\frac{\partial\rho}{\partial x_d}\big(\bar{x}-\bar{y},x_d-\gamma(\bar{y})\big)\theta\psi\,ds.\end{aligned}$$

Now, integrating by parts in the first term we get

$$\begin{aligned}\lim_{\eta\to 0}\lim_{\delta\to 0}\tilde{I}_{4,1}=&-\lim_{\eta\to 0}\lim_{\delta\to 0}\tilde{I}_{4,2}\\&-\mathbb{E}\int_0^T\iint_{\mathcal{O}_x\times U_y}\mathbb{B}\big(u,u_b(s,\bar{y},\gamma(\bar{y}))\big)N\otimes N\frac{\partial\rho}{\partial x_d}\big(\bar{x}-\bar{y},x_d-\gamma(\bar{y})\big)\theta\psi\,ds.\end{aligned}$$

Thus, we have that

$$\begin{aligned}&\lim_{\eta\to 0}\lim_{\delta\to 0}I_4\\&=-\mathbb{E}\int_0^T\iint_{\mathcal{O}_x\times U_y}\mathbb{B}\big(u,u_b(s,\bar{y},\gamma(\bar{y}))\big)N\otimes N\frac{\partial\rho}{\partial x_d}\big(\bar{x}-\bar{y},x_d-\gamma(\bar{y})\big)\theta\psi\,ds+M_{4,m,n},\end{aligned}\tag{3.25}$$

where $\lim_{n\to\infty}\lim_{m\to\infty}M_{4,m,n}=0$.

By a similar reasoning, we also have

$$\begin{aligned}&\lim_{\eta\to 0}\lim_{\delta\to 0}I_5\\&=-\mathbb{E}\int_0^T\iint_{U_x\times\mathcal{O}_y}\mathbb{B}\big(u_b(s,\bar{x},\gamma(\bar{x})),v\big)N\otimes N\frac{\partial\rho}{\partial y_d}\big(\bar{x}-\bar{y},\gamma(\bar{x})-y_d\big)\theta\psi\,ds+M_{5,m,n},\end{aligned}\tag{3.26}$$

where $\lim_{n\to\infty}\lim_{m\to\infty}M_{5,m,n}=0$.

Taking (3.21)-(3.23), (3.25) and (3.26) into account, let us denote

$$\begin{aligned}
&L_1 := \lim_{\eta\to 0}\lim_{\delta\to 0} I_1 + \lim_{\eta\to 0}\lim_{\delta\to 0} I_2,\\
&L_2 := \lim_{\eta\to 0}\lim_{\delta\to 0} I_3,\\
&L_3 := \lim_{\eta\to 0}\lim_{\delta\to 0} I_4 - M_{4,m,n} + \lim_{\eta\to 0}\lim_{\delta\to 0} I_5 - M_{5,m,n}.
\end{aligned}$$

Our goal now is to show that

$$\lim_{n\to\infty}\lim_{m\to\infty}(L_1+L_2+L_3)=0.$$

*Term $L_1$:* Following [36] it is easy to prove that

$$\lim_{n\to\infty}\lim_{m\to\infty} L_1 = -\frac{1}{2}\mathbb{E}\big\langle \mathcal{K}(u,u_B)\cdot\nu,\theta\psi_2\big\rangle - \frac{1}{2}\mathbb{E}\big\langle \mathcal{K}(v,u_B)\cdot\nu,\theta\psi_1\big\rangle. \tag{3.27}$$

*Term $L_2$:* Note that $L_2$ is the sum of two terms, the first of which is

$$L_{2,1} = -\mathbb{E}\int_0^T \iint\limits_{(\partial\mathcal{O})_x\times\mathcal{O}_y} F\big(u_b(s,x),v(s,y)\big)\cdot\nu(x)\rho\theta\psi\, ds.$$

Changing variables and taking the limit as $m\to\infty$, we see that

$$\begin{aligned}
&\lim_{m\to\infty} L_{2,1}\\
&= -\mathbb{E}\int_0^T \int\limits_{U_y}\int_{\gamma(\bar y)-\frac{1}{n}}^{\gamma(\bar y)} F\big(u_b(s,\bar y,\gamma(\bar y)),v(s,\bar y,y_d)\big)\cdot N(\bar y)\rho_n\big(\gamma(\bar y)-y_d\big)\theta\psi\, dy_d\, d\bar y\, ds\\
&= -\mathbb{E}\int_0^T \int\limits_{U_y}\int_{\gamma(\bar y)-\frac{1}{n}}^{\gamma(\bar y)} F\big(u_B(s,\bar y,y_d),v(s,\bar y,y_d)\big)\cdot N(\bar y)\rho_n\big(\gamma(\bar y)-y_d\big)\theta\psi\, dy_d\, d\bar y\, ds + \tilde L_{2,1}^{(n)},
\end{aligned}$$

where

$$\begin{aligned}
\tilde L_{2,1}^{(n)} := \mathbb{E}\int_0^T \int\limits_{U_y}\int_{\gamma(\bar y)-1/n}^{\gamma(\bar y)} &\Big(F\big(u_B(s,y),v(s,y)\big) - F\big(u_b(s,\bar y,\gamma(\bar y)),v(s,y)\big)\Big)\\
&\times N(\bar y)\rho_n\big(\gamma(\bar y)-y_d\big)\theta\psi\, dy_d\, d\bar y\, ds.
\end{aligned}$$

Now,

$$|\tilde{L}_{2,1}^{(n)}| \le C\mathbb{E}\int_0^T \int_{U_y} \int_{\gamma(\bar{y})-1/n}^{\gamma(\bar{y})} |u_B(s,y)-u_b(s,\bar{y},\gamma(\bar{y}))| \rho_n(\gamma(\bar{y})-y_d)\theta\psi dy_d d\bar{y} ds,$$

and writing

$$u_B(s,y)-u_b(s,\bar{y},\gamma(\bar{y})) = \int_{y_d}^{\gamma(\bar{y})} \frac{\partial u_B}{\partial y_d}(s,\bar{y},\tau)d\tau,$$

we deduce that

$$|\tilde{L}_{2,1}^{(n)}| \le \frac{C}{n}\mathbb{E}\int_0^T \int_{U_y} \int_{\gamma(\bar{y})-1/n}^{\gamma(\bar{y})} \left|\frac{\partial u_B}{\partial y_d}(s,\bar{y},y_d)\right| \rho_n(\gamma(\bar{y})-y_d)\theta\psi dy_d d\bar{y} ds,$$

which tends to 0 as $n\to\infty$. A similar reasoning may be applied to the second term integral of (3.23) and we conclude that

$$\begin{aligned}
&\lim_{m\to\infty} L_2 \qquad (3.28)\\
&= -\mathbb{E}\int_0^T \int_{U_y} \int_{\gamma(\bar{y})-\frac{1}{n}}^{\gamma(\bar{y})} F(u_B(s,y),v(s,y))\cdot N(\bar{y})\rho_n(\gamma(\bar{y})-y_d)\theta\psi dy_d d\bar{y} ds\\
&\quad -\mathbb{E}\int_0^T \int_{U_x} \int_{\gamma(\bar{x})-\frac{1}{n}}^{\gamma(\bar{x})} (u_B(s,x),u(s,x))\cdot N(\bar{x})\rho_n(x_d-\gamma(\bar{x}))\theta\psi dx_d d\bar{x} ds + L_{2,n},
\end{aligned}$$

where $\lim_{n\to\infty} L_{2,n} = 0$.

*Term L3:* Let us denote

$$\begin{aligned}
L_4 &:= \lim_{\eta\to 0}\lim_{\delta\to 0} I_4 - M_{4,m,n},\\
L_5 &:= \lim_{\eta\to 0}\lim_{\delta\to 0} I_5 - M_{5,m,n},
\end{aligned}$$

so that $L_3 = L_4 + L_5$. Note that

$$\begin{aligned}
&\lim_{m\to\infty} L_4\\
&= -\mathbb{E}\int_0^T \int_{U_x} \int_{\gamma(\bar{x})-\frac{1}{n}}^{\gamma(\bar{x})} \mathbb{B}(u(s,x),u_b(s,\bar{x},\gamma(\bar{x})))N\otimes N\rho_n'(x_d-\gamma(\bar{x}))\theta\psi dx_d d\bar{x} ds\\
&= -\mathbb{E}\int_0^T \int_{U_x} \int_{\gamma(\bar{x})-\frac{1}{n}}^{\gamma(\bar{x})} \mathbb{B}(u(s,x),u_B(s,x))N\otimes N\rho_n'(x_d-\gamma(\bar{x}))\theta\psi dx_d d\bar{x} ds + \tilde{L}_4^{(n)},
\end{aligned}$$

where

$$\tilde{L}_4^{(n)} := \mathbb{E}\int_0^T \int_{U_x} \int_{\gamma(\bar{x})-\frac{1}{n}}^{\gamma(\bar{x})} \left[\mathbb{B}(u(s,x),u_B(s,x)) - \mathbb{B}(u(s,x),u_b(s,\bar{x},\gamma(\bar{x})))\right] \\ \times N\otimes N\rho_n'(x_d-\gamma(\bar{x}))\theta\psi dx_d d\bar{x} ds.$$

Reasoning as for $\tilde{L}_{2,1}^{(n)}$, we note that

$$\begin{aligned} \left|\tilde{L}_4^{(n)}\right| &\le C\mathbb{E}\int_0^T \int_{U_x} \int_{\gamma(\bar{x})-\frac{1}{n}}^{\gamma(\bar{x})} \left|u_B(s,x)-u_b(s,\bar{x},\gamma(\bar{x}))\right| \rho_n'(x_d-\gamma(\bar{x}))\theta\psi dx_d d\bar{x} ds \\ &\le \frac{C}{n}\mathbb{E}\int_0^T \int_{U_x} \int_{\gamma(\bar{x})-\frac{1}{n}}^{\gamma(\bar{x})} \left|\frac{\partial u_B}{\partial x_d}(s,x)\right| \rho_n'(x_d-\gamma(\bar{x}))\theta\psi dx_d d\bar{x} ds, \end{aligned}$$

where the integral on the last line converges to 0 as $n\to\infty$ since $\frac{\partial u_B}{\partial x_d}=0$ on $\partial\cap B$, so that $\lim_{n\to\infty}\tilde{L}_{4,n}=0$.

With a similar reasoning for the term $L_5$ we conclude that

$$\begin{aligned} &\lim_{m\to\infty} L_3 \qquad (3.29) \\ =&-\mathbb{E}\int_0^T \int_{U_x} \int_{\gamma(\bar{x})-\frac{1}{n}}^{\gamma(\bar{x})} \mathbb{B}(u(s,x),u_B(s,x))N\otimes N\rho_n'(x_d-\gamma(\bar{x}))\theta\psi dx_d d\bar{x} ds \\ &-\mathbb{E}\int_0^T \int_{U_y} \int_{\gamma(\bar{y})-\frac{1}{n}}^{\gamma(\bar{y})} \mathbb{B}(u_B(s,y),v(s,y))N\otimes N\rho_n'(\gamma(\bar{y})-y_d)\theta\psi dy_d d\bar{y} ds + L_{3,n}, \end{aligned}$$

where $\lim_{n\to\infty}L_{3,n}=0$.

At this point, with (3.27)-(3.29) at hand, the arguments from the proof of uniqueness of solutions for the deterministic case considered in [19] (see also [36]) may be carried out line by line to show that

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}L_2+\lim_{m\to\infty}L_3\right) = -\lim_{m\to\infty}\lim_{m\to\infty}L_1. \tag{3.30}$$

More precisely, it can be shown that the function

$$w_n := 2\int_{x_d-\gamma(\bar{x})}^0 \rho_n(\tau)d\tau$$

is a boundary layer sequence and that we can write

$$\begin{aligned} &\lim_{m\to\infty}L_2+\lim_{m\to\infty}L_3 \\ =&\frac{1}{2}\int_0^T\int_{\mathcal{O}_x}\left\{\mathbb{B}(u,u_B)\cdot\nabla^2 w_n + F(u_B,u)\cdot\nabla w_n\right\}\theta\psi dx ds \\ &+\frac{1}{2}\int_0^T\int_{\mathcal{O}_y}\left\{\mathbb{B}(u_B,v)\cdot\nabla^2 w_n + F(u_B,v)\cdot\nabla w_n\right\}\theta\psi dy ds + N_n, \end{aligned}$$

where $\lim_{n\to\infty} N_n = 0$, which clearly implies (3.30). We omit the details.

Thus, taking the limit as $\delta,\eta \to 0$ first and then as $m,n\to\infty$ in (3.20) we obtain that

$$\begin{aligned}
&-\mathbb{E}\int_0^T\int_{\mathcal{O}} \big|u^{\pm}(s,x)-v^{\pm}(s,x)\big|\theta'(s)\psi_1(x)\,dx\,dy\\
\le&\,\mathbb{E}\int_0^T\int_{\mathcal{O}} F\big(u(s,x),v(s,x)\big)\cdot\nabla_x\psi_1(x)\,dx\,dy\,ds\\
&-\mathbb{E}\int_0^T\int_{\mathcal{O}} \nabla_x\cdot\mathbb{B}\big(u(s,x),v(s,x)\big)\nabla_x\psi_1(x)\,dx\,dy\,ds.
\end{aligned}\tag{3.31}$$

To conclude, we see that we may take a covering $\{B_j\}_{j=0}^N$ of $\overline{\mathcal{O}}$, where $B_j\in\mathcal{B}$ for $1\le j\le N$ and $B_0 \subset\subset \mathcal{O}$ and a partition of unity $\{\tilde\psi_j\}_{j=0}^N$ subordinated, so that we have the inequality (3.31) with $\psi_1=\tilde\psi_j$, for each $j=1,\dots,N$. Regarding $\tilde\psi_0$, we see that (3.31) may also be deduced to hold with $\psi_1=\tilde\psi_0$ much more easily as there is no boundary analysis in this case. Thus, adding the inequalities (3.31) corresponding to each $\tilde\psi_j$ we obtain

$$-\mathbb{E}\int_0^T\int_{\mathcal{O}} \big|u^{\pm}(s,x)-v^{\pm}(s,x)\big|\theta'(s)\,dx\,dy\le 0,$$

and choosing $\theta(s)=\theta_l(s)$ conveniently and letting $l\to\infty$ we finally deduce (3.18), which concludes the proof. □

As a consequence of Theorem 3.3 we have the following result whose proof may be found in [25, Corollary 3.4].

**Corollary 3.2** (Continuity in time)**.** *Let $u$ be a kinetic solution to* (1.1)-(1.3)*. Then there exists a representative of $u$ which has almost surely continuous trajectories in $L^p(\Omega)$ for all $p\in[1,\infty)$.*

# 4 Consistency

As aforementioned, the existence of kinetic solutions of (1.1)-(1.3) will be obtained as a limit of solutions of nondegenerate approximations, namely, as the limit of the solutions $u^\varepsilon$ of the regularized problem

$$du^\varepsilon+\nabla\cdot\mathbf{A}(u^\varepsilon)dt-D^2:\mathbf{B}(u^\varepsilon)dt-\varepsilon\Delta u^\varepsilon dt=\Phi(u^\varepsilon)dW(t),\tag{4.1}$$
$$u^\varepsilon(0,x)=u_0(x),\quad x\in\mathcal{O},\tag{4.2}$$
$$u^\varepsilon=u_b(t,x),\qquad x\in\partial\mathcal{O},\quad t>0\tag{4.3}$$

as $\varepsilon\to 0$. In this section we prove that if a predictable function $u\in L^\infty(\Omega\times[0,T]\times\mathcal{O})$ is such that

$$u^\varepsilon\to u \quad \text{a.s. and a.e. in } [0,T]\times\mathcal{O}, \tag{4.4}$$

then $u$ satisfies the Dirichlet boundary condition (1.3) in the sense of Definition 1.2. Since our only concern in this section is the verification of the Dirichlet boundary conditions, in view of (1.12), we omit the dependence of the $g_k$ with respect to $x$.

In Section 5, we prove the existence and uniqueness of solutions of (4.1)-(4.3) in the space $L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H^1(\mathcal{O}))$, satisfying the boundary condition (4.2) in the sense of traces. In the process, we show that if $u_{\min}\le u_0,u_b\le u_{\max}$, then the unique weak solution $u^\varepsilon$ satisfies the estimates $u_{\min}\le u^\varepsilon(t,x)\le u_{\max}$ a.s. and a.e. in $[0,T]\times\mathcal{O}$ and

$$\mathbb{E}\int_0^T\int_{\mathcal{O}}\Big(|\nabla b(u^\varepsilon(t,x))|^2+\varepsilon|\nabla u(t,x)|^2\Big)\,dx\,dt\le C,$$

where $C>0$ is independent of $\varepsilon$. In particular, there is a subsequence $\varepsilon_n\to 0$ such that

$$\varepsilon_n\nabla u^{\varepsilon_n}\to 0, \quad \text{a.s. in } L^2([0,T]\times\mathcal{O}).$$

Note, also, that $\varepsilon_n$ may be taken so that $b(u^{\varepsilon_n})-b(\tilde{u}_b)$ converges to $b(u)-b(\tilde{u}_b)$ weakly in $L^2(\Omega\times(0,T);H_0^1(\mathcal{O}))$, where $\tilde{u}_b$ is any $H^1$ extension of $u_b$ to $\mathcal{O}$ ($\tilde{u}_b$ may be constructed using partition of unity and flattening out the boundary, for example). In particular, $u$ satisfies (1.27).

**Proposition 4.1.** *Let $u^\varepsilon$ be the solution of* (4.1)-(4.3) *and assume that* (4.4) *holds. Then $u$ satisfies* (1.25).

*Proof.* Let $B\in\mathcal{B}$ satisfy (1.17) and let $u_B$ be an extension of $u_b$ to $[0,T]\times\mathcal{O}$ satisfying (1.19)-(1.22). Note that $u^\varepsilon-u_B$ satisfies a.s. the equation

$$\begin{aligned}&d(u^\varepsilon-u_B)+\operatorname{div}\big(\mathbf{A}(u^\varepsilon)-\mathbf{A}(u_B)\big)\,dt\\ &=D^2:\big(\mathbf{B}^\varepsilon(u^\varepsilon)-\mathbf{B}^\varepsilon(u_B)\big)\,dt+\big(\Phi(u^\varepsilon)-\Phi(u_B)\big)\,dW(t)+G_\varepsilon dt,\end{aligned}$$

where

$$\mathbf{B}^\varepsilon(\xi)=\mathbf{B}(\xi)+\varepsilon\xi\mathrm{Id}, \quad G_\varepsilon=D^2:\mathbf{B}(u_B)+\varepsilon\Delta u_B-\operatorname{div}\mathbf{A}(u_B)+\Delta^2 u_B.$$

Let $S_\theta(\xi)$ be a $C^2$ convex approximation of $|\xi|$, such that $S_\theta'(\xi)$ is monotone nondecreasing, $S_\theta'(\xi)=1$ for $\xi>\theta$, and $S_\theta'(\xi)=-1$ for $\xi\le-\theta$, so that $S_\theta''(\xi)\xi$ converges to 0 pointwise as $\theta\to 0$. Given any nonnegative test function $\varphi(t,x)\in$

$C_c^\infty((0,T)\times(\mathcal{O}\cap B))$, by the Itô formula (see [14]) we have a.s. that

$$\begin{aligned}
&\int_0^T\int_{\mathcal{O}} S_\theta(u^\varepsilon-u_B)\partial_t\varphi+S_\theta'(u^\varepsilon-u_B)\big(\mathbf{A}(u^\varepsilon)-\mathbf{A}(u_B)\big)\cdot\nabla\varphi dxdt\\
&\quad-\int_0^T\int_{\mathcal{O}} S_\theta''(u^\varepsilon-u_B)(\nabla u^\varepsilon-\nabla u_B)\cdot\nabla\big(\mathbf{B}^\varepsilon(u^\varepsilon)-\mathbf{B}^\varepsilon(u_B)\big)\varphi dxdt\\
&\quad-\int_0^T\int_{\mathcal{O}} S_\theta'(u^\varepsilon-u_B)\nabla\big(\mathbf{B}^\varepsilon(u^\varepsilon)-\mathbf{B}^\varepsilon(u_B)\big)\cdot\nabla\varphi dxdt\\
&\quad+\sum_{k\ge1}\int_0^T\int_{\mathcal{O}} S_\theta'(u^\varepsilon-u_B)\big(g_k(u^\varepsilon(t))-g_k(u_B(t))\big)\varphi dxd\beta_k(t)\\
&\quad+\frac{1}{2}\int_0^T\int_{\mathcal{O}} S_\theta''(u^\varepsilon-u_B)\sum_{k\ge1}|g_k(u^\varepsilon)-g_k(u_B)|^2\varphi dxdt\\
=&-\int_0^T\int_{\mathcal{O}} S_\theta'(u^\varepsilon-u_B)G_\varepsilon\varphi dxdt. 
\end{aligned}\tag{4.5}$$

Concerning the second integral on the left hand side of (4.5), we have

$$\begin{aligned}
&\int_0^T\int_{\mathcal{O}} S_\theta''(u^\varepsilon-u_B)(\nabla u^\varepsilon-\nabla u_B)\cdot\nabla\big(\mathbf{B}^\varepsilon(u^\varepsilon)-\mathbf{B}^\varepsilon(u_B)\big)\varphi dxdt\\
=&\int_0^T\int_{\mathcal{O}} S_\theta''(u^\varepsilon-u_B)\big((\nabla u^\varepsilon-\nabla u_B)\cdot(\mathbf{B}^\varepsilon)'(u^\varepsilon)\big)\cdot(\nabla u^\varepsilon-\nabla u_B)\varphi dxdt\\
&+\int_0^T\int_{\mathcal{O}} S_\theta''(u^\varepsilon-u_B)\big((\nabla u^\varepsilon-\nabla u_B)\cdot((\mathbf{B}^\varepsilon)'(u^\varepsilon)-(\mathbf{B}^\varepsilon)'(u_B))\big)\cdot\nabla u_B\varphi dxdt\\
\ge&\int_0^T\int_{\mathcal{O}} S_\theta''(u^\varepsilon-u_B)\big((\nabla u^\varepsilon-\nabla u_B)\cdot((\mathbf{B}^\varepsilon)'(u^\varepsilon)-(\mathbf{B}^\varepsilon)'(u_B))\big)\cdot\nabla u_B\varphi dxdt,
\end{aligned}$$

where the integral in the last line tends to 0 a.s. as $\theta\to0$, due to (1.4) and (1.14).

Similarly, in regards to the third integral on the left hand side of (4.5), we have

$$\begin{aligned}
&\int_0^T\int_{\mathcal{O}} S_\theta'(u^\varepsilon-u_B)\nabla\big(\mathbf{B}^\varepsilon(u^\varepsilon)-\mathbf{B}^\varepsilon(u_B)\big)\cdot\nabla\varphi dxdt\\
=&-\int_0^T\int_{\mathcal{O}} S_\theta''(u^\varepsilon-u_B)\big((\nabla u^\varepsilon-\nabla u_B)\cdot(\mathbf{B}^\varepsilon(u^\varepsilon)-\mathbf{B}^\varepsilon(u_B))\big)\cdot\nabla\varphi dxdt\\
&-\int_0^T\int_{\mathcal{O}} S_\theta'(u^\varepsilon-u_B)\big(\mathbf{B}^\varepsilon(u^\varepsilon)-\mathbf{B}^\varepsilon(u_B)\big):D^2\varphi dxdt,
\end{aligned}$$

where

$$\lim_{\theta\to0}\int_0^T\int_{\mathcal{O}} S_\theta''(u^\varepsilon-u_B)\big((\nabla u^\varepsilon-\nabla u_B)\cdot(\mathbf{B}^\varepsilon(u^\varepsilon)-\mathbf{B}^\varepsilon(u_B))\big)\cdot\nabla\varphi dxdt=0,$$

a.s. Finally, we also observe that, due to (1.11), a.s. we have

$$\lim_{\theta\to 0}\int_0^T\int_{\mathcal{O}} S_\theta''(u^\varepsilon-u_B)\sum_{k\geq 1}|g_k(u^\varepsilon)-g_k(u_B)|^2\varphi\,dx\,dt=0.$$

Thus, taking the limit as $\theta\to 0$ in (4.5) we obtain the following inequality which is satisfied a.s.

$$\begin{aligned}
&\int_0^T\int_{\mathcal{O}}|u^\varepsilon-u_B|\partial_t\varphi\,dx\,dt+F\big(u^\varepsilon(t,x),u_B(t,x)\big)\cdot\nabla\varphi\,dx\,dt\\
&\quad+\int_0^T\int_{\mathcal{O}}\Big(\mathbb{B}(u^\varepsilon(t,x),u_B(t,x)):D^2\varphi+\varepsilon|u^\varepsilon-u_B|\Delta\varphi\Big)\,dx\,dt\\
&\quad+\sum_{k\geq 1}\int_0^T\int_{\mathcal{O}}\mathrm{sgn}(u^\varepsilon-u_B)\big(g_k(u^\varepsilon(t))-g_k(u_B(t))\big)\varphi\,dx\,d\beta_k(t)\\
&\geq -\|G_\varepsilon\|_{L^2([0,T]\times\mathcal{O})}\|\varphi\|_{L^2([0,T]\times\mathcal{O})}.
\end{aligned}\tag{4.6}$$

Now, by approximation we take $\varphi=\zeta_\delta(x)\tilde{\varphi}$, where $\zeta_\delta$ is a canonical local boundary layer sequence and $0\leq\tilde{\varphi}\in C_c^\infty((0,T)\times B)$ to obtain

$$\begin{aligned}
&\int_0^T\int_{\mathcal{O}}|u^\varepsilon-u_B|\zeta_\delta\partial_t\tilde{\varphi}\,dx\,dt+F\big(u^\varepsilon(t,x),u_B(t,x)\big)\cdot\zeta_\delta\nabla\tilde{\varphi}\,dx\,dt\\
&\quad+\int_0^T\int_{\mathcal{O}}\Big(\mathbb{B}(u^\varepsilon(t,x),u_B(t,x)):\zeta_\delta D^2\tilde{\varphi}+\varepsilon|u^\varepsilon-u_B|\zeta_\delta\Delta\tilde{\varphi}\Big)\,dx\,dt\\
&\quad+\sum_{k\geq 1}\int_0^T\int_{\mathcal{O}}\mathrm{sgn}(u^\varepsilon-u_B)\big(g_k(u^\varepsilon(t))-g_k(u_B(t))\big)\zeta_\delta\tilde{\varphi}\,dx\,d\beta_k(t)+\sum_{j=1}^3 I_j\\
&\geq -\|G_\varepsilon\|_{L^2([0,T]\times\mathcal{O})}\|\zeta_\delta\tilde{\varphi}\|_{L^2([0,T]\times\mathcal{O})},
\end{aligned}\tag{4.7}$$

where

$$\begin{aligned}
\sum_{j=1}^3 I_j=&\int_0^T\int_{\mathcal{O}}F\big(u^\varepsilon(t,x),u_B(t,x)\big)\cdot\nabla\zeta_\delta\tilde{\varphi}\,dx\,dt\\
&+2\int_0^T\int_{\mathcal{O}}\big(\mathbb{B}(u^\varepsilon(t,x),u_B(t,x))\nabla\zeta_\delta\nabla\tilde{\varphi}+\varepsilon|u^\varepsilon-u_B|\nabla\zeta_\delta\cdot\nabla\tilde{\varphi}\big)\,dx\,dt\\
&+\int_0^T\int_{\mathcal{O}}\Big(\tilde{\varphi}\mathbb{B}\big(u^\varepsilon(t,x),u_B(t,x)\big):D^2\zeta_\delta+\varepsilon|u^\varepsilon-u_B|\tilde{\varphi}\Delta\zeta_\delta\Big)\,dx\,dt.
\end{aligned}$$

In view of (2.7), we note that $I_1$ and $I_2$ tend to zero as $\delta\to 0$, since $F(u^\varepsilon,u_B)$, $\mathbb{B}(u^\varepsilon,u_B)$ and $|u^\varepsilon-u_B|$ vanish on $(0,T)\times\partial\mathcal{O}$. Also, as noted in [19], we observe that the first term in the integral $I_3$ is the sum of two terms, the first one of which is

$$-\frac{\tilde{\varphi}\lambda}{\delta}\operatorname{sgn}(u^\varepsilon-u_B)\big(\mathbf{B}(u^\varepsilon)-\mathbf{B}(u_B)\big):\nu(x)\otimes\nu(x)d\mathcal{H}^{d-1}dt,$$

which is nonpositive, since $\mathbf{B}'(\xi):\nu\otimes\nu\ge 0$, for any $\xi\in\mathbb{R}$; and the second term which converges to zero as $\delta\to 0$, as $\mathbf{B}(u^\varepsilon,u_B)$ vanishes on $(0,T)\times\partial\mathcal{O}$. Similarly, using (2.7) once again, the second term in the integral $I_3$ can also be decomposed into the sum of a nonpositive term and a term which converges to zero as $\delta\to 0$.

Thus, noting that $G_\varepsilon$ in bounded in $L^2([0,T]\times\mathcal{O})$ uniformly with respect to $\varepsilon$, we may take the limit as $\delta\to 0$ in (4.7) in order to obtain that

$$\begin{aligned}
&\int_0^T\int_{\mathcal{O}}|u^\varepsilon-u_B|\partial_t\tilde{\varphi}dxdt+F\big(u^\varepsilon(t,x),u_B(t,x)\big)\cdot\nabla\tilde{\varphi}dxdt\\
&\quad+\int_0^T\int_{\mathcal{O}}\Big(\mathbb{B}\big(u^\varepsilon(t,x),u_B(t,x)\big):D^2\tilde{\varphi}+\varepsilon|u^\varepsilon-u_B|\Delta\tilde{\varphi}\Big)dxdt\\
&\quad+\sum_{k\ge 1}\int_0^T\int_{\mathcal{O}}\operatorname{sgn}(u^\varepsilon-u_B)\big(g_k(u^\varepsilon(t))-g_k(u_B(t))\big)\tilde{\varphi}dxd\beta_k(t)\\
&\ge -C_*\|\tilde{\varphi}\|_{L^2([0,T]\times\mathcal{O})}
\end{aligned}\tag{4.8}$$

for a certain constant $C_*\ge 0$, upon which, integrating by parts in the second integral on the left hand side and taking $\varepsilon=\varepsilon_n\to 0$ we obtain (1.25). □

**Proposition 4.2.** *Let $u^\varepsilon$ be the solution of* (4.1)-(4.3) *and assume that* (4.4) *holds. Then, $u$ satisfies* (1.26).

*Proof.* Let us first observe that the arguments of the doubling of variables that lead to inequality (3.14) (proof of Theorem 3.2 may be easily adapted to the case where $u$ satisfies Eq. (1.1) with $\mathbf{A}=\mathbf{A}_1$ and $\mathbf{B}=\mathbf{B}_1$, and $v$ satisfies Eq. (1.1) with $\mathbf{A}=\mathbf{A}_2$ and $\mathbf{B}=\mathbf{B}_2$ such that, possibly, $\mathbf{A}_1\ne\mathbf{A}_2$ and $\mathbf{B}_1\ne\mathbf{B}_2$ (for instance $\mathbf{B}_2(\xi)=\mathbf{B}_1(\xi)+\xi\mathrm{Id}$). In this case, in the resulting inequality instead of $F_+(u(s,x),v(s,y))\cdot\nabla_{x+y}$ we have

$$F_{1+}\big(u(s,x),v(s,y)\big)\cdot\nabla_x\phi(x,y)+F_{2+}\big(u(s,x),v(s,y)\big)\cdot\nabla_y\phi(x,y),$$

where $F_{1+}$ and $F_{2+}$ correspond to $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively; and also instead of $\mathbb{B}_+(u(s,x),v(s,y)):(D_x^2+D_y^2)$ we have

$$\mathbb{B}_{1+}\big(u(s,x),v(s,y)\big):D_x^2\phi(x,y)+\mathbb{B}_{2+}\big(u(s,x),v(s,y)\big):D_y^2\phi(x,y),$$

where $\mathbb{B}_{1+}$ and $\mathbb{B}_{2+}$ correspond to $\mathbf{B}_1$ and $\mathbf{B}_2$, respectively. Moreover, the inequality obtained for $(u^\pm - v^\pm)_+$ can be put together with the one for $(v^\pm - u^\pm)_+$ in order to obtain the corresponding inequality for $|u^\pm - v^\pm|$.

Let $u^\varepsilon$ be the solution of (4.1)-(4.3) and $v$ be a kinetic solution of Eq. (1.1) in the sense of Definition 1.2(ii). Since $u^\varepsilon \in L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H^1(\mathcal{O}))$, it can be easily shown to be the entropy solution of Eq. (4.1), and consequently, also a kinetic solution. Then we may use (the variant of) Theorem 3.2, as described above, to conclude that for $0\le t\le T$ and any $0\le \phi\in C_c^\infty(\mathcal{O}^2)$ we have a.s. that

$$
\begin{aligned}
&\int_{\mathcal{O}^2} |u^\varepsilon(t,x) - v^\pm(t,y)|\phi(x,y)\,dx\,dy \le \int_{\mathcal{O}^2} |u_0(x) - v_0(y)|\phi(x,y)\,dx\,dy \\
&\quad + \int_0^t \int_{\mathcal{O}^2} F\big(u^\varepsilon(s,x), v(s,y)\big)\cdot \nabla_{x+y}\phi(x,y)\,dx\,dy\,ds \\
&\quad + \int_0^t \int_{\mathcal{O}^2} \mathbb{B}\big(u^\varepsilon(s,x), v(s,y)\big) : \big(D_x^2 + D_y^2\big)\phi(x,y)\,dx\,dy\,ds \\
&\quad - \varepsilon \int_0^t \int_{\mathcal{O}^2} |u^\varepsilon(s,x) - v(s,y)|\Delta_x \phi(x,y)\,dx\,dy\,ds \\
&\quad + \sum_{k\ge 1} \int_0^t \int_{\mathcal{O}^2} \operatorname{sgn}\big(u^\varepsilon(s,x) - v(s,y)\big)\big(g_k(u^\varepsilon(s,x)) - g_k(v(s,y))\big)\phi(x,y)\,dx\,dy\,d\beta_k(s).
\end{aligned} \tag{4.9}
$$

Now, by a similar reasoning to the one leading to (4.6), but using a test function of the form $\varphi(x) = \phi(x,y)$, with $y\in\mathcal{O}$ fixed, we have a.s. and for a.e. $y\in\mathcal{O}$ that

$$
\begin{aligned}
&\int_{\mathcal{O}} |u^\varepsilon(t,x) - u_B(t,x)|\phi(x,y)\,dx \\
\le &\int_{\mathcal{O}} |u_0(x) - u_{B0}(x)|\phi(x,y)\,dx + \int_0^t \int_{\mathcal{O}} F\big(u^\varepsilon(s,x), u_B(s,x)\big)\cdot \nabla_x \phi(x,y)\,dx\,ds \\
&\quad + \int_0^t \int_{\mathcal{O}} \Big(\mathbb{B}\big(u^\varepsilon(s,x), u_B(s,x)\big) : D_x^2 \phi(x,y) + \varepsilon |u^\varepsilon - u_B| \Delta_x \phi(x,y)\Big)\,dx\,ds \\
&\quad + \sum_{k\ge 1} \int_0^t \int_{\mathcal{O}} \operatorname{sgn}\big(u^\varepsilon(s,x) - u_B(s,x)\big)\big(g_k(u^\varepsilon(s,x)) - g_k(u_B(s,x))\big)\phi(x,y)\,dx\,d\beta_k(s) \\
&\quad + \|G_\varepsilon\|_{L^2([0,T]\times\mathcal{O})} \|\phi(\cdot,y)\|_{L^2([0,T]\times\mathcal{O})}.
\end{aligned} \tag{4.10}
$$

Let us recall the notations (1.15) and (1.16) and denote

$$
\begin{aligned}
&\mathcal{F}(u^\varepsilon, v, u_B) := F(u^\varepsilon, v) + F(u^\varepsilon, u_B) - F(u_B, v), \\
&\mathbb{B}^*(u^\varepsilon, v, u_B) := \mathbb{B}(u^\varepsilon, v) + \mathbb{B}(u^\varepsilon, u_B) - \mathbb{B}(u_B, v).
\end{aligned}
$$

Then, integrating (4.10) over $\mathcal{O}$ with respect to $y$, adding the resulting inequality to (4.9) and adding and subtracting a few terms, we obtain

$$
\begin{aligned}
&\int_{\mathcal{O}^2} \mathcal{A}\big(u^\varepsilon(t,x),v(t,y),u_B(t,x)\big)\phi(x,y)\,dxdy \qquad\qquad (4.11)\\
\leq &\int_{\mathcal{O}^2} \mathcal{A}\big(u_0(x),v_0(y),u_{B0}(x)\big)\phi(x,y)\,dxdy\\
&+\int_0^t\int_{\mathcal{O}^2} \mathcal{F}\big(u^\varepsilon(s,x),v(s,y),u_B(s,x)\big)\cdot\nabla_x\phi(x,y)\,dxdyds\,[J_1]\\
&+\int_0^t\int_{\mathcal{O}^2} \mathbb{B}^*\big(u^\varepsilon(s,x),v(s,y),u_B(s,x)\big):D_x^2\phi(x,y)\,dxdyds\,[J_2]\\
&+\varepsilon\int_0^t\int_{\mathcal{O}^2} \mathcal{A}\big(u^\varepsilon(s,x),v(s,y),u_B(s,x)\big)\Delta_x\phi(x,y)\,dxdyds\,[J_3]\\
&+\sum_{k\geq 1}\int_0^t\int_{\mathcal{O}^2} \operatorname{sgn}\big(u^\varepsilon(s,x)-v(s,y)\big)\big(g_k(u^\varepsilon(s,x))-g_k(v(s,y))\big)\phi(x,y)\,dxd\beta_k(s)\\
&+\sum_{k\geq 1}\int_0^t\int_{\mathcal{O}^2} \operatorname{sgn}\big(u^\varepsilon(s,x)-u_B(s,x)\big)\big(g_k(u^\varepsilon(s,x))-g_k(u_B(s,x))\big)\phi(x,y)\,dxd\beta_k(s)\\
&+C\int_{\mathcal{O}}\|\phi(\cdot,y)\|_{L^2([0,T]\times\mathcal{O})}dy+\int_{\mathcal{O}^2}|u_B(t,x)-v(t,y)|\phi(x,y)\,dxdy\\
&+\int_0^t\int_{\mathcal{O}^2}\phi(x,y)\,d\mu_y(x)\,dyds\\
&+\int_0^t\int_{\mathcal{O}^2}\Big(F\big(u^\varepsilon(s,x),v(s,y)\big)\cdot\nabla_y\phi(x,y)+\mathbb{B}\big(u^\varepsilon(s,x),v(s,y)\big):D_y^2\phi(x,y)\Big)\,dxdyds,
\end{aligned}
$$

where $\mu_y=|\operatorname{div}K_x(u_B(s,x),v(s,y))|$.

Having (4.11) at hand, the rest of the proof is as follows. Fix $B\in\mathcal{B}$ (that is, $B$ is a ball centered at some point in $\partial\mathcal{O}$ satisfying (1.17)). In (1.26) we need to be able to take test functions in $C_c^\infty(B\times\mathcal{O})$, which possibly do not vanish at $\partial\mathcal{O}\times\mathcal{O}$. To achieve this, we take $\phi=\tilde{\zeta}_\delta(x)\psi(x,y)$, where $\tilde{\zeta}_\delta$ is a canonical local boundary layer sequence and $\psi\in C_c^\infty(B\times\mathcal{O})$, take $\delta\to 0$ and perform an analysis similar to that of the proof of Proposition 4.1 in order to get rid of some of the terms that involve derivatives in $x$ of the test function.

Then, from the resulting inequality we will prove (1.26) by first taking the limit as $\varepsilon\to 0$ and then substituting $\psi(x,y)=(1-\zeta_\delta(x))\tilde{\phi}(x,y)$, where $\zeta_\delta$ is now any boundary layer sequence and, upon taking the limit as $\delta\to 0$, most of the remaining terms vanish and we recover (1.26).

Let us, then, take $\phi=\tilde{\zeta}_\delta(x)\psi(x,y)$ in (4.11) by approximation. Note that all of the terms that do not involve derivatives in $x$ of the test function converge nicely as $\delta\to 0$. The remaining terms are the following:

$$
\begin{aligned}
&J_1+J_2+J_3 \qquad\qquad (4.12)\\
=&\int_0^t\int_{\mathcal{O}^2}\Big(\mathcal{F}(u^\varepsilon,v,u_B)\cdot\nabla_x\psi(x,y)\tilde{\zeta}_\delta(x)+\mathbb{B}^*(u^\varepsilon,v,u_B):D_x^2\psi(x,y)\tilde{\zeta}_\delta(x)\Big)\,dxdyds\\
&+\varepsilon\int_0^t\int_{\mathcal{O}^2}\mathcal{A}(u^\varepsilon,v,u_B)\Delta_x\psi(x,y)\tilde{\zeta}_\delta(x)\,dxdyds\\
&+\int_0^t\int_{\mathcal{O}^2}\big(\mathcal{F}(u^\varepsilon,v,u_B)\cdot\nabla_x\tilde{\zeta}_\delta\psi(x,y)+2\big(\mathbb{B}^*(u^\varepsilon,v,u_B)\nabla\tilde{\zeta}_\delta(x)\big)\cdot\nabla_x\psi(x,y)\big)\,dxdyds\\
&+2\varepsilon\int_0^t\int_{\mathcal{O}^2}\big(\mathcal{A}(u^\varepsilon,v,u_B)\nabla_x\tilde{\zeta}_\delta(x)\big)\cdot\nabla\psi(x,y)\,dxdyds\\
&+\int_0^t\int_{\mathcal{O}^2}\Big(\psi(x,y)\mathbb{B}^*(u^\varepsilon,v,u_B):D_x^2\tilde{\zeta}_\delta(x)+\psi(x,y)\mathcal{A}(u^\varepsilon,v,u_B)\Delta_x\tilde{\zeta}_\delta(x)\Big)\,dxdyds.
\end{aligned}
$$

Note that the first two integrals in (4.12) converge nicely as $\delta\to 0$. Recalling (2.7), we see that the third and fourth integrals converge to 0 as $\delta\to 0$, since $\mathcal{F}(u^\varepsilon(s,x),v(s,y),u_B(s,x))$, $\mathbf{B}^*(u^\varepsilon(s,x),v(s,y),u_B(s,x))$, $\mathcal{A}(u^\varepsilon(s,x),v(s,y),u_B(s,x))$ vanish on $\partial\mathcal{O}\times\mathcal{O}$. Finally, using the same arguments as in the proof of Proposition 4.1, the integral on the fifth line of (4.12) can be written as a sum of a nonpositive term and a term that tends to zero as $\delta\to 0$. In this last argument, we use the fact that $\mathbb{B}^*(u,v,w)$ is a positive definite matrix by virtue of (1.4), which is easy to verify (see [19, Lemma 1.2]).

As a consequence, after taking the limit as $\delta\to 0$ in (4.12) we get

$$
\begin{aligned}
&\int_{\mathcal{O}^2}\mathcal{A}\big(u^\varepsilon(t,x),v(t,y)u_B(t,x)\big)\psi(x,y)\,dxdy\\
\le&\int_{\mathcal{O}^2}\mathcal{A}\big(u_0(x),v_0(y),u_{B0}(x)\big)\psi(x,y)\,dxdy\\
&+\int_0^t\int_{\mathcal{O}^2}\mathcal{F}\big(u^\varepsilon(s,x),v(s,y),u_B(s,x)\big)\cdot\nabla_x\psi(x,y)\,dxdyds\\
&+\int_0^t\int_{\mathcal{O}^2}\mathbb{B}^*\big(u^\varepsilon(s,x),v(s,y),u_B(s,x)\big):D_x^2\psi(x,y)\,dxdyds\\
&+\varepsilon\int_0^t\int_{\mathcal{O}^2}\mathcal{A}\big(u^\varepsilon(s,x),v(s,y),u_B(s,x)\big)\Delta_x\psi(x,y)\,dxdyds\\
&+\sum_{k\ge1}\int_0^t\int_{\mathcal{O}^2}\operatorname{sgn}\big(u^\varepsilon(s,x)-v(s,y)\big)\big(g_k(u^\varepsilon(s,x))-g_k(v(s,y))\big)\psi(x,y)\,dxd\beta_k(s)\\
&+\sum_{k\ge1}\int_0^t\int_{\mathcal{O}^2}\operatorname{sgn}\big(u^\varepsilon(s,x)-u_B(s,x)\big)\big(g_k(u^\varepsilon(s,x))-g_k(u_B(s,x))\big)\psi(x,y)\,dxd\beta_k(s)
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_{\mathcal{O}}\|\psi(\cdot,y)\|_{L^2([0,T]\times\mathcal{O})}dy+\int_{\mathcal{O}^2}|u_B(t,x)-v(t,y)|\psi(x,y)\,dx\,dy\\
&+\int_0^t\int_{\mathcal{O}^2}\psi(x,y)d\mu_y(x)\,dy\,ds\\
&+\int_0^t\int_{\mathcal{O}^2}\Big(F\big(u^\varepsilon(s,x),v(s,y)\big)\cdot\nabla_y\psi(x,y)+\mathbb{B}\big(u^\varepsilon(s,x),v(s,y)\big):D_y^2\psi(x,y)\Big)\,dx\,dy\,ds.
\end{aligned}
$$

Now we integrate by parts in the integral on the third line and then take the limit as $\varepsilon\to 0$ to arrive at

$$
\begin{aligned}
&\int_{\mathcal{O}^2}\mathcal{A}\big(u(t,x),v(t,y)u_B(t,x)\big)\psi(x,y)\,dx\,dy \\
\le&\int_{\mathcal{O}^2}\mathcal{A}\big(u_0(x),v_0(y),u_{B0}(x)\big)\psi(x,y)\,dx\,dy\\
&-\int_0^t\int_{\mathcal{O}^2}H_x\big(u(s,x),v(s,y),u_B(s,x)\big)\cdot\nabla_x\psi(x,y)\,dx\,dy\,ds\\
&+\sum_{k\ge1}\int_0^t\int_{\mathcal{O}^2}\operatorname{sgn}\big(u(s,x)-v(s,y)\big)\big(g_k(u(s,x))-g_k(v(s,y))\big)\psi(x,y)\,dx\,d\beta_k(s)\\
&+\sum_{k\ge1}\int_0^t\int_{\mathcal{O}^2}\operatorname{sgn}\big(u(s,x)-u_B(s,x)\big)\big(g_k(u(s,x))-g_k(u_B(s,x))\big)\psi(x,y)\,dx\,d\beta_k(s)\\
&+C\int_{\mathcal{O}}\|\psi(\cdot,y)\|_{L^2([0,T]\times\mathcal{O})}dy+\int_{\mathcal{O}^2}|u_B(t,x)-v(t,y)|\psi(x,y)\,dx\,dy\\
&+\int_0^t\int_{\mathcal{O}^2}\psi(x,y)d\mu_y(x)\,dy\,ds\\
&+\int_0^t\int_{\mathcal{O}^2}\Big(F\big(u(s,x),v(s,y)\big)\cdot\nabla_y\psi(x,y)+\mathbb{B}\big(u(s,x),v(s,y)\big):D_y^2\psi(x,y)\Big)\,dx\,dy\,ds.
\end{aligned}
\tag{4.13}
$$

At last, we take $\psi(x,y)=(1-\zeta_\delta(x))\tilde{\varphi}(x,y)$ in (4.13), where $\zeta_\delta$ is any boundary layer sequence and $\tilde{\varphi}\in C_c^\infty(B\times\mathcal{O})$, take expectation and then take the limit as $\delta\to 0$ to conclude that

$$
0\le\liminf_{\delta\to0}\mathbb{E}\int_0^t\int_{\mathcal{O}^2}H_x\big(u(s,x),v(s,y),u_B(s,x)\big)\cdot\nabla_x\zeta_\delta(x)\tilde{\varphi}(x,y)\,dx\,dy\,ds, \tag{4.14}
$$

which concludes the proof. □

# 5 Existence part 1: nondegenerate case

In this section we solve the following approximation of problem (1.1)-(1.3):

$$
du^\varepsilon+\nabla\cdot\mathbf{A}^\varepsilon(u^\varepsilon)dt-D^2:\mathbf{B}^\varepsilon(u^\varepsilon)dt-\varepsilon\Delta u^\varepsilon dt=\Phi^\varepsilon(u^\varepsilon)dW(t), \tag{5.1}
$$

$$u^\varepsilon(0,x)=u_0^\varepsilon(x), \quad x\in\mathcal{O}, \tag{5.2}$$
$$u^\varepsilon=u_b^\varepsilon(t,x), \qquad x\in\partial\mathcal{O}, \quad t>0, \tag{5.3}$$

where $u_0^\varepsilon$ is a smooth approximation of $u_0$, $u_0^\varepsilon\in L^\infty(\Omega,C_c^\infty(\mathcal{O}))$, $u_{\min}\le u_0^\varepsilon\le u_{\max}$, a.s., $u_b^\varepsilon$ is a smooth approximation of $u_b$, $\Phi^\varepsilon$ is a suitable Lipschitz approximation of $\Phi$ satisfying (1.9) uniformly, as well as (1.13), with $g_k^\varepsilon$ and $G^\varepsilon$ as in the case $\varepsilon=0$, $g_k^\varepsilon(x,\xi)$ smooth with compact support in $\xi$ contained in $(-M,M)$, for some $M>\max\{|u_{\min}|,|u_{\max}|\}$. Moreover, $g_k^\varepsilon\equiv 0$ for $k\ge 1/\varepsilon$. Finally, $\mathbf{A}^\varepsilon\in C_c^2(\mathbb{R};\mathbb{R}^d)$, $\mathbf{B}^\varepsilon\in C_c^2(\mathbb{R};\mathbb{M}^d)$, $\mathbf{A}^\varepsilon(u)=\mathbf{A}(u)$, $\mathbf{B}^\varepsilon(u)=\mathbf{B}(u)$ for $u\in[a,b]$. The latter assumptions will be justified later on when we prove that the solution of (5.1)-(5.3), $u^\varepsilon(t,x)$, satisfies $u_{\min}\le u^\varepsilon(t,x)\le u_{\max}$, $(t,x)\in(0,T)\times\mathcal{O}$.

In order to solve (5.1)-(5.3), we will proceed as follows. First we consider the following approximation for (5.1)-(5.3):

$$du^{\varepsilon,\mu}+\nabla\cdot\mathbf{A}^\varepsilon(u^{\varepsilon,\mu})dt-D^2:\widetilde{\mathbf{B}}^\varepsilon(u^{\varepsilon,\mu})dt+\mu\Delta^2u^{\varepsilon,\mu}dt=\Phi^\varepsilon(u^{\varepsilon,\mu})dW(t), \tag{5.4}$$
$$u^{\varepsilon,\mu}(0,x)=u_0^\varepsilon(x), \quad x\in\mathcal{O}, \tag{5.5}$$
$$u^{\varepsilon,\mu}=u_b^\varepsilon(t,x), \qquad x\in\partial\mathcal{O}, \quad t>0, \tag{5.6}$$
$$\partial_\nu u^{\varepsilon,\mu}=0, \qquad x\in\partial\mathcal{O}, \quad t>0, \tag{5.7}$$

where $\widetilde{\mathbf{B}}^\varepsilon(u)=\mathbf{B}^\varepsilon(u)+\varepsilon u\mathbf{I}$. Notice the additional boundary condition (5.7), introduced due to the biharmonic operator $\Delta^2$. Once we prove the existence and uniqueness of solutions of (5.4)-(5.7), we can pass to the limit $\mu\to0$ applying a reasoning similar to the one used in [23], thus solving the nondegenerated problem (5.1)-(5.3).

## 5.1 Two-level approximations

We are going to solve problem (5.4)-(5.7) by a fixed point argument. To that end, we consider the following functional

$$\begin{aligned}(Kv)(t)=(K^{\varepsilon,\mu}v)(t)=&S(t)(u_0^\varepsilon-w_0)+\int_0^t S(t-s)D^2:\mathbf{B}^\varepsilon\big(v(s)\big)\,ds\\&-\int_0^t S(t-s)\nabla\cdot\mathbf{A}^\varepsilon\big(v(s)\big)\,ds+\int_0^t S(t-s)\Phi^\varepsilon\big(x,v(s)\big)dW(s)+\widetilde{u_b^\varepsilon}(t).\end{aligned} \tag{5.8}$$

Here, $S(t)$ denotes the strongly continuous semigroup generated by the operator $A=A_{\mu,\varepsilon}:D(A)\subset L^2(\mathcal{O})\to L^2(\mathcal{O})$ given by

$$\begin{cases}D(A)=H_0^2(\mathcal{O})\cap H^4(\mathcal{O})\\ \qquad\quad=\big\{u\in H^4(\mathcal{O});\, u=\partial_\nu u=0 \text{ (in the sense of traces) on } \partial\mathcal{O}\big\},\\ Au=-\varepsilon\Delta u+\mu\Delta^2u,\end{cases}$$

which by Theorem B.5 is nonnegative and self-adjoint, thus $S(t)=\exp\{-tA\}$. The last term in (5.8) is the solution to the following initial-boundary value problem:

$$\begin{cases}\dfrac{\partial \widetilde{u_b^\varepsilon}}{\partial t}(t,x)+\mu\Delta^2\widetilde{u_b^\varepsilon}(t,x)-\varepsilon\Delta\widetilde{u_b^\varepsilon}(t,x)=0 & \text{in } \{0<t<T\}\times\mathcal{O},\\ \widetilde{u_b^\varepsilon}(0,x)=w_0(x) & \text{on } \{t=0\}\times\mathcal{O},\\ \widetilde{u_b^\varepsilon}(t,x)=u_b^\varepsilon(t,x) & \text{on } \{0<t<T\}\times\partial\mathcal{O},\\ \dfrac{\partial \widetilde{u_b^\varepsilon}}{\partial \nu}(t,x)=0 & \text{on } \{0<t<T\}\times\partial\mathcal{O},\end{cases} \tag{5.9}$$

for some suitable $w_0$, whose well-posedness is shown in Lemma 5.5 below.

In this fashion, we are able to prove the following theorem.

**Theorem 5.1.** *Assume that $u_0\in L^2(\Omega\times\mathcal{O})$ and $u_b^\varepsilon\in L^2(\Omega\times[0,T];H^2(\partial\mathcal{O}))\cap L^2(\Omega;H^1((0,T);L^2(\partial\mathcal{O})))$ is predictable. Let also $w_0\in L^2(\Omega,H^2(\mathcal{O}))$ be an extension of $u_b|_{t=0}$ to $\partial\mathcal{O}$. If*

$$\mathcal{E}=L^2\big(\Omega;C([0,T];L^2(\mathcal{O}))\big)\cap L^2\big(\Omega;L^2(0,T;H^2(\mathcal{O}))\big),$$

*then $K$, defined in* (5.8)*, can be understood as a continuous mapping from $\mathcal{E}$ into $\mathcal{E}$, which possesses a unique fixed point $u^{\varepsilon,\mu}$. This fixed point $u^{\varepsilon,\mu}$ is the unique weak solution to* (5.4)-(5.7) *in $\mathcal{E}$ and satisfies the following energy estimate:*

$$\begin{aligned}\mathbb{E}\bigg\{\sup_{0\le t\le T}\|u^{\varepsilon,\mu}(t)\|^2_{L^2(\mathcal{O})}+\varepsilon\int_0^T\|\nabla u^{\varepsilon,\mu}(t)\|^2_{L^2(\mathcal{O})}dt\\ +\mu\int_0^T\|\Delta u^{\varepsilon,\mu}(t)\|^2_{L^2(\mathcal{O})}dt\bigg\}\le C,\end{aligned} \tag{5.10}$$

*where $C$ depends on $\varepsilon$, $u_0^\varepsilon$ and $u_b$, but otherwise remains bounded as $\mu\to 0$.*

Passing $\mu\to 0$, we obtain the expected convergence of the $u^{\varepsilon,\mu}$ to the unique solution $u^\varepsilon$ to (5.1)-(5.3), as stated below.

**Theorem 5.2.** *If $\mu\to 0$, then the functions $u^{\mu,\varepsilon}$ converge in probability in the space $L^2(0,T;L^2(\mathcal{O}))\cap C([0,T];H^{-4}(\mathcal{O}))$ to the unique weak solution $u^\varepsilon\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H^1(\mathcal{O}))$ to* (5.1)-(5.3)*. Moreover, one has that*

$$\begin{cases}a\le u^\varepsilon(t,x)\le b, \quad \text{a.s., a.e. in } (0,T)\times\mathcal{O},\\ \mathbb{E}\displaystyle\int_0^T\int_{\mathcal{O}}|\nabla b_\varepsilon(u^\varepsilon(t,x))|^2dxds\le C\end{cases} \tag{5.11}$$

*for some constant $C$, which depends on $a$, $b$, $u^b$, but not on $0<\varepsilon<1$.*

We will prove these two theorems in the next few subsections.

## 5.2 Proof of Theorem 5.1.

### 5.2.1 The analysis of $K:\mathcal{E}\to\mathcal{E}$.

Let us begin by showing that the operator $K$, given by (5.8) for $v\in\mathcal{E}$, is well defined and continuous. To that end, let us decompose $K$ into five parts which will be analyzed individually

$$(Kv)(t)=(K_0v)(t)+(K_1v)(t)-(K_2v)(t)+(K_3v)(t)+(K_4v)(t),$$

where

$$\begin{cases}(K_0v)(t)=S(t)(u_0^\varepsilon-w_0),\\ (K_1v)(t)=\displaystyle\int_0^t S(t-s)D^2:\mathbf{B}^\varepsilon\big(v(s)\big)\,ds,\\ (K_2v)(t)=\displaystyle\int_0^t S(t-s)\nabla\cdot\mathbf{A}^\varepsilon\big(v(s)\big)\,ds,\\ (K_3v)(t)=\displaystyle\int_0^t S(t-s)\Phi^\varepsilon\big(x,v(s)\big)\,dW(s),\\ (K_4v)(t)=\widetilde{u_b}(t).\end{cases}$$

In order to simplify the notation, in this subsection we drop the superindices $\varepsilon$ and $\mu$.

### 5.2.2 On $(K_0v)(t)$.

Note that $K_0v$ is independent of $v$. Then, the following result may be easily deduced from the theory of semigroups of linear operators (see Appendix B).

**Lemma 5.1.** *Let $u_0\in L^2(\Omega;L^2(\mathcal{O}))$, then the expression $S(t)(u_0-w_0)$ defines an element of $\mathcal{E}$. Additionally, the correspondence $u_0\in L^2(\Omega;L^2(\mathcal{O}))\mapsto S(t)(u_0-w_0)\in\mathcal{E}$ is continuous.*

### 5.2.3 On $(K_1v)(t)$.

Regarding the term $K_1v$ we can use the spectral analysis of the operator $A$ (see Section B.2) to deduce the following.

**Lemma 5.2.** *The mapping $K_1:\mathcal{E}\to\mathcal{E}$ is well-defined and continuous.*

*Proof.* Note that if $h\in L^2(\Omega;L^2(0,T;L^2(\mathcal{O})))$, then, as distributions,

$$\frac{\partial h}{\partial x_j}\in L^2\big(\Omega;L^2(0,T;H^{-1}(\mathcal{O}))\big),\quad \frac{\partial^2 h}{\partial x_j\partial x_k}\in L^2\big(\Omega;L^2(0,T;H^{-2}(\mathcal{O}))\big)$$

for $1\le j,k\le d$ with

$$\mathbb{E}\int_0^T \left\|\frac{\partial h}{\partial x_j}(s)\right\|^2_{H^{-1}(\mathcal{O})} ds \le \mathbb{E}\int_0^T \|h(s)\|^2_{L^2(\mathcal{O})} ds,$$
$$\mathbb{E}\int_0^T \left\|\frac{\partial^2 h}{\partial x_j \partial x_k}(s)\right\|^2_{H^{-2}(\mathcal{O})} ds \le \mathbb{E}\int_0^T \|h(s)\|^2_{L^2(\mathcal{O})} ds.$$

In particular, given $v\in\mathcal{E}$ we have that $D^2: B(v(s))\in H_A^{-1/2}=H^{-2}(\mathcal{O})$ (see Theorem B.6). Consequently, using Theorem B.2 we have a.s. that

$$\begin{aligned}
&\left\|\int_0^t S(t-s)D^2:\mathbf{B}(v(s))ds\right\|_{L^2(\mathcal{O})}\\
\le&\int_0^t \|S(t-s)D^2:\mathbf{B}(v(s))\|_{L^2(\mathcal{O})}ds \le C\int_0^t\left(1+\frac{1}{(t-s)^{1/2}}\right)\|\mathbf{B}(v(s))\|_{L^2(\mathcal{O})}ds\\
\le& C\sup_{0\le s\le T}\|\mathbf{B}(v(s))\|_{L^2(\mathcal{O})},
\end{aligned}\tag{5.12}$$

from which we conclude that $(K_1v)\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))$, by taking the supremum over $0\le t\le T$, squaring and taking the expectation.

Likewise, but now using Theorem B.3 with $\alpha=-1/2$,

$$\begin{aligned}
&\mathbb{E}\int_0^T\left\|\int_0^t S(t-s)D^2:\mathbf{B}(v(s))ds\right\|^2_{H_0^2(\mathcal{O})}dt\\
\le& C\mathbb{E}\int_0^T\|D^2:\mathbf{B}(v(s))\|^2_{H^{-2}}ds\le C\mathbb{E}\int_0^T\|\mathbf{B}(v(s))\|^2_{L^2(\mathcal{O})}ds,
\end{aligned}\tag{5.13}$$

thus again we see that $(K_1v)\in L^2(\Omega;L^2(0,T;H^2(\mathcal{O})))$.

Finally, minor modifications in the reasoning leading to the inequalities (5.12)-(5.13) demonstrate that the mapping $v\in\mathcal{E}\mapsto K_1v\in\mathcal{E}$ is indeed continuous. □

### 5.2.4 On $(K_2v)(t)$.

The analysis of $K_2v$ is similar to that of $K_1v$. Indeed, note that for each $v\in\mathcal{E}$, $\nabla\cdot\mathbf{A}(v(s))\in H_A^{-1/4}$ so we can use Theorems B.2 and B.3 to obtain the following.

**Lemma 5.3.** *The mapping $K_2:\mathcal{E}\to\mathcal{E}$ is well-defined and continuous.*

#### 5.2.5 On $(K_3v)(t)$.

Regarding the stochastic convolution operator $K_3$, we may state the following result.

**Lemma 5.4.** *The mapping $K_3:\mathcal{E}\to\mathcal{E}$ is well-defined and continuous.*

*Proof.* Let $v\in\mathcal{E}$. Then, by the maximal estimate for stochastic convolutions [5,13, 24,31,32,43], which asserts that

$$\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t S(t-s)\Phi(x,v(s))\,dW(s)\right\|^2_{L^2(\mathcal{O})}\le C\mathbb{E}\int_0^T\|\Phi(x,v(s))\|^2_{L_2(\mathfrak{U},L^2(\mathcal{O}))}ds$$

for a constant $C$ depending only on $T>0$, we have $K_3v\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))$.

Furthermore, by Theorem B.4 (with $\alpha=0$), and Theorem B.6,

$$\mathbb{E}\int_0^T\left\|\int_0^t S(t-s)\Phi(x,v(s))\,dW(s)\right\|^2_{H_0^2(\mathcal{O})}dt\le C\mathbb{E}\int_0^T\|\Phi(x,v(s))\|^2_{L_2(\mathfrak{U},L^2(\mathcal{O}))}ds,$$

allowing us to argue that $K_5v\in L^2(\Omega;L^2(0,T;H^2(\mathcal{O})))$. Finally, observe that, replacing $\Phi(x,v)$ with $\Phi(x,v_1(s))-\Phi(x,v_2(s))$ in the above inequalities also shows that $K_3:\mathcal{E}\to\mathcal{E}$ is continuous. $\square$

#### 5.2.6 On $(K_4v)(t)$

Note that $K_4v$ is independent of $v$. So, we only have to show that Eq. (5.9) has a unique solution in the space $\mathcal{E}$.

Now, straightening out the boundary, using partition of unity, etc., one can extend $u_b$ to some predictable $h\in L^2(\Omega\times[0,T];H^2(\mathcal{O}))\cap L^2(\Omega;H^1((0,T);L^2(\mathcal{O})))$ in a way that a.s.

$$\partial_\nu h=0\quad\text{on }\ (0,T)\times\partial\mathcal{O}.$$

Note that we may choose $w_0=h|_{t=0}$. Hence, $z=\widetilde{u_b^\varepsilon}-h$ should satisfy

$$\begin{aligned}
&\frac{\partial z}{\partial t}-\varepsilon\Delta z+\mu\Delta^2 z=-\left(\frac{\partial}{\partial t}-\varepsilon\Delta+\mu\Delta^2\right)h\quad\text{in }\ (0,T)\times\mathcal{O},\\
&z(0,x)=0\qquad\text{on }\ \{t=0\}\times\mathcal{O},\\
&z(t,x)=0\qquad\text{on }\ (0,T)\times\mathcal{O},\\
&\frac{\partial z}{\partial \nu}(t,x)=0\quad\text{on }\ (0,T)\times\partial\mathcal{O}.
\end{aligned}\tag{5.14}$$

Concerning the well-posedness of (5.14), we recall the following result by J.-L. Lions, as stated in [4]. We refer to [33] for the proof.

**Theorem 5.3.** *Let $H$ be a Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|_H$, and identify $H^*$ with $H$. Let also $V$ be another Hilbert space with norm $\|\cdot\|_V$, for which $V\subset H$ with dense and continuous injection, so that we have the triplet*

$$V\subset H\subset V^*.$$

*Let $T>0$ be fixed; and suppose that for a.e. $0<t<T$ we are given a bilinear form $a(t;u,z):V\times V\to\mathbb{R}$ satisfying the following properties:*

1. *For every $u,z\in V$, the function $t\mapsto a(t;u,z)$ is measurable;*
2. *$|a(t;u,z)|\le M\|u\|_V\|z\|_V$ for a.e. $0<t<T$, $\forall u,z\in V$, and where $M$ is a constant;*
3. *$a(t;u,u)\ge\alpha\|u\|_V^2-C\|u\|_H^2$ for a.e. $0<t<T$, $\forall u\in V$, and where $\alpha$ and $C$ are positive constants.*

*Then for every $F\in L^2(0,T;V^*)$ and $u_0\in H$, there exists a unique function*

$$u\in L^2(0,T;V)\cap C([0,T];H)\cap H^1(0,T;V^*),$$

*such that $u(0)=u_0$ and*

$$\left\langle\frac{du}{dt}(t),z\right\rangle_{V^*,V}+a(t;u(t),z)=\langle F(t),z\rangle_{V^*,V}\quad\text{for a.e.}\quad 0<t<T,\quad\forall z\in V. \tag{5.15}$$

**Lemma 5.5.** *Assume that $u_b^\varepsilon\in L^2(\Omega\times[0,T];H^2(\partial\mathcal{O}))\cap L^2(\Omega;H^1((0,T);L^2(\partial\mathcal{O})))$ is predictable. Then the problem* (5.9) *has a unique weak solution $\widetilde{u_b^\varepsilon}\in L^2(\Omega\times(0,T);H^2(\mathcal{O}))$ $\cap L^2(\Omega;H^1(0,T;H^{-2}(\mathcal{O})))$. In particular, $\widetilde{u_b^\varepsilon}\in\mathcal{E}$. Moreover, $\widetilde{u_b^\varepsilon}$ is predictable.*

*Proof.* We apply Theorem 5.3 as follows. Let $H=L^2(\mathcal{O})$, $V=H_A^{1/2}=H_0^2(\mathcal{O})$ (cf. Theorem B.6),

$$a(t;u,z)=\varepsilon\int_{\mathcal{O}}\nabla u\cdot\nabla z\,dx+\mu\int_{\mathcal{O}}\Delta u\Delta z\,dx,$$

and

$$\langle F(t),\varphi\rangle_{H_A^{-\frac12},H_A^{\frac12}}:=\left\langle-\left(\frac{\partial}{\partial t}-\varepsilon\Delta+\mu\Delta_{x''}^2\right)h,\varphi\right\rangle_{H^{-2}(\mathcal{O}),H_0^2(\mathcal{O})}. \tag{5.16}$$

For each fixed $\omega\in\Omega$, the conditions in Theorem 5.3 are immediately verified, so existence and uniqueness of $\widetilde{u_b^\varepsilon}$ for each fixed $\omega\in\Omega$ follows. Now the proof of Theorem 5.3 can be made by the Galerkin method. So, since $h$ is predictable, $F$ given by (5.16) is also predictable and so are its finite dimensional projections.

Therefore, the Galerkin approximations, which are solutions of finite dimensional ODEs obtained as projections of (5.15), are also predictable. The convergence of the Galerkin approximations is obtained alongside a uniform estimate in $L^2(\Omega\times(0,T),\mathcal{P};H_A^{1/2})$ so that in the limit we obtain a predictable weak solution. $\square$

**Remark 5.1.** Note that by virtue of (5.15) we have that the solution of (5.14) satisfies a.s. and for $0\le t\le T$ the estimate

$$\int_{\mathcal{O}}|z(t,x)|^2dx+\int_0^t\int_{\mathcal{O}}\left(\varepsilon|\nabla z|^2+\mu|\Delta z|^2\right)\le C\|h\|^2_{H^1(0,T;L^2(\mathcal{O}))\cap L^2(0,T;H^2(\mathcal{O})}$$

for some $C>0$ independent of $\mu$, which implies, in particular, that

$$\|\widetilde{u_b^\varepsilon}\|_{L^2(\Omega\times(0,T);H^2(\mathcal{O}))\cap L^2(\Omega;H^1(0,T;H^{-2}(\mathcal{O})))}\le\tilde{C} \tag{5.17}$$

for some $\tilde{C}>0$ independent of $\mu$.

### 5.2.7 Conclusion.

Gathering Lemmas 5.1-5.5 we have the following.

**Theorem 5.4.** *The operator $K:\mathcal{E}\to\mathcal{E}$ is well-defined, continuous, depends continuously on initial data $u_0\in L^2(\Omega;L^2(\mathcal{O}))$ and on boundary data $u_b\in L^2(\Omega;H^{\frac{3}{8},\frac{7}{4}}((0,T)\times\partial\mathcal{O}))$.*

In order to prove that $K$ has a unique fixed point, we will exhibit an equivalent norm to $\mathcal{E}$, under which $K:\mathcal{E}\to\mathcal{E}$ is a contraction. First we need the following estimate.

**Lemma 5.6.** *Given any $u^1$ and $u^2\in\mathcal{E}$, it holds a.s. that, for all $0\le t\le T$,*

$$\begin{aligned}&\left\|(Ku^1)(t)-(Ku^2)(t)\right\|^2_{L^2(\mathcal{O})}dx+2\mu\int_0^t\left\|\Delta(Ku^1)(s)-\Delta(Ku^2)(s)\right\|^2_{L^2(\mathcal{O})}dxds\\
=&2\int_0^t\int_{\mathcal{O}}\left(\widetilde{\mathbf{B}}(u^1)-\widetilde{\mathbf{B}}(u^2)\right):D^2\left(Ku^1-Ku^2\right)dxds\\
&+2\int_0^t\int_{\mathcal{O}}\left(\mathbf{A}(u^1)-\mathbf{A}(u^2)\right)\cdot\nabla\left(Ku^1-Ku^2\right)dxds\\
&+2\sum_{k=1}^\infty\int_0^t\int_{\mathcal{O}}\left(g_k(x,u^1)-g_k(x,u^2)\right)\left(Ku^1-Ku^2\right)dxd\beta_k(s)\\
&+\sum_{k=1}^\infty\int_0^t\int_{\mathcal{O}}\left|g_k(x,u^1(s,x))-g_k(x,u^2(s,x))\right|^2dxds\quad a.s.\end{aligned} \tag{5.18}$$

*Proof.* Let us keep employing the notations of the Appendix B. First of all, let us observe that $Ku^1-Ku^2$ lies in $L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H_0^2(\mathcal{O}))$ and obeys the equation

$$\begin{aligned}
&Ku^1(t)-Ku^2(t)\\
=&-\int_0^t A\big((Ku^1)(s)-(Ku^2)(s)\big)\,ds+\int_0^t D^2:\big(\mathbf{B}(u^1(s))-\mathbf{B}(u^1(s))\big)\,ds\\
&-\int_0^t \nabla\big(\mathbf{A}(u^1(s))-\mathbf{A}(u^1(s))\big)\,ds+\int_0^t\big(\Phi(x,u^1(s))-\Phi(x,u^2(s))\big)\,dW(s)
\end{aligned}$$

a.s. in $H^{-2}(\mathcal{O})=H_A^{-1/2}$.

The idea here would be to apply the usual Itô's formula in the expression above; however, given that the equation is only satisfied in a negative Sobolev space, we are impeded to do so. To circumvent this difficulty, we introduce the "approximations of the identity" $J_\lambda=(I+\lambda A)^{-1}$ for $\lambda>0$, so that

1. for any $-\infty<\alpha<\infty$, $J_\lambda\in L(H_A^\alpha;H_A^{\alpha+1})$ with norm $\leq\frac{1}{\lambda}$;
2. for any $-\infty<\alpha<\infty$ and $h\in H_A^\alpha$, $J_\lambda h\to h$ in $H_A^\alpha$ as $\lambda\to 0$.

With this in mind, we see that $J_\lambda(Ku^1-Ku^2)$ satisfies

$$\begin{aligned}
&J_\lambda\big(Ku^1(t)-Ku^2(t)\big)\\
=&-\int_0^t AJ_\lambda\big((Ku^1)(s)-(Ku^2)(s)\big)\,ds\\
&+\int_0^t J_\lambda D^2:\big(\mathbf{B}(u^1(s))-\mathbf{B}(u^1(s))\big)\,ds-\int_0^t J_\lambda\nabla\cdot\big(\mathbf{A}(u^1(s))-\mathbf{A}(u^1(s))\big)\,ds\\
&+\int_0^t J_\lambda\big(\Phi(x,u^1(s))-\Phi(x,u^2(s))\big)\,dW(s)\text{ a.s. in }H_A^0=L^2(\mathcal{O}).
\end{aligned}$$

Hence, the usual Itô formula (with $\eta(u)=\|u\|^2_{L^2(\mathcal{O})}$) combined with some integrations by parts implies that

$$\begin{aligned}
&\int_{\mathcal{O}}\big|J_\lambda\big(Ku^1-Ku^2\big)(t)\big|^2dx+2\mu\int_0^t\int_{\mathcal{O}}\big|\Delta J_\lambda\big(Ku^1-Ku^2\big)\big|^2dxds\\
&\quad+2\varepsilon\int_0^t\int_{\mathcal{O}}\big|\nabla J_\lambda\big(Ku^1-Ku^2\big)\big|^2dxds\\
=&2\int_0^t\int_{\mathcal{O}}\Big[J_\lambda D^2:\big(\mathbf{B}(u^1)-\mathbf{B}(u^2)\big)\Big]J_\lambda\big(Ku^1-Ku^2\big)\,dxds
\end{aligned}$$

$$
\begin{aligned}
&-2\int_0^t\int_{\mathcal{O}} J_\lambda\Big[\nabla\cdot\big(\mathbf{A}(u^1)-\mathbf{A}(u^2)\big)\Big]J_\lambda\big(Ku^1-Ku^2\big)\,dxds\\
&+2\sum_{k=1}^\infty\int_0^t\int_{\mathcal{O}} J_\lambda\big(g_k(x,u^1)-g_k(x,u^2)\big)J_\lambda\big(Ku^1-Ku^2\big)\,dxd\beta_k(s)\\
&+\sum_{k=1}^\infty\int_0^t\int_{\mathcal{O}}\big|J_\lambda\big(g_k(x,u^1)-g_k(x,u^2)\big)\big|^2dxds \text{ a.s.}
\end{aligned}
$$

Now we make $\lambda\to 0$ and, given the two properties (1) and (2) stated above, we see that there is a subsequence along which all the terms converge almost surely to their respective correspondents. Indeed, the convergence of the term involving the stochastic integral follows by the continuity of the stochastic integral as an operator from the space of square integrable adapted processes to the space of square integrable martingales, due to the Itô isometry, and the fact that mean square convergence implies convergence in probability which, in turn, implies almost sure convergence along a subsequence. Now, fixed a subsequence $\{\lambda_k\}_{k\in\mathbb{N}}$ along which the term involving the stochastic integral converges almost surely, the convergence of the remaining terms is straightforward, except maybe for $J_\lambda D^2:(\mathbf{B}(u^1)-\mathbf{B}(u^2))$ (this is because $\mathbf{B}(u^j)$ may not lie in $L^2(\Omega\times[0,T];H^2(\mathcal{O}))$). However, interpreting the parcel on which it appears as

$$
\begin{aligned}
&2\int_0^t\int_{\mathcal{O}}\Big[J_\lambda D^2:\big(\mathbf{B}(u^1)-\mathbf{B}(u^2)\big)\Big]J_\lambda\big(Ku^1-Ku^2\big)\,dxds\\
=&2\int_0^t\Big\langle J_\lambda D^2:\big(\mathbf{B}(u^1)-\mathbf{B}(u^2)\big),J_\lambda\big(Ku^1-Ku^2\big)\Big\rangle_{H_A^{-1/2},H_A^{1/2}}dxds,
\end{aligned}
$$

this technicality is overcome. In any case, arranging the terms correctly (recall that $\widetilde{\mathbf{B}}=\mathbf{B}+\varepsilon I$), (5.18) is readily established. $\square$

**Lemma 5.7.** *$K:\mathcal{E}\to\mathcal{E}$ has a unique fixed-point $u=u^{\varepsilon,\mu}\in\mathcal{E}$.*

*Proof.* Let $u^1$ and $u^2\in\mathcal{E}$ and $0\le\tau\le T$. Taking the supremum of (5.18) in the interval $0\le t\le\tau$ and subsequently integrating in $\omega\in\Omega$, we obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le\tau}\left(\|(Ku^1)(t)-(Ku^2)(t)\|^2_{L^2(\mathcal{O})}+2\mu\int_0^t\|\Delta(Ku^1)(s)-\Delta(Ku^2)(s)\|^2_{L^2(\mathcal{O})}ds\right)\\
\le&C\mathbb{E}\int_0^\tau\|u^1(s)-u^2(s)\|_{L^2(\mathcal{O})}\|(Ku^1)(s)-(Ku^2)(s)\|_{H^2(\mathcal{O})}ds\\
&+C\mathbb{E}\int_0^\tau\|u^1(s)-u^2(s)\|^2_{L^2(\mathcal{O})}ds
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}\sup_{0\le t\le\tau}\bigg|\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}}\big(g_k(x,u^1(s))-g_k(x,u^2(s))\big)\\
&\qquad\qquad\qquad\times\big(Ku^1(s)-Ku^2(s)\big)\,dx\,d\beta_k(s)\bigg|.
\end{aligned}
\tag{5.19}
$$

In the estimate above, the only terms on the right-hand side that pose some problem are the first and the last. We will treat them as follows.

To handle the term

$$
\mathbb{E}\int_0^\tau \|u^1(s)-u^2(s)\|_{L^2}\|(Ku^1)(s)-(Ku^2)(s)\|_{H^2(\mathcal{O})}ds,
$$

we notice that $Ku^1-Ku^2\in L^2(\Omega;L^2(0,T;H_0^2(\mathcal{O})))$. Therefore, a joint application of the identity

$$
\sum_{i,j=1}^{N}\int_{\mathcal{O}}\left|\frac{\partial^2 h}{\partial x_i\partial x_j}(x)\right|^2dx=\int_{\mathcal{O}}|\Delta h(x)|^2dx
$$

(valid for functions $h\in H_0^2(\mathcal{O})$) and the Poincaré and Young inequalities shows that

$$
\begin{aligned}
&\mathbb{E}\int_0^\tau \|u^1(s)-u^2(s)\|_{L^2(\mathcal{O})}\|(Ku^1)(s)-(Ku^2)(s)\|_{H^2(\mathcal{O})}ds\\
\le&\frac{1}{4}\mathbb{E}\sup_{0\le t\le\tau}\bigg(\|(Ku^1)(t)-(Ku^2)(t)\|^2_{L^2(\mathcal{O})}\\
&\qquad\qquad+2\mu\int_0^t\|\Delta(Ku^1)(s)-\Delta(Ku^2)(s)\|^2_{L^2(\mathcal{O})}ds\bigg)\\
&+C\mathbb{E}\int_0^\tau\|u^1(s)-u^2(s)\|^2_{L^2(\mathcal{O})}ds.
\end{aligned}
\tag{5.20}
$$

On the other hand, to deal with the last term of (5.19) we invoke Burkholder's inequality as to so obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le\tau}\bigg|\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}}\big(g_k(x,u^1)-g_k(x,u^2)\big)\big(Ku^1-Ku^2\big)\,dx\,d\beta_k(s)\bigg|\\
\le&C\mathbb{E}\left[\left(\int_0^\tau\sum_{k=1}^{\infty}\left\{\int_{\mathcal{O}}\big(g_k(x,u^1)-g_k(x,u^2(s))\big)\big(Ku^1-Ku^2\big)\,dx\right\}^2ds\right)^{\frac{1}{2}}\right]\\
\le&\frac{1}{4}\mathbb{E}\sup_{0\le t\le\tau}\bigg(\|(Ku^1)(t)-(Ku^2)(t)\|^2_{L^2(\mathcal{O})}
\end{aligned}
$$

$$
\begin{aligned}
&+2\mu\int_0^t \big\|\Delta(Ku^1)(s)-\Delta(Ku^2)(s)\big\|^2_{L^2(\mathcal{O})}dxds\Big)\\
&+C\mathbb{E}\int_0^\tau \big\|u^1(s)-u^2(s)\big\|^2_{L^2(\mathcal{O})}ds. \qquad (5.21)
\end{aligned}
$$

Consequently, inserting (5.20)-(5.21) into (5.19) gives

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le\tau}\Big(\big\|(Ku^1)(t)-(Ku^2)(t)\big\|^2_{L^2(\mathcal{O})}+2\mu\int_0^t\big\|\Delta(Ku^1)(s)-\Delta(Ku^2)(s)\big\|^2_{L^2(\mathcal{O})}ds\Big)\\
&\le C_*\mathbb{E}\int_0^\tau\big\|u^1(s)-u^2(s)\big\|^2_{L^2(\mathcal{O})}ds \qquad (5.22)
\end{aligned}
$$

for some constant $C_*>0$ independent of $u^1$ and $u^2$. On the basis of (5.22), we will introduce the equivalent norm in $\mathcal{E}$ given by

$$
\|u\|^2_{*\mathcal{E}}=\sup_{0\le\tau\le T}e^{-C_*\frac{t}{\alpha}}\mathbb{E}\sup_{0\le t\le\tau}\Big(\|u(t)\|^2_{L^2(\mathcal{O})}+2\mu\int_0^t\|\Delta u(s)\|^2_{L^2(\mathcal{O})}ds\Big),
$$

where $0<\alpha<1$ may be arbitrarily chosen. Since

$$
\begin{aligned}
&C_*\mathbb{E}\int_0^\tau\big\|u^1(s)-u^2(s)\big\|^2_{L^2(\mathcal{O})}ds\\
&\le C_*\int_0^\tau e^{C_*\frac{s}{\alpha}}e^{-C_*\frac{s}{\alpha}}\mathbb{E}\sup_{0\le t\le s}\big\|u^1(t)-u^2(t)\big\|^2_{L^2(\mathcal{O})}ds\\
&\le\alpha e^{C_*\frac{\tau}{\alpha}}\big\|u^1-u^2\big\|^2_{*\mathcal{E}},
\end{aligned}
$$

(5.22) implies that

$$
\big\|Ku^1-Ku^2\big\|^2_{*\mathcal{E}}\le\alpha\big\|u^1-u^2\big\|^2_{*\mathcal{E}}.
$$

This proves that $K$ is a contraction and thus the desired result. □

At last, all is left to finish the proof of Theorem 5.1 is the energy estimate (5.10), whose justification we provide below.

**Lemma 5.8.** *Fixing $\varepsilon>0$, denote by $u^{\varepsilon,\mu}\in\mathcal{E}$ the solution given by Lemma* 5.7. *Then there exists a constant $C$, independent of $\mu>0$, for which* (5.10) *indeed holds.*

*Proof.* Similar to the proof of Lemma 5.6, we note that the relation $u^{\varepsilon,\mu}-\widetilde{u_b^\varepsilon}\in L^2(\Omega\times[0,T];H_0^2(\mathcal{O}))$ guarantees that it holds a.s. and for all $0\le t\le T$

$$
\int_{\mathcal{O}}\big|u^{\mu,\varepsilon}(t,x)-\widetilde{u_b^\varepsilon}(t,x)\big|^2dx+2\mu\int_0^t\int_{\mathcal{O}}\big|\Delta u^{\mu,\varepsilon}(s,x)-\Delta\widetilde{u_b^\varepsilon}(s,x)\big|^2dxds
$$

$$
\begin{aligned}
&+2\varepsilon\int_0^t\int_{\mathcal{O}}\left|\nabla u^{\mu,\varepsilon}(s,x)-\nabla\widetilde{u_b^\varepsilon}(s,x)\right|^2dxds\\
&+2\int_0^t\int_{\mathcal{O}}\mathbf{B}'\left(u^{\mu,\varepsilon}(s,x)\right)\nabla\left(u^{\mu,\varepsilon}(s,x)-\widetilde{u_b^\varepsilon}(s,x)\right)\cdot\nabla\left(u^{\mu,\varepsilon}(s,x)-\widetilde{u_b^\varepsilon}(s,x)\right)dxds\\
=&-2\int_0^t\int_{\mathcal{O}}\mathbf{B}'\left(u^{\mu,\varepsilon}(s,x)\right)\nabla\widetilde{u_b^\varepsilon}(s,x)\cdot\nabla\left(u^{\mu,\varepsilon}(s,x)-\widetilde{u_b^\varepsilon}(s,x)\right)dxds\\
&+2\int_0^t\int_{\mathcal{O}}\mathbf{A}\left(u^{\mu,\varepsilon}(s,x)\right)\cdot\nabla\left(u^{\mu,\varepsilon}(s,x)-\widetilde{u_b^\varepsilon}(s,x)\right)dxds\\
&+2\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}}g_k\left(x,u^{\mu,\varepsilon}(s,x)\right)\left(u^{\mu,\varepsilon}(s,x)-\widetilde{u_b^\varepsilon}(s,x)\right)dxd\beta_k(s)\\
&+\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}}\left|g_k(x,u^{\mu,\varepsilon}(s,x))\right|^2dxds.
\end{aligned}
$$

Recalling that $\mathbf{A}=\mathbf{A}^\varepsilon$, $\mathbf{B}=\mathbf{B}^\varepsilon$ and $g_k=g_k^\varepsilon$ all have compact support in $\xi$, with $g_k^\varepsilon\equiv 0$ for $k>1/\varepsilon$, and taking into account (5.17), we see that (5.10) can be established using Young inequality with $\varepsilon$ and the Burkholder inequality, as it was done in the proof of Lemma 5.7; also using the fact that $\mathbf{B}'(u^{\mu,\varepsilon})$ is positive semi-definite. $\square$

## 5.3 Proof of Theorem 5.2.

Before we turn into the limit $\mu\to 0$, let us first study the properties of solutions $u\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H^1(\mathcal{O}))$ to the nondegenerate problem (5.1)-(5.3). Let us begin by showing that this equation possesses an entropy, thus also a kinetic formulation, and that a relative entropy estimate is available.

**Proposition 5.1.** *Let $\varphi\in C_c^1(\mathcal{O})$ and $\eta\in C^2(\mathbb{R})$ with $\eta''\in L^\infty(\mathbb{R})$.*

1. *If $u\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H^1(\mathcal{O}))$ solves* (5.1)-(5.3), *then, a.s. for all $0\le t\le T$,*

$$
\begin{aligned}
&\int_{\mathcal{O}}\eta\left(u(t,x)\right)\varphi(x)dx\\
=&\int_{\mathcal{O}}\eta(u_0^\varepsilon)\varphi dx+\int_0^t\int_{\mathcal{O}}\mathbf{A}^\varepsilon(u)\cdot\nabla\left(\eta'(u)\varphi\right)dxds\\
&-\int_0^t\int_{\mathcal{O}}\operatorname{div}\mathbf{B}^\varepsilon(u)\cdot\nabla\left(\eta'(u)\varphi\right)dxds+\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}}\eta'(u)g_k(x,u)\varphi dxd\beta_k(s)\\
&+\frac{1}{2}\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}}\eta''(u)g_k(x,u)^2\varphi dxds. \qquad (5.23)
\end{aligned}
$$

2. *Assume that* $v\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H^1(\mathcal{O}))$ *solves* (5.1)-(5.3) *but with possibly different initial* $v(0,x)=v_0^\varepsilon(x)$ *and boundary* $v(t,x)_{|(0,T)\times\partial\mathcal{O}}=v^b(t,x)$ *datum. Then, a.s. for all* $0\le t\le T$,

$$\begin{aligned}\int_{\mathcal{O}}\eta\big(u(t,x)-v(t,x)\big)\varphi(x)dx=&\int_{\mathcal{O}}\eta\big(u_0^\varepsilon(x)-v_0^\varepsilon(x)\big)\varphi(x)dx\\&+\int_0^t\int_{\mathcal{O}}\big(\mathbf{A}^\varepsilon(u)-\mathbf{A}^\varepsilon(v)\big)\cdot\nabla\big(\eta'(u-v)\varphi\big)dxds\\&-\int_0^t\int_{\mathcal{O}}\operatorname{div}\big(\mathbf{B}^\varepsilon(u)-\mathbf{B}^\varepsilon(v)\big)\cdot\nabla\big(\eta'(u-v)\varphi\big)dxds\\&+\sum_{k=1}^\infty\int_0^t\int_{\mathcal{O}}\eta'(u-v)\big(g_k(x,u)-g_k(x,v)\big)\varphi dxd\beta_k(s)\\&+\frac12\sum_{k=1}^\infty\int_0^t\int_{\mathcal{O}}\eta''(u^\varepsilon-v^\varepsilon)\big(g_k(x,u)-g_k(x,v)\big)^2\varphi dxds.\end{aligned}\tag{5.24}$$

*Proof.* This is nothing more than a localized version of the generalized Itô's formula proven in [14]. □

**Remark 5.2.** Let us point out that if $\eta'(u-v)\in L^2(\Omega\times[0,T];\,L^2(\mathcal{O}';H_0^1(\mathcal{O})))$, then in (5.24), $\varphi$ can be chosen to be in $C^1(\overline{\mathcal{O}})$. This can be shown by standard Sobolev spaces arguments (see, e.g., the proof of [16, Theorem 2, Section 5.5]).

As a result, we can prove the following comparison principle.

**Lemma 5.9.** *Let* $u$ *and* $v\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H^1(\mathcal{O}))$ *be two solutions to* (5.1)-(5.3) *with possibly different initial* $u(0)=u_0^\varepsilon$ *and* $v(0)=v_0^\varepsilon\in L^2(\Omega;L^2(\mathcal{O}))$ *and boundary* $u_{|(0,T)\times\partial\mathcal{O}}=u^b$ *and* $v_{|(0,T)\times\partial\mathcal{O}}=v^b\in L^2(\Omega\times[0,T];H^{1/2}(\mathcal{O}))$ *datum. Then, if* $u^b\le v^b$ *a.s. in the sense of the distributions,*

$$\mathbb{E}\int_{\mathcal{O}}\big(u(t,x)-v(t,x)\big)_+dx\le\mathbb{E}\int_{\mathcal{O}}\big(u_0^\varepsilon(x)-v_0^\varepsilon(x)\big)_+dx\tag{5.25}$$

*for all* $0\le t\le T$.

*Proof.* Consider $\psi\in C_c^\infty(-\infty,\infty)$ to be such that $\psi\ge0$, $\operatorname{supp}\psi\subset(-1,1)$ and

$$\int_{-\infty}^\infty\psi(t)dt=1.$$

If

$$\psi_\delta(\xi)=\frac1\delta\psi(\delta^{-1}\xi),\quad\delta>0,$$

then put

$$\mathrm{sgn}_\delta^+(\xi) = \int_{-\infty}^{t} \psi_\delta(\zeta - \delta)\,d\zeta = \int_{-\infty}^{\xi} \psi\left(\frac{\zeta - 1}{\delta}\right)\frac{d\zeta}{\delta}.$$

Define also

$$\eta_\delta(\xi) := \int_{-\infty}^{\xi} \mathrm{sgn}_\delta^+(\zeta)\,d\zeta.$$

Notice that $\eta_\delta$ is a smooth convex approximation of the "positive part" function $\xi \mapsto \xi_+$.

Since $u^b \le v^b$, it is not hard to see that $\eta_\delta'(u-v) \in L^2(\Omega \times [0,T];\, H_0^1(\mathcal{O}))$. Consequently, as noted in Remark 5.2, we are allowed to choose $\varphi \equiv 1$ on (5.24). The proof of the lemma will be then completed by passing $\delta \to 0$.

First of all, take the expected value in (5.24) to eliminate the term involving the stochastic integrals. Next, let us consider the diffusive term featuring $\mathbf{B}^\varepsilon$. Notice that it may be written as

$$\begin{aligned} &-\int_0^t \int_{\mathcal{O}} \eta_\delta''(u-v)(\nabla u - \nabla v)\cdot(\mathbf{B}^\varepsilon)'(u)(\nabla u - \nabla v)\,dx\,ds \\ &\quad - \int_0^t \int_{\mathcal{O}} \eta_\delta''(u-v)(\nabla u - \nabla v)\cdot\big((\mathbf{B}^\varepsilon)'(u) - (\mathbf{B}^\varepsilon)'(v)\big)\nabla v\,dx\,ds = -(\mathrm{i}) + (\mathrm{ii}) \end{aligned}$$

(recall that $\varphi \equiv 1$). Of course $(\mathrm{i}) \ge 0$, whereas

$$(\mathrm{ii}) \le \|(\mathbf{B}^\varepsilon)'\|_\infty \int_0^t \int_{\mathcal{O}} \psi\left(\frac{u-v-1}{\delta}\right)\left|\frac{u-v}{\delta}\right| |\nabla u - \nabla v|\,|\nabla v|\,dx\,ds.$$

The integrand above is uniformly bounded by an $L^1$–function and converges pointwisely to 0. Hence $\mathbb{E}(\mathrm{ii}) = o(1)$.

Likewise the hyperbolic term is $o(1)$. Finally, for the last term, we notice that

$$\begin{aligned} &\frac{1}{2}\int_0^t \int_{\mathcal{O}} \eta_\delta''(u-v) \sum_{k=1}^{\infty} |g_k(x,u) - g_k(x,v)|^2\,dx\,ds \\ &\le D\delta \int_0^t \int_{\mathcal{O}} \psi\left(\frac{u-v-1}{\delta}\right)\left(\frac{u-v}{\delta}\right)^2 dx\,ds = \mathcal{O}(\delta). \end{aligned}$$

Gathering all these remarks, we see that (5.25) is established letting $\delta \to 0$. The lemma is proved. $\square$

With Lemma 5.9 at hand, we can deduce a maximum principle, a uniqueness property, and an $L^\infty$–bound related to (5.1)-(5.3). The proof is immediate once we notice that the constant functions $u_{\min}$ and $u_{\max}$ are solutions to (5.1)-(5.3).

**Theorem 5.5.** *Let $u$ and $v \in L^2(\Omega;C([0,T];L^2(\mathcal{O}))) \cap L^2(\Omega\times[0,T];H^1(\mathcal{O}))$ be two solutions to* (5.1)-(5.3) *with possibly different initial $u(0)=u_0^\varepsilon$ and $v(0)=v_0^\varepsilon \in L^2(\Omega; L^2(\mathcal{O}))$ and boundary $u_{|(0,T)\times\partial\mathcal{O}}=u^b$ and $v_{|(0,T)\times\partial\mathcal{O}}=v^b \in L^2(\Omega\times[0,T];H^{1/2}(\mathcal{O}))$ data. The following conclusions hold.*

1. *(Comparison principle). If $u^b\le v^b$ and $u_0^\varepsilon\le v_0^\varepsilon$ a.s. in the sense of the distributions, then $u\le v$ a.s. in the sense of distributions in $(0,T)\times\mathcal{O}$.*
2. *(Uniqueness). If $u^b\equiv v^b$ and $u_0^\varepsilon\equiv v_0^\varepsilon$, then $u\equiv v$.*
3. *($L^\infty$–bound). If $u_{\min}\le u^b, u_0\le u_{\max}$ a.s. in the sense of distributions, then $u_{\min}\le u\le u_{\max}$ a.s. in the sense of distributions in $(0,T)\times\mathcal{O}$.*

Furthermore, regarding the regularity properties claims in the statement of Theorem 5.2, all that is left is the following energy estimate.

**Lemma 5.10.** *Let $u^\varepsilon \in L^2(\Omega;C([0,T];L^2(\mathcal{O}))) \cap L^2(\Omega\times[0,T];H^1(\mathcal{O}))$ be a solution to* (5.1)-(5.3)*, where $u_{\min}\le u_0$, $u^b\le u_{\max}$ a.s. and with $u_b\in L^2(\Omega\times[0,T];H^2(\partial\mathcal{O}))\cap L^2(\Omega;(H^1((0,T));L^2(\partial\mathcal{O})))$ being predictable. Then*

$$\mathbb{E}\int_0^T\int_{\mathcal{O}}|\nabla b_\varepsilon(u^\varepsilon(t,x))|^2\,dx\,ds\le C \tag{5.26}$$

*for some constant $C$, which depends on $a$, $b$, $u^b$, but not on $0<\varepsilon<1$.*

*Proof.* The argument is quite similar to the proof of Lemma 5.8; we will only point out the differences. Consider now $z\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H^2(\mathcal{O}))$ be the solution to

$$\begin{cases} dz-\Delta z\,dt=\Phi(z)dW(t) & \text{in } \{0<t<T\}\times\mathcal{O},\\ z(0,x)=w_0(x) & \text{on } \{t=0\}\times\partial\mathcal{O},\\ z(t,x)=u_b(t,x) & \text{on } \{0<t<T\}\times\partial\mathcal{O},\end{cases}$$

where $w_0\in L^2(\Omega;H^2(\mathcal{O}))$ is a suitable extension of $u_b|_{t=0}$ to $\partial\mathcal{O}$ (see Appendix A). Notice that $u_{\min}\le w_0\le u_{\max}$ and, thus, $u_{\min}\le f\le u_{\max}$ a.s. as well.

Evidently, we can rewrite the heat equation observed by $f$ as

$$dz-D^2:\mathbf{B}^\varepsilon(z)dt+\nabla\cdot\mathbf{A}^\varepsilon(z)dt=\Phi(z)dW(t)+F_\varepsilon dt$$

with $F_\varepsilon=\Delta z-D^2:\mathbf{B}^\varepsilon(z)+\nabla\cdot\mathbf{A}^\varepsilon(z)$, which is bounded in $L^2(\Omega\times[0,T];L^2(\mathcal{O}))$ for $0<\varepsilon<1$. Hence, reprising the arguments of the proof of Proposition 5.1 and using

Remark 5.2, the fact that $u^\varepsilon - z \in L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$ says that, choosing $\eta(\xi)=\xi^2$ and $\varphi\equiv 1$,

$$\begin{aligned}
&\int_{\mathcal{O}} |u^\varepsilon(t,x)-z(t,x)|^2 dx + \int_0^t\int_{\mathcal{O}} \operatorname{div}\big(\mathbf{B}^\varepsilon(u^\varepsilon)-\mathbf{B}^\varepsilon(z)\big)\cdot\nabla(u^\varepsilon-z)\,dxds\\
=&\int_{\mathcal{O}} |u_0^\varepsilon(x)-w_0(x)|^2 dx + \int_0^t\int_{\mathcal{O}} \big(\mathbf{A}^\varepsilon(u^\varepsilon)-\mathbf{A}^\varepsilon(z)\big)\cdot\nabla(u^\varepsilon-z)\,dxds\\
&+\sum_{k=1}^\infty \int_0^t\int_{\mathcal{O}} (u^\varepsilon-z)\big(g_k(x,u^\varepsilon)-g_k(x,z)\big)\,dxd\beta_k(s)\\
&+\frac12\sum_{k=1}^\infty \int_0^t\int_{\mathcal{O}} |g_k(x,u^\varepsilon)-g_k(x,z)|^2\,dxds\\
&-\int_0^t\int_{\mathcal{O}} F_\varepsilon(u^\varepsilon-z)\,dxds \quad \text{a.s. and for all} \quad 0\le t\le T.
\end{aligned}$$

Because of (1.5), the $L^\infty$–bound for $u^\varepsilon$ and $f$, and the fact that $z\in L^2(\Omega\times[0,T];H^2(\mathcal{O}))$, some algebraic manipulations and applications of the Gauss-Green theorem on the equation above provides an estimate for

$$\mathbb{E}\int_0^t [(b_\varepsilon)'(u^\varepsilon)]^2|\nabla u^\varepsilon|^2 dxds = \mathbb{E}\int_0^t |\nabla b_\varepsilon(u^\varepsilon)|^2 dxds$$

which is uniform on $0<\varepsilon<1$. □

Finally, in order to obtain all ingredients necessary to prove Theorem 5.2, we need just another technical proposition.

**Proposition 5.2.** *If $u_b\in L^2(\Omega\times[0,T];H^2(\partial\mathcal{O}))\cap L^2(\Omega;(H^1((0,T));L^2(\partial\mathcal{O})))$ is predictable and $u\in L^2(\Omega;L^\infty(0,T;L^2(\mathcal{O})))\cap L^1(\Omega;C([0,T];H^{-4}(\mathcal{O})))\cap L^2(\Omega\times[0,T];H^1(\mathcal{O}))$ is a weak solution to* (5.1)-(5.4)*, then $u\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))$.*

*Proof.* The line of reasoning will be sketched, as it closely follows the ideas we expressed so far. By weak solution, we mean that, a.s.,

$$\begin{aligned}
&-\int_0^T\int_{\mathcal{O}} u\frac{\partial\varphi}{\partial t}\,dxdt + \int_{\mathcal{O}} u_0^\varepsilon\varphi(0)\,dx\\
=&\int_0^T\int_{\mathcal{O}} \mathbf{A}^\varepsilon(u)\cdot\nabla\varphi\,dxdt - \int_0^T\int_{\mathcal{O}} \operatorname{div}\mathbf{B}^\varepsilon(u)\cdot\nabla\varphi\,dxdt + \sum_{k=1}^\infty\int_0^T g_k(x,u)\varphi\,dxd\beta_k(s)
\end{aligned}$$

for any $\varphi\in C_c^\infty((-\infty,T)\times\mathcal{O})$, and that $u_{|_{(0,T)\times\partial\mathcal{O}}}=u^b$ in the sense of traces. If $z$ is as in the proof of the previous lemma, it is not hard to see that $v=u-z$ has a representative that can be expressed via the Duhamel formula (see, e.g., [13])

$$v(t)=Z(t)(u_0^\varepsilon-w_0)+(1-\varepsilon)\int_0^t Z(t-s)\Delta z(s)\,ds-\int_0^t Z(t-s)\big(\operatorname{div}\mathbf{A}(u(s))\big)\,ds$$
$$+\int_0^t Z(t-s)\big(\operatorname{div}\operatorname{div}\mathbf{B}(u(s))\big)\,ds+\int_0^t Z(t-s)\big(\Phi(u)-\Phi(z)\big)\,dW(s),$$

where $Z(t)=\exp\{-Bt\}$ is now the semigroup associated to the Dirichlet Laplacian

$$\begin{cases} D(B)=H^2(\mathcal{O})\cap H_0^1(\mathcal{O})\subset L^2(\mathcal{O}),\\ Bu=-\varepsilon\Delta u. \end{cases}$$

Maintaining the notations of the Appendix B, it is not hard to see that $H_B^{1/2}=H_0^1(\mathcal{O})$, so that $H_B^{-1/2}=H^{-1}(\mathcal{O})$. As a result, identical arguments employed in the last subsection yield that $v\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))$. As $z\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))$ and $u=v+f$, the result now follows. □

*Proof of Theorem 5.2:* Essentially, the proof is now indistinguishable from the one in [14, Section 4.3]. Below is a brief summary of the most crucial ideas.

Per the energy estimate (5.10) and the Kolmogorov continuity criterion, one can show that the laws of $(u^{\mu,\varepsilon})_{0<\mu<1}$ (with $0<\varepsilon<1$ kept fixed) are tight in $L^2(0,T;L^2(\mathcal{O}))\cap C([0,T];H^{-4}(\mathcal{O}))\stackrel{\text{def}}{=}X$. Thus, using the Prohorov and Skohorod theorems, one can thus construct a weak martingale solution $\widetilde{u}\in L^2(\widetilde{\Omega};L^\infty(0,T;L^2(\mathcal{O})))\cap L^1(\widetilde{\Omega};C([0,T];H^{-4}(\mathcal{O})))\cap L^2(\widetilde{\Omega}\times[0,T];H^1(\mathcal{O}))$ to (5.1)-(5.3).

On the other hand, Proposition 5.2 asserts that $\widetilde{u}\in L^2(\widetilde{\Omega};C([0,T];L^2(\mathcal{O})))$, hence, according to Theorem 5.5, it is the unique martingale solution in the class $L^2(\widetilde{\Omega};C([0,T];L^2(\mathcal{O})))\cap L^2(\widetilde{\Omega}\times[0,T];H^1(\mathcal{O}))$. As a byproduct of this method of constructing martingale solutions and their a priori uniqueness, one can invoke the Gyöngi-Krylov lemma to deduce that, as $\mu\to 0$, $u^{\mu,\varepsilon}$ converges in probability in $X$ to the unique weak solution $u^\varepsilon\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H^1(\mathcal{O}))$ to (5.1)-(5.3).

Once the (5.11) were verified in Theorem 5.5 and Lemma 5.10, the proof of Theorem 5.2 is hereby concluded. □

# 6 Existence part 2: degenerate case

We finally discuss the existence of a kinetic solution to problem (1.1)-(1.3). Here, we follow the compactness argument in [25], with the decisive help of the space regularity result established in [21]. We use the Yamada-Watanabe scheme [45], with application of Gyöngy-Krylov's criterion [22]. Concerning the latter, we recall that the uniqueness of the kinetic solution to problem (1.1)-(1.3) has been established in Theorem 3.3. Now, let $u^\varepsilon$ be the solution to the problem (5.1)-(5.3) and let $\mathfrak{f}^\varepsilon(t,x,\xi) := \chi^\varepsilon(t,x,\xi) = 1_{(-\infty,u^\varepsilon(t,x))}(\xi) - 1_{(-\infty,0)}(\xi)$. We can prove, by an argument similar to the one in [14], that $\mathfrak{f}^\varepsilon$ satisfies

$$\begin{aligned}
&\partial_t \mathfrak{f}^\varepsilon + \mathbf{a}(\xi)\cdot\nabla_x \mathfrak{f}^\varepsilon + \mathbf{b}^\varepsilon(\xi) : D_x^2 \mathfrak{f}^\varepsilon \\
&= \left( m^\varepsilon - \frac{1}{2} G^2(x,\xi)\delta_{u^\varepsilon(t,x)}(\xi) \right)_\xi + \sum_{k=1}^\infty g_k(x,\xi)\dot\beta_k(t)\delta_{u^\varepsilon(t,x)}(\xi) \\
&= q^\varepsilon_\xi - \sum_{k=1}^\infty g_k(x,\xi)(\partial_\xi \mathfrak{f}^\varepsilon)\dot\beta_k(t) + \sum_{k=1}^\infty \delta_0(\xi) g_k(x,\xi)\dot\beta_k(t), 
\end{aligned} \tag{6.1}$$

where

$$\begin{aligned}
&\mathbf{b}^\varepsilon(\xi) := \mathbf{b}(\xi) + \varepsilon I_{d\times d}, \\
&q^\varepsilon = m^\varepsilon - \frac{1}{2} G^2(x,\xi)\delta_{u^\varepsilon(t,x)}(\xi), \\
&dm^\varepsilon(t,x,\xi) = |\sigma^\varepsilon(u^\varepsilon)\nabla u^\varepsilon|^2 d\delta_{u^\varepsilon=\xi}\, dx\, d\xi,
\end{aligned}$$

and $\sigma^\varepsilon(u) \in \mathbb{M}^d$ is such that $\sigma^\varepsilon(u)^2 = \mathbf{b}(u) + \varepsilon I_{d\times d}$. Reasoning as in [21], we see that the symbol

$$\mathcal{L}_0^\varepsilon(i\tau, i\kappa, \xi) := i\big(\tau + \mathbf{a}(\xi)\cdot\kappa\big) + \kappa^\top \mathbf{b}^\varepsilon(\xi)\kappa,$$

$(\tau,\kappa) \in \mathbb{R}\times\mathbb{R}^d$ satisfies condition (1.8), uniformly in $\varepsilon$. Moreover, given any $V \Subset \mathcal{O}$ and $\phi \in C_c^\infty(\mathcal{O})$, with $\phi(x) = 1$, for $x \in V$, we see that $\mathfrak{f}^{\phi,\varepsilon} := \phi \mathfrak{f}^\varepsilon$ satisfies

$$\begin{aligned}
&\partial_t \mathfrak{f}^{\phi,\varepsilon} + \mathbf{a}(\xi)\cdot\nabla_x \mathfrak{f}^{\phi,\varepsilon} + \mathbf{b}^\varepsilon(\xi) : D_x^2 \mathfrak{f}^{\phi,\varepsilon} \\
&= q^{\phi,\varepsilon}_\xi + \sigma^\varepsilon(\xi)\nabla_x\phi\cdot\sigma^\varepsilon(\xi)\nabla_x \mathfrak{f}^\varepsilon + \sum_{k=1}^\infty \phi(x) g_k(x,\xi)(\partial_\xi \mathfrak{f}^\varepsilon)\dot\beta_k(t) \\
&\quad + \sum_{k=1}^\infty \delta_0(\xi)\phi(x) g_k(x,\xi)\dot\beta_k(t),
\end{aligned} \tag{6.2}$$

where

$$q^{\phi,\varepsilon} = \phi\left( m^\varepsilon - \frac{1}{2} G^2(x,\xi)\delta_{u^\varepsilon(t,x)}(\xi) \right) - \mathfrak{f}^\varepsilon \mathbf{a}(\xi)\cdot\nabla_x\phi.$$

Now, we also have that $n^{\phi,\varepsilon} := \sigma^\varepsilon(\xi)\nabla_x\phi\cdot\sigma^\varepsilon(\xi)\nabla_x \mathfrak{f}^\varepsilon$ are a.s. finite total variation measures on $(0,T)\times\mathcal{O}\times(-M,M)$ such that $\mathbb{E}|n^{\phi,\varepsilon}| \le C$ because of (1.6), (1.8) and Lemma 5.10. After extending $\mathfrak{f}^{\phi,\varepsilon}$ periodically in the space variable $x$ with a period

$\Pi \supset \operatorname{supp}\phi$, we can apply the averaging lemma by Gess and Hofmanová in [21] to deduce that

$$\|u^\varepsilon\|_{L^r(\Omega\times(-T_0,T_0);W^{s,r}(V))} \leq C_V \tag{6.3}$$

for some $C_V$ independent of $\varepsilon$ and some $1<r<2$, $0<s<1$, for any $V \Subset \mathcal{O}$. Again, using Kolmogorov's continuity theorem as in [14, Proposition 4.4] we get that

$$\mathbb{E}\|u^\varepsilon\|_{C^\lambda([0,T];H^{-2}(\mathcal{O}))} \leq C \tag{6.4}$$

for any $\lambda \in (0,1/2)$ for some $C>0$, independent of $\varepsilon$. Define, $\mathcal{X}_u = L^2([0,T];L^2(\mathcal{O})) \cap C([0,T];H^{-3}(\mathcal{O}))$, $\mathcal{X}_W = C([0,T];\mathfrak{U}_0)$ and $\mathcal{X} = \mathcal{X}_u \times \mathcal{X}_W$. Let $\mu_{u^\varepsilon}$ be the law of $u^\varepsilon$ in $\mathcal{X}_u$, $\mu_W$ be the law of $W$ in $\mathcal{X}_W$, and $\mu_\varepsilon$ be the joint law of $(\mu_{u^\varepsilon},\mu_W)$ in $\mathcal{X}$. From (6.3) and (6.4), as in [20], we conclude the tightness of the $\mu_\varepsilon$ in $\mathcal{X}$, and so the pre-compactness of these laws in $\mathcal{X}$. Then one applies Skorokhod's representation theorem to obtain a new probability space $(\tilde{\Omega};\tilde{\mathbb{P}})$ and a subsequence of random variables $(\tilde{u}_{\varepsilon_j},\tilde{W}) : \tilde{\Omega} \to \mathcal{X}$, whose laws $\tilde{\mu}_{\varepsilon_j}$ are equivalent to $\mu_{\varepsilon_j}$ such that $\tilde{u}_{\varepsilon_j}$ converges in measure to some $\tilde{u} : \tilde{\Omega} \to \mathcal{X}$. In particular, $\tilde{u}_{\varepsilon_j}$ converges a.s. in $L^2((0,T)\times\mathcal{O}\times(-M,M))$ to a certain $\tilde{u} : \tilde{\Omega} \to \mathcal{X}_u$. We can then apply the consistency results in Section 4, Proposition 4.1 and Proposition 4.2, to get that the conditions in Definition 1.2 (iii) are satisfied by $\tilde{u}$. Then, one can reason as in [20, 25], to prove that $\tilde{u}$ is a martingale solution to (1.1)-(1.3), that is, $\tilde{u}$ is a kinetic solution of (1.1)-(1.4), with $\tilde{\Omega}$ and $\tilde{W}$ instead of $\Omega$ and $W$. Hence, because of the uniqueness of the kinetic solution of (1.1)-(1.3) established by Theorem 3.3, we may apply Gyönly-Krylov's criterion to conclude that the whole sequence $u^\varepsilon$ converges to a kinetic solution of (1.1)-(1.3), which concludes the prove of the existence of a kinetic solution to (1.1)-(1.3).

# Appendix A. A boundary value problem for a fourth order stochastic PDE

Let $u_b \in L^2(\Omega\times[0,T];H^4(\partial\mathcal{O})) \cap L^2(\Omega;H^1(0,T;L^2(\partial\mathcal{O}))) \cap L^4(\Omega\times[0,T];W^{1,4}(\mathcal{O}))$, be predictable. We claim that for $B \in \mathcal{B}$ satisfying (1.17), the problem

$$du_B = -\Delta^2 u_B dt + \Phi(u_B) dW(t), \quad x\in\mathcal{O}, \quad t\in(0,T), \tag{A.1}$$

$$u_B(0) = u_{B0}, \tag{A.2}$$

$$u_B(t)\big|_{\partial\mathcal{O}} = u_b(t), \tag{A.3}$$

$$\frac{\partial u_B}{\partial x_d}(t)\big|_{\partial\mathcal{O}\cap B} = 0, \tag{A.4}$$

where $u_{B0}$ is a smooth extension of $u_b(0,\cdot)$ to $B$ such that $\frac{\partial u_{B0}}{\partial x_d}|\partial\mathcal{O}\cap B=0$, has a strong solution $u_B\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H^4(\mathcal{O}))$. In other words, $u_b|\partial\mathcal{O}\cap B$ is the restriction to the boundary of $\mathcal{O}$ of a solution of (A.1)-(A.4).

In order to show this, first, we extend $u_b$ to some $\tilde{u}_B\in L^2(\Omega\times[0,T];H^4(\mathcal{O}))\cap L^2(\Omega;H^1((0,T);L^2(\mathcal{O})))$ such that $\frac{\partial\tilde{u}_B}{\partial x_d}(t)|\partial\mathcal{O}\cap B=0$ (by using partition of unity and setting $u_B(\bar{x},x_d)=u_b(\bar{x})$, for $x=(\bar{x},x_d)\in B\cap\overline{\mathcal{O}}$). In particular, we choose $u_{b0}=\tilde{u}_b(0,\cdot)$. Then, we take a solution $f$ of

$$\begin{aligned}
&\partial_t f=\Delta^2 f, \quad x\in\mathcal{O}, \quad t\in(0,T),\\
&f(0)=u_{B0},\\
&f(t)\big|\partial\mathcal{O}=u_b(t),\\
&\frac{\partial f}{\partial x_d}(t)\big|\partial\mathcal{O}\cap B=0.
\end{aligned}$$

For instance, we may take $f=\tilde{f}+\tilde{u}_B$, where $\tilde{f}$ satisfies the equation

$$\begin{aligned}
&\partial_t\tilde{f}=\Delta^2\tilde{f}-\partial_t\tilde{u}_B+\Delta^2\tilde{u}_B, \quad x\in\mathcal{O}, \quad t\in(0,T),\\
&\tilde{f}(0)=0,\\
&\tilde{f}(t)|\partial\mathcal{O}=0,\\
&\frac{\partial\tilde{f}}{\partial\nu}=0,
\end{aligned}$$

which has a solution $\tilde{f}\in L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H_0^2(\mathcal{O})\cap H^4(\mathcal{O}))$, since $-\partial_t\tilde{u}_B+\Delta^2\tilde{u}_B\in L^2(\Omega;L^2([0,T]\times\mathcal{O}))$.

Then, a solution $u_B$ of problem (A.1)-(A.4) can be found as a fixed point of the operator

$$Kz(t)=\int_0^t S(t-s)\Phi\big(z(s)\big)dW(s)+f(t)$$

defined on the space

$$\mathcal{E}:=f+L^2\big(\Omega;C([0,T];L^2(\mathcal{O}))\big)\cap L^2\big(\Omega\times[0,T];H_0^2(\mathcal{O})\big),$$

where $S(t)$ is the semigroup generated by the operator $-\Delta^2$ with domain $H_0^2(\mathcal{O})\cap H^4(\mathcal{O})$.

Indeed, the fact that the operator is well defined and continuous may be shown as in the proof of Lemma 5.4. Also, as in the proof of Lemma 5.7 it is possible to define a convenient norm on $L^2(\Omega;C([0,T];L^2(\mathcal{O})))\cap L^2(\Omega\times[0,T];H_0^2(\mathcal{O}))$

so that the operator is contractive. This argument yields a weak solution of (A.1)-(A.4). Since we are looking for a strong solution of we have to show that the solution is more regular.

Assuming, additionally that $u_b \in L^4(\mathcal{O};L^2(0,T;W^{1,4}(\mathcal{O})))$ we have that the solution $u_B$ obtained above also belongs to the space $L^4(\Omega;C([0,T];\, W^{1,4}(\mathcal{O})))$. Indeed, this fact follows by the same lines as the proof of [26, Proposition 4.2], taking into account Theorem B.4.

Finally, we show that $u_B \in L^2(\mathcal{O}\times[0,T];H_0^2(\mathcal{O})\cap H^4(\mathcal{O}))$. It suffices to show that

$$\int_0^t S(t-s)\Phi\big(u_B(s)\big)dW(s)\in L^2\big(\mathcal{O}\times[0,T];H_0^2(\mathcal{O})\cap H^4(\mathcal{O})\big).$$

As shown in Appendix B, we have that the operator $A=\Delta^2$ with domain $D(A)=H_0^2(\mathcal{O})\cap H^4(\mathcal{O})$ is a positive self-adjoint operator (in particular, 0 belongs to the resolvent of $A$). Moreover, in the notations of Appendix B, we have that $A^\beta$ is a linear isometry between its intermediate spaces $H_A^\alpha$ and $H_A^{\alpha-\beta}$, for any $\alpha,\beta\in\mathbb{R}$. Now, as shown in the proof of Theorem B.6 we have, in particular, that $H_A^{1/2}=H_0^2(\mathcal{O})$ and $H_A^{-1/2}=H^{-2}(\mathcal{O})$. Now, given $z\in\mathcal{E}$ we have that

$$\int_0^t S(t-s)\Phi\big(z(s)\big)dW(s)=A^{-\frac{1}{2}}\int_0^t S(t-s)A^{\frac{1}{2}}\Phi(z)dW(s), \tag{A.5}$$

where, we are considering $A^{1/2}$ as an operator between the spaces $H_A^0$ and $H_A^{-1/2}$, that is, between $L^2(\mathcal{O})$ and $H^{-2}(\mathcal{O})$. Now, by virtue of (1.9) and since $u_B\in L^4(\Omega;C([0,T];\, W^{1,4}(\mathcal{O})))\cap L^2(\mathcal{O}\times[0,T];H_0^2(\mathcal{O}))$, we actually have that $\Phi(u_B)\in L^2(\Omega\times[0,T];L_2(\mathfrak{U};H^2(\mathcal{O})))$ (cf. [26, Proposition 3.1]). In particular, $A^{1/2}\Phi(u_B)$ may be represented by an element of $L^2(\Omega\times[0,T];L_2(\mathfrak{U};L^2(\mathcal{O})))$. Thus, from (A.5) and using Theorem B.4 we conclude that

$$\begin{aligned}&\int_0^t S(t-s)\Phi\big(u_B(s)\big)dW(s)\in L^2\big(\Omega;L^2(0,T;H_A^1)\big)\\ &:=L^2\big(\Omega;L^2(0,T;H_0^2(\mathcal{O})\cap H^4(\mathcal{O}))\big),\end{aligned}$$

which proves the claim.

Note that the boundary conditions (A.3) and (A.4) are satisfied in the sense of traces.

# Appendix B. The diagonalization technique

In this supplementary paragraph, we discourse on the so-called "diagonalization technique", which is nothing more than the application of the spectral theorem to trivialize the action of some operators. This procedure, which is essential for establishing the existence of solutions to the regularized problem (5.5)-(5.7), is classical and most notably delved into in the classic work of J.-L. Lions and E. Magenes [33].

## B.1 On nonnegative self-adjoint operators and their semigroups

First of all, let us recall the celebrated spectral theorem in its multiplicative operator form, whose statement we reproduce from [39].

**Theorem B.1.** *Let $A$ be a self-adjoint operator on a separable Hilbert space $H$ with domain $D(A)$. Then there is a measure space $(M,\mu)$ with $\mu$ a finite measure, a unitary operator $U:H\to L^2(M,d\mu)$, and a real-valued function $f$ on $M$ which is finite a.e. so that*

1. *$\psi\in D(A)$ if and only if $f(\cdot)(U\psi)(\cdot)\in L^2(M,d\mu)$.*

2. *If $\varphi\in U(D(A))$, then $(UAU^{-1}\varphi)(m)=f(m)\varphi(m)$.*

In the remainder of this section, we will preserve the notations and assumptions of this spectral theorem. Moreover, we will also assume that the operator $A$ is positive, allowing us to characterize its generated contraction semigroup $S(t)=\exp\{-tA\}$ by means of the operational calculus as

$$\big(U\exp\{-tA\}\psi\big)(m)=\exp\big\{-tf(m)\big\}(U\psi)(m).$$

The nonnegativity hypothesis also allows us to understand the "intermediate" spaces $H^\alpha_A:=D(A^\alpha)$ as

$$H^\alpha_A=U^{-1}\big(L^2(M,(1+f(m))^{2\alpha}d\mu)\big)=:U^{-1}(V^\alpha)$$

provided that $\alpha\geq 0$. We will endow these subspaces with the norm

$$\|u\|^2_{H^\alpha_A}=\int_M\big(1+f(m)\big)^{2\alpha}|(Uu)(m)|^2d\mu(m), \tag{B.1}$$

which is evidently equivalent to the graph norm $u\mapsto\sqrt{\|u\|^2_H+\|A^\alpha u\|^2_H}$. However, if $\alpha<0$, we will put $H_A^\alpha=(H_A^{-\alpha})^*$, which may still be naturally identified with $V^\alpha=L^2(M,(1+f(m))^{2\alpha}d\mu)$ through the duality form

$$[u,v]\mapsto\int_M(Uu)(m)\overline{(Uv)(m)}d\mu(m).$$

To conclude this section, let us state some regularizing effects associated with the propagator $S(t)=\exp\{tA\}$.

**Theorem B.2.** *For any real number $s\geq 0$ and $t>0$, the expression $A^sS(t)$ defines a bounded linear operator in $H$. Moreover,*

$$\|A^sS(t)\|_{\mathscr{L}(H)}\leq\frac{c_s}{t^s}. \tag{B.2}$$

*Consequently, for $\alpha<\beta$ and $t>0$, $S(t)$ is a bounded linear operator from $H_A^\alpha$ into $H_A^\beta$ whose norm may be majorized by*

$$\|S(t)\|_{\mathscr{L}(H_A^\alpha;H_A^\beta)}\leq c_{\alpha,\beta}\left(1+\frac{1}{t^{\beta-\alpha}}\right).$$

*Proof.* Given $u\in H$, the spectral Theorem B.1 implies that

$$[U(S(t)u)](m)=\exp\{-tf(m)\}(Uu)(m).$$

Thus, on account of the exponential decay of $\exp\{-tf(m)\}$, and on the fact that $f\geq 0$, it follows that $S(t)u\in H_A^s$. Furthermore, for $f(m)^{2s}\exp\{-2tf(m)\}\leq c_s/t^{2s}$, we have that

$$\begin{aligned}\|A^sS(t)u\|_H^2&=\int_M f(m)^{2s}\exp\{-2tf(m)\}|(Uu)(m)|^2d\mu(m)\\&\leq\frac{c_s}{t^{2s}}\int_M|(Uu)(m)|^2d\mu(m)=\frac{c_s}{t^{2s}}\|u\|_H^2.\end{aligned}$$

This essentially proves the result. ☐

The proof of the next two convolution inequalities may be found in a previous work of ours [20].

**Theorem B.3.** *For any $-\infty<\alpha<\infty$, define the Duhamel convolution operator*

$$(\mathcal{I}h)(t)=\int_0^t S(t-s)h(s)ds$$

*for* $h\in L^2(0,T;H_A^\alpha)$. *Then* $\mathcal{I}$ *maps* $L^2(0,T;H_A^\alpha)$ *into* $L^2(0,T;H_A^{\alpha+1})$ *and*

$$\int_0^T \|\mathcal{I}h(s)\|^2_{H_A^{\alpha+1}}ds \le C\int_0^T \|h(s)\|^2_{H_A^\alpha}ds \tag{B.3}$$

*for some absolute constant* $C$ *depending only on* $T$.

For the next theorem, which is probabilistic in nature, $(\Omega,\mathscr{F},(\mathscr{F})_{t\ge 0},\mathbb{P})$ will denote stochastic basis with a complete and right-continuous filtration. Moreover, $W$ will stand for a cylindrical Wiener process, i.e.,

$$W(t)=\sum_{k=1}^\infty \beta_k(t)e_k,$$

where the $\beta_k$ are mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)_{t\ge 0}$, and $(e_k)$ is an orthonormal basis of another separable Hilbert space $\mathfrak{U}$.

**Theorem B.4.** *For some* $-\infty<\alpha<\infty$, *assume that* $\Psi\in L^2((0,T)\times\Omega;L_2(\mathfrak{U};H_A^\alpha))$ *is predictable. Then, if* $(\mathcal{I}_W\Psi)(t)$ *is the stochastic convolution*

$$(\mathcal{I}_W\Psi)(t)=\int_0^t S(t-s)\Psi(s)dW(s),$$

*then* $\mathcal{I}_W\Psi\in L^2(\Omega\times[0,T];H_A^{\alpha+1/2})$ *and*

$$\|\mathcal{I}_W\Psi\|_{L^2(\Omega;L^2(0,T;H_A^{\alpha+1/2}))}\le C\|\Psi\|_{L^2(\Omega;L^2(0,T;L_2(\mathfrak{U};H_A^\alpha))} \tag{B.4}$$

*for some* $C>0$ *depending only on* $T$.

## B.2 The spectral analysis of an elliptic operator of fourth order

For $\mu>0$ and $\varepsilon\ge 0$, let us consider the unbounded operator $A:D(A)\subset L^2(\mathcal{O})\to L^2(\mathcal{O})$ given by

$$\begin{cases} D(A)=H_0^2(\mathcal{O})\cap H^4(\mathcal{O}),\\ Au=\mu\Delta^2u-\varepsilon\Delta u.\end{cases}$$

**Theorem B.5.** *The operator* $A$ *is self-adjoint and nonnegative.*

*Proof.* Since $C_c^\infty(\mathcal{O})\subset D(A)$, one can see that $A$ is densely defined. According to [33, Vol. 1, Chapter 2], (more specifically Theorem 5.1 and Remark 1.3), there exists a constant $C>0$ such that

$$\|u\|_{H^4(\mathcal{O})}\leq C\left(\|Au\|_{L^2(\mathcal{O})}+\|u\|_{L^2(\mathcal{O})}\right),$$

which immediately implies that $A$ is closed. Moreover, a simple integration by parts implies that $A$ is symmetric, strictly positive and, thus, injective.

Consequently, since the surjectivity of $A$ is equivalent to the solvability of boundary problem

$$\text{Given } f\in L^2(\mathcal{O}),\text{ find } u\in H^4(\mathcal{O}) \text{ such that}\begin{cases}\mu\Delta^2u-\varepsilon\Delta u=f & \text{in } \mathcal{O},\\ u=\dfrac{\partial u}{\partial \nu}=0 & \text{on } \partial\mathcal{O},\end{cases}$$

a Fredholm alternative-type theorem (see [33, Vol. 1, Chapter 2, Theorem 5.2]) asserts that $A:D(A)\to H$ is onto, since it is into. On the other hand, the bijectiveness of $A$ implies that $0\in\rho(A)$, i.e., a real number lies on the resolvent set of $A$, hence the self-adjointness of $A$ (see, e.g., the second corollary to [39, Vol. 2, Theorem X.1] or [40, Theorem 13.11 (b)]). $\square$

We may then apply the theory discussed so far to further study $A$. For instance, we have the following characterization of some of the intermediate spaces $H_A^\alpha$.

**Theorem B.6.** *Let $A$ be as in Theorem B.5 and let $H_A^\alpha$ be its interpolation spaces. Then, with equivalent norms*

$$\begin{cases}H_A^{\frac12}=H_0^2(\mathcal{O}),\\ H_A^{\frac14}=H_0^1(\mathcal{O}),\\ H_A^{-\frac12}=H^{-2}(\mathcal{O}),\\ H_A^{-\frac14}=H^{-1}(\mathcal{O}).\end{cases}$$

*Proof.* Let us substitute $A$ with a more simpler operator $B:D(B)\subset L^2(\mathcal{O})\to L^2(\mathcal{O})$, namely

$$\begin{cases}D(B)=H_0^2(\mathcal{O})\cap H^4(\mathcal{O}),\\ Bu=\mu\Delta^2u.\end{cases}$$

Notice that, by the very same Theorem B.5, $B$ is self-adjoint and nonnegative. Furthermore, since, for $0<\alpha<1$, $H_A^\alpha=[L^2(\mathcal{O}),D(A)]_\alpha$ is independent of the "spectral decomposition" $(M,\mu,f)$ (cf. [33]),

$$H_A^\alpha=\big[L^2(\mathcal{O}),D(A)\big]_\alpha=\big[L^2(\mathcal{O}),D(B)\big]_\alpha=H_B^\alpha.$$

In particular, $H_A^{1/2}=H_B^{1/2}$ and $H_A^{1/4}=H_B^{1/4}$ (with equivalent norms); thus, by duality, $H_A^{-1/2}=H_B^{-1/2}$ and $H_A^{-1/4}=H_B^{-1/4}$ (with equivalent norms). Hence, it suffices to prove the theorem for $B$.

**Step 1.** We first will prove that $H_B^{1/2}=H_0^2(\mathcal{O})$ with equivalent norms. Let us define the unbounded operator $T:D(T)\subset L^2(\mathcal{O})\to L^2(\mathcal{O})$ by

$$\begin{cases} D(T)=H_0^2(\mathcal{O}),\\ Tu=\mu^{\frac{1}{2}}\Delta u. \end{cases}$$

In light of the fact that $H_0^2(\mathcal{O})$ may be regarded as the closed linear subspace of the $u$'s in $H^2(\mathbb{R}^d)$ whose supports (in the sense of the distributions) are contained in $\mathcal{O}$, an argument involving the Fourier transform deduces that $T$ is a closed, densely defined, symmetric operator. Moreover, if $u\in D(B)=H_B^1$, we have that

$$\Big(B^{\frac{1}{2}}u,B^{\frac{1}{2}}u\Big)_{L^2}=\mu\int_{\mathcal{O}}\Delta^2u\overline{u}dx=\mu\int_{\mathcal{O}}|\Delta u|^2dx=(Tu,Tu)_{L^2}. \tag{B.5}$$

Thus, for

1. $H_B^1\subset H_B^{1/2}$ with continuous and dense injection;
2. $H_B^1\subset D(T)$ with continuous and dense injection (due to $H_0^2(\mathcal{O})$ being the closure of $C_c^\infty(\mathcal{O})$ under the $H^2-$norm),

the isometry relation (B.5) asserts that $u\in H_B^{1/2}\Longleftrightarrow u\in D(T)$. In this case, one has that $\|B^{1/2}u\|_{L^2}=\|Tu\|_{L^2}$, corroborating the equality $H_B^{1/2}=H_0^2(\mathcal{O})$.

**Step 2.** Next, let us show that $H_B^{1/4}=H_0^1(\mathcal{O})$. Since the interpolation spaces $[L^2(\mathcal{O}),H_B^{1/2}]_{1/2}$ and $H_B^{1/4}$ are independent of $(M,\mu,f)$, [33, Vol. 1, Chapter 1, Theorem 11.6], asserts that

$$H_B^{\frac{1}{4}}=\Big[L^2(\mathcal{O}),H_B^{\frac{1}{2}}\Big]_{\frac{1}{2}}=\big[L^2(\mathcal{O})H_0^2(\mathcal{O})\big]_{\frac{1}{2}}=H_0^1(\mathcal{O}),$$

as we wanted to show.

**Step 3.** Finally, the last two equalities for the negative exponents $s=-\frac{1}{2}$ and $-\frac{1}{4}$ are unmistakable duality relations.

The proof is complete. $\square$

## Acknowledgments

H. Frid gratefully acknowledges the support from CNPq (Grant 305097/2019-9) and FAPERJ (Grant E-26/202.900/2017). Y. Li gratefully acknowledges the support from NSF of China (Grants 11831011, 11571232) and also the partial support from the Institute of Modern Analysis-A Shanghai Frontier Research Center. D. Marroquin thankfully acknowledges the support from CNPq (Grant 150118/2018-0). J.F.C. Nariyoshi appreciatively acknowledges the support from CNPq (Grant 140600/2017-5) and from FAPESP (Grant 2021/01800-7). Z. Zeng gratefully acknowledges the support from NSF of China (Grant 11571232).

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