Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2022-0003

Vol. **1**, No. 2, pp. 319-344
May 2022

On a New Type of Chemotaxis Model with Acceleration

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Received 26 January 2022; Accepted 2 February 2022

Abstract. In this paper, we consider a new type of chemotaxis model with acceleration, which assumes that the advective acceleration, instead of velocity in the classical chemotaxis model, of species is proportional to the chemical signal concentration gradient. This new model has an additional equation governing the velocity field with more delicate boundary conditions. We show that this new type of chemotaxis model with acceleration precludes the blow-up and admits globally bounded solutions in two and three dimensions, in contrast to the classical chemotaxis models which may have blow-up in three dimensions or in two dimensions with a critical mass. Moreover, we numerically illustrate that this new type of chemotaxis model can typically generate aggregation patterns. Some open questions are discussed for further studies.

AMS subject classifications: 35M33, 35A01, 35B40, 35K57, 35Q92

Key words: Chemotaxis, acceleration, global boundedness, global stability, pattern formation.

1 Introduction

The reaction-diffusion systems are widely used to describe various biological processes such as propagation of genetics [5,22], sundry pattern formation [9,19,24],

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ecological invasions [2,4,12], tumour growth [3], wound healing [37], and so on. In these models, random diffusion was used as the only dispersal strategy [17,38], which however cannot explain some more complex ecological processes involving the rational movement (directed motion of individuals dispersing so as to increase the chance of survival [8]), nor accurately reflect the non-Brownian motion of individuals [34]. For example, for the Lotka-Volterra type predator-prey system, if the movement is assumed to be random diffusion then no spatial patterns shall be observed [15,44], which can not support the spatio-temporal heterogeneity patterns observed in the field experiment [17,41]. One common way to model the rational movement is the use of taxis mechanism, such as chemotaxis [18] or preytaxis [14,17]. A well-known classical chemotaxis model was the following so-called minimal model [28]:

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v), \\ \tau v_t = \Delta v - v + u, \end{cases} \tag{1.1}$$

where u and v denote the cell density and the concentration of the chemical signal emitted from cells, respectively, $\tau \in \{0,1\}$. The chemotaxis term $\nabla\cdot(u\nabla v)$ accounts for the rational movement. The preytaxis system also has the similar rational movement structure (cf. [14,17]). In these models, the advective velocity of species is assumed to be directly proportional to the concentration gradient of signals/resources, and the details of how species respond to signal concentration gradient are not reflected. On the other hand, there are many observations of the dependence of individual acceleration on the signal gradient, such as acceleration vectors of individuals in fish schools (cf. [30]) and in swarms of flying insects (cf. [32]) are directed towards the centroid of such dynamically stable formations, the moving flea-beetles modify their acceleration in response to food patch quality (cf. [16]) and individual fish in schools adjust their variation of velocity according to the difference between ambient and preferred temperatures (cf. [6]). In these observations, the directed movement of individuals is not determined by the velocity itself but by the velocity variation (i.e., acceleration) which is proportional to the resource gradient. This type of rational movement with advective velocity accelerating with respect to resource gradient has been mathematically modelled in the works [1,35] which found that the model can generate the spatio-temporal patterns consistent with the experimental observations while the conventional rational movement modelled by taxis directly can not do (cf. [15,44]). Such kind of models with acceleration incorporate more detailed responses of migrants to the signal instead of simply advecting to the signal/resource gradient. The purpose of this paper is to apply this type of rational movement to chemotaxis and explore what will be different from the classical chemotaxis model (1.1).

To this end, we consider the following model:

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\mathbf{w}), & x\in\Omega, \quad t>0, & (1.2a)\\ v_t = \Delta v - v + u, & x\in\Omega, \quad t>0, & (1.2b)\\ \tau\mathbf{w}_t = \varepsilon\Delta\mathbf{w} - \mathbf{w} + \chi\nabla v, & x\in\Omega, \quad t>0, & (1.2c)\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\Omega, & (1.2d) \end{cases}$$

where $\Omega\subset\mathbb{R}^n (n\geq 1)$ is a bounded domain with smooth boundary, $u(x,t)$ denotes the density of a species, $v(x,t)$ is the concentration of the signal emitted by the species, and $\mathbf{w}(x,t)$ denotes the species velocity. The model (1.2) entails that the species migrates with advective velocity $\mathbf{w}$ which is accelerated by the signal gradient with a small fluctuation of magnitude $\varepsilon>0$. When $\mathbf{w}$ is in its equilibrium without fluctuation ($\varepsilon=0$), then $\mathbf{w}=\gamma\nabla v$, which upon the substitution into Eq. (1.2a) gives rise to the minimal chemotaxis model (1.1). The purpose of this paper is to investigate whether the dynamics of chemotaxis system (1.2) with acceleration will have substantial differences from the classical/minimal chemotaxis model (1.1). It is well-known that the minimal chemotaxis system (1.1) with Neumann boundary conditions admits globally bounded classical solutions in one-dimensional space (cf. [11, 31]), but has a critical mass phenomenon in two dimensions: there is a number m_* such that the solution of (1.1) blows up in finite or infinite time if $\int_\Omega u_0 > m_*$ while globally exists if $\int_\Omega u_0 < m_*$, where $m_* = 8\pi$ if Ω is radially symmetric and $m_* = 4\pi$ otherwise. This was first proved in [25, 26] for the parabolic-elliptic case $\tau=0$, and then extended to the fully parabolic case $\tau=1$ in [10,13,26,27]. In three dimensions, the solution may blow up for any positive initial mass (cf. [25] for the parabolic-elliptic case $\tau=0$ and [43] for the full parabolic case $\tau=1$). The blow-up behavior exhibited in the classical chemotaxis model can not fully explain the typical chemotactic aggregation patterns which are condensed but do not explode. In this paper, among other things, we shall explore whether the new type of chemotaxis model (1.2) has similar or different behaviors compared to the classical minimal chemotaxis system (1.1).

To investigate the global well-posedness of (1.2), the suitable boundary conditions are more complicated than the classical chemotaxis model due to the presence of an additional velocity field. A natural one is the no-flux boundary condition (i.e. no individuals can cross the boundary)

$$\nabla u\cdot\mathbf{n} = \nabla v\cdot\mathbf{n} = \mathbf{w}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega,$$

where $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$. The above boundary conditions are comparable with the zero Neumann boundary conditions used for the minimal chemotaxis system (1.1). However, the mere condition $\mathbf{w}\cdot\mathbf{n}|_{\partial\Omega}=0$ for $\mathbf{w}$

seems inadequate to assert the global well-posedness of (1.2) (e.g., see [29]). In this paper, we shall consider the following strengthened boundary conditions:

$$\nabla u\cdot\mathbf{n}=\nabla v\cdot\mathbf{n}=\mathbf{w}\cdot\mathbf{n}=0,\quad \partial_{\mathbf{n}}\mathbf{w}\times\mathbf{n}=\mathbf{0}\quad \text{on } \partial\Omega, \tag{1.3}$$

where $\partial_{\mathbf{n}}\mathbf{w}:=(\partial_{\mathbf{n}}w_1,\partial_{\mathbf{n}}w_2,\cdots,\partial_{\mathbf{n}}w_n)$ for $\mathbf{w}=(w_1,w_2,\cdots,w_n)$, and as usual, for the vectors $\mathbf{a}=(a_1,a_2,\cdots,a_n)$ and $\mathbf{b}=(b_1,b_2,\cdots,b_n)$ in $\mathbb{R}^n$, the cross product of $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\mathbf{a}\times\mathbf{b}:=\begin{cases}0, & \text{if} \quad n=1,\\ a_1b_2-a_2b_1, & \text{if} \quad n=2,\\ (a_2b_3-a_3b_2,a_1b_3-a_3b_1,a_1b_2-a_2b_1) & \text{if} \quad n=3.\end{cases}$$

Indeed the boundary condition of $\mathbf{w}$ in (1.3) means that $\partial_{\mathbf{n}}\mathbf{w}$ is parallel to $\mathbf{n}$ on $\partial\Omega$, which along with $\mathbf{w}\cdot\mathbf{n}\,|_{\partial\Omega}=0$ implies that

$$\mathbf{w}\cdot\partial_{\mathbf{n}}\mathbf{w}=0,\quad x\in\partial\Omega. \tag{1.4}$$

We note that one can simply impose the zero Dirichlet boundary condition $\mathbf{w}|_{\partial\Omega}=0$, which indicates the boundary conditions in (1.3) for $\mathbf{w}$, but we want to keep the boundary condition for $\mathbf{w}$ as weak as possible. We also remark that the conditions in (1.3) are more or less equivalent to the Neumann boundary conditions. Let us take an example to look at this. Consider a disk $\Omega=B_r(0)$ with radius $r>0$. Then for the radial symmetric solution $(u,v,\mathbf{w})$ of (1.2)-(1.3), the boundary condition $\partial_{\mathbf{n}}\mathbf{w}\times\mathbf{n}\,|_{\partial B_r(0)}=\mathbf{0}$ is equivalent to $\nabla\times\mathbf{w}\,|_{\partial B_r(0)}=\mathbf{0}$ since $\partial_{\mathbf{n}}\mathbf{w}\times\mathbf{n}=-r^2(\nabla\times\mathbf{w})$ on $\partial B_r(0)$. Denoting the vorticity function $\omega:=\nabla\times\mathbf{w}$ for $(x,t)\in B_r(0)\times(0,\infty)$, then by (1.2)-(1.3) we have

$$\begin{cases}\tau\omega_t-\varepsilon\Delta\omega+\omega=0, & x\in B_r(0), \quad t>0, \qquad (1.5a)\\ \omega=0, & x\in\partial B_r(0), \quad t>0. \qquad (1.5b)\end{cases}$$

Multiplying Eq. (1.5a) by ω and using the boundary condition $\omega\,|_{\partial B_r(0)}=0$, one has

$$\tau\frac{d}{dt}\|\omega\|_{L^2(\Omega)}+\varepsilon\|\nabla\omega\|_{L^2(\Omega)}+\|\omega\|_{L^2(\Omega)}=0\quad \text{for all}\quad t>0,$$

which implies that $\omega\equiv0$ if $\tau=0$ or $\tau=1$ and $\omega(x,0)=0$. This means that $\mathbf{w}$ is curl free, and hence $\mathbf{w}=\nabla\phi$ for some potential function ϕ. Thus the system (1.2)-(1.3) turns into

$$\begin{cases}u_t=\Delta u-\nabla\cdot(u\nabla\phi), & x\in B_r(0), \quad t>0,\\ v_t=\Delta v-v+u, & x\in B_r(0), \quad t>0,\\ \tau\phi_t=\varepsilon\Delta\phi+\lambda-\phi+\chi v, & x\in B_r(0), \quad t>0,\\ \nabla u\cdot\mathbf{n}=\nabla v\cdot\mathbf{n}=\nabla\phi\cdot\mathbf{n}=0, & x\in\partial B_r(0), \quad t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in B_r(0)\end{cases} \tag{1.6}$$

for some constant $\lambda \in \mathbb{R}$. The equivalent system (1.6) is nothing but an indirect chemotaxis model. We can easily show that the solution of (1.6) is globally bounded and no blow-up will occur in two and three dimensions (see Remark 1.2). This is dissimilar to the minimal chemotaxis model (1.1), and hence allows us to speculate that the new chemotaxis system (1.2)-(1.3) may admit globally bounded solutions in two and three dimensions. Specifically, we shall prove the following results in this paper.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \in \{1,2,3\}$, be a bounded domain with smooth boundary. Assume that $u_0,v_0 \in W^{1,p}(\Omega)$ with $p>n$ and $u_0,v_0 \gneqq 0$. Then the system* (1.2)-(1.3) *with $\tau=0$ admits a unique classical solution $(u,v,\mathbf{w})$ satisfying*

$$u,v \in C^0\left(\bar{\Omega}\times[0,\infty)\right)\cap C^{2,1}\left(\bar{\Omega}\times(0,\infty)\right), \quad \mathbf{w}\in\left[C^{2,1}\left(\bar{\Omega}\times(0,\infty)\right)\right]^2,$$

and $u,v>0$ in $\Omega\times(0,\infty)$. Moreover, there exists some constant $C>0$ independent of t such that

$$\|u(\cdot,t)\|_{W^{1,p}(\Omega)}+\|v(\cdot,t)\|_{W^{1,p}(\Omega)}+\|\mathbf{w}(\cdot,t)\|_{W^{1,p}(\Omega)} \le C \quad \text{for all} \quad t>0. \tag{1.7}$$

Remark 1.1. Theorem 1.1 asserts that the model (1.2)-(1.3) has no critical mass phenomenon in two dimensions and no blow-up in three dimensions, which is in sharp contrast to the classical chemotaxis model (1.1). This indicates that modeling ideas in (1.2)-(1.3) might be more biologically relevant in some sense. We conjecture the results of Theorem 1.1 hold true for $\tau=1$, but we have to leave the proof open in this paper due to the unavailability of parabolic regularity under the boundary condition in (1.3) for $\mathbf{w}$.

Remark 1.2. Below we briefly argue that the solution of (1.6) is globally bounded. Let us first consider the case $\tau=0$. First the boundary conditions immediately yield the mass-preserving property of u:

$$\int_{B_r(0)} u = \int_{B_r(0)} u_0(x),$$

which implies $\|v(\cdot,t)\|_{W^{1,q_1}(B_r(0))}$ is uniformly-in-time bounded for some number $q_1 \in [1,2)$ (cf. Lemma 2.2), which along with L^p-estimate for the elliptic equation (cf. [7, Section 9.5]) implies that $\|\phi(\cdot,t)\|_{W^{2,q_2}(B_r(0))}$ is uniformly-in-time bounded for some number $q_2 \in [1,\infty)$, and hence $\|\nabla\phi(\cdot,t)\|_{W^{1,\infty}(B_r(0))}$ is uniformly-in-time bounded. Then by the standard arguments based on the well-known Neumann heat semigroup (cf. [42, Lemma 1.3]) or Moser iteration (cf. [39]), the global-in-time boundedness of solutions can be immediately obtained. The case $\tau=1$ can be proved in a similar way by resorting to the parabolic regularity given in Lemma 2.4.

Next, we proceed to explore the large-time behavior of solutions to (1.2)-(1.3). Clearly, the system (1.2) has only one possible homogeneous steady state $(u_*,v_*,\mathbf{0})$ with

$$u_*=v_*=\frac{1}{|\Omega|}\int_\Omega u_0(x)=:m. \tag{1.8}$$

Our second result on the asymptotic dynamics of (1.2)-(1.3) is stated in the following.

Theorem 1.2. *Suppose that the conditions in Theorem* 1.1 *hold. There exists a number* $\tilde{\chi}>0$ *such that if* $\chi\in(0,\tilde{\chi})$*, then the solution* $(u,v,\mathbf{w})$ *of the system* (1.2)-(1.3) *with* $\tau=0$ *obtained in Theorem* 1.1 *satisfies*

$$\|u(\cdot,t)-m\|_{W^{1,\infty}(\Omega)}+\|v(\cdot,t)-m\|_{W^{1,\infty}(\Omega)}+\|\mathbf{w}(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le Ce^{-\sigma t} \quad \textit{for all} \quad t>1,$$

where C and σ are positive constants independent of t.

The rest of this paper is organized as follows. In Section 2, we shall establish the global existence and boundedness of solutions to (1.2)-(1.3). The global stability stated in Theorem 1.2 is proved in Section 3 by constructing a Lyapunov functional. In Section 4, we conduct linear stability analysis to find the possible parameter regime of pattern formation and perform numerical simulations to illustrate that the system (1.2)-(1.3) may generate aggregation patterns.

2 Global existence and uniform boundedness

Before proceeding, we introduce some notations used throughout the paper.

Notations:

- For brevity, we abbreviate $\int_0^t\int_\Omega f(\cdot,s)dxds$ and $\int_\Omega f(\cdot,t)dx$ as $\int_0^t\int_\Omega f$ and $\int_\Omega f$, respectively. In addition, C or C_i, $i=1,2,3,\dots$ stands for a generic positive constant which may vary from line to line.
- $W^{k,p}(\Omega):=\{u\in L^p(\Omega)|D^\alpha u\in L^p(\Omega) \text{ for } 0\le|\alpha|\le k\}$ denotes the usual Sobolev space, where $D^\alpha u$ is the weak partial derivative. We use $H^k(\Omega):=W^{k,2}(\Omega)$ when $p=2$.

- We shall use C_P to denote the following Poincaré constant:

$$C_P := \left\{ \inf_{C>0} C \,\middle|\, \|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \text{ for all } u \in H^1(\Omega) \text{ with } \int_\Omega u = 0 \right\}. \tag{2.1}$$

- In the sequel we always assume $\tau = 0$ in (1.2) without further comments.

This section is devoted to establishing the global existence and boundedness of solutions to (1.2)-(1.3). To begin with, we recall the regularities of the solution $\mathbf{w}$ to the following system:

$$\begin{cases} -\varepsilon\Delta\mathbf{w} + \mathbf{w} = \mathbf{f}, & x \in \Omega, \\ \mathbf{w}\cdot\mathbf{n} = 0, \;\; \partial_{\mathbf{n}}\mathbf{w} \times \mathbf{n} = \mathbf{0}, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

where $\mathbf{f} \in (L^q(\Omega))^n$ for some $1 < q < +\infty$.

Lemma 2.1 (cf. [29, Theorem 3.2] and [36, Theorem 3])**.** *Let $\Omega \subset \mathbb{R}^n$, $n = 1,2,3$, be a bounded domain with smooth boundary. For $f \in (L^q(\Omega))^n$ with $1 < q < \infty$,* (2.2) *has a unique solution in $(W^{2,q}(\Omega))^n$ such that*

$$\|\mathbf{w}\|_{W^{2,q}} \leq \frac{C(q,n,\Omega)}{\varepsilon}\|f\|_q,$$

where $C(q,n,\Omega)$ is a positive constant depending only on q, n and Ω.

With Lemma 2.1, the local existence of solutions to the system (1.2)-(1.3) can be established under appropriate initial conditions.

Lemma 2.2. *Suppose that the conditions in Theorem* 1.1 *hold. Then there exists some $T_{\max} \in (0,\infty]$ such that the system* (1.2)-(1.3) *has a unique classical solution $(u,v,\mathbf{w})$ satisfying*

$$\begin{cases} u,v \in C^0\big(\bar{\Omega}\times[0,T_{\max})\big) \cap C^{2,1}\big(\bar{\Omega}\times(0,T_{\max})\big), \\ \mathbf{w} \in \big(C^{2,1}\big(\bar{\Omega}\times(0,T_{\max})\big)\big)^n, \end{cases} \tag{2.3}$$

and $u,v > 0$ in $\Omega\times(0,T_{\max})$. Moreover, if $T_{\max} < \infty$, then

$$\lim_{t\to T_{\max}} \Big(\|u(\cdot,t)\|_{W^{1,p}(\Omega)} + \|v(\cdot,t)\|_{W^{1,p}(\Omega)} \Big) = \infty. \tag{2.4}$$

Proof. The proof follows from standard arguments such as the contraction mapping principle (see [39, Lemma 3.1] for instance). For completeness, we shall sketch the proof below. Define $R := \|u_0\|_{L^\infty(\Omega)} + 1$. For $T \in (0,1)$ to be specified below, the closed convex set

$$X_T := \left\{ \varphi \in C^0(\bar{\Omega} \times [0,T]) \,\middle|\, \sup_{t\in[0,T]} \|\varphi(\cdot,t)\|_{L^\infty(\Omega)} \le R \right\}$$

is a complete metric space with the metric

$$d_{X_T}(\varphi,\psi) = \sup_{t\in[0,T]} \|\varphi(\cdot,t) - \psi(\cdot,t)\|_{L^\infty(\Omega)}, \quad \forall \varphi,\psi \in X_T.$$

For any $\tilde{u} \in X_T$, introduce the mapping $\Phi : X_T \to X_T$ with $\Phi(\tilde{u}) := \Phi_1 \circ \Phi_2 \circ \Phi_3(\tilde{u})$, where $u := \Phi(\tilde{u}) = \Phi_1(\mathbf{w})$ is the solution to

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\mathbf{w}), & x\in\Omega, \quad t\in(0,T), \\ \nabla u\cdot\mathbf{n} = 0, & x\in\partial\Omega, \quad t\in(0,T), \\ u(x,0) = u_0(x), & x\in\Omega \end{cases} \tag{2.5}$$

with $\mathbf{w}(\cdot,t) := \Phi_2(v) \in W^{2,p}(\Omega)$ being the unique solution of (cf. Lemma 2.1)

$$\begin{cases} -\varepsilon\Delta\mathbf{w}(\cdot,t) + \mathbf{w}(\cdot,t) = \chi\nabla v(\cdot,t), & x\in\Omega, \\ \mathbf{w}\cdot\mathbf{n} = 0, \ \partial_{\mathbf{n}}\mathbf{w}\times\mathbf{n} = \mathbf{0}, & x\in\partial\Omega \end{cases} \tag{2.6}$$

for $t \in (0,T)$, here $v := \Phi_3(\tilde{u})$ represents the solution of (cf. [21] for the existence theories of linear parabolic equation)

$$\begin{cases} v_t = \Delta v - v + \tilde{u}, & x\in\Omega, \quad t\in(0,T), \\ \nabla v\cdot\mathbf{n} = 0, & x\in\partial\Omega, \quad t\in(0,T), \\ v(x,0) = v_0(x), & x\in\Omega. \end{cases} \tag{2.7}$$

We next prove that Φ has a unique fixed point on X_T if T is small enough. Noting that $\tilde{u} \in W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ due to $p > n$, then by the standard L^p and Schauder theories of linear parabolic equation (cf. [21]), we know that there is a unique solution $v(x,t) \in C^{1+\theta_1,\frac{1+\theta_1}{2}}(\bar{\Omega}\times(0,T))$ to the system (2.7) for each $\theta_1 \in (0,1)$, which implies that $\|\nabla v\|_{L^\infty(\Omega)} \le C_1(R)$ (controlled by $\|\tilde{u}\|_{L^\infty(\Omega)} \le R$). Then it follows from Lemma 2.1, the Sobolev embedding $W^{2,n}(\Omega) \hookrightarrow L^\infty(\Omega)$ and Hölder's inequality that the solution $\mathbf{w}$ to (2.6) satisfies

$$\|\mathbf{w}\|_{L^\infty(\Omega)} \le C\|\mathbf{w}\|_{W^{2,n}(\Omega)} \le C\|\nabla v\|_{L^n(\Omega)} \le C\|\nabla v\|_{L^\infty(\Omega)} \le C_2(R).$$

Since $u_0 \in W^{1,p}(\Omega)$ with $p > n$, there exist some $\theta_2 \in (0,1)$ such that $u_0 \in C^{\theta_2}(\bar{\Omega})$. Therefore, we have from [21] (see also [33, Theorem 1.3, Remarks 1.3 and 1.4]) that there exists a unique solution $u \in C^{\theta_2,\frac{\theta_2}{2}}(\bar{\Omega}\times[0,T])$ (controlled by $\|\mathbf{w}\|_{L^\infty(\Omega)} \le C_2(R)$) to (2.5) such that

$$\|u\|_{C^{\theta_2,\frac{\theta_2}{2}}(\bar{\Omega}\times[0,T])} \le C_3(R).$$

Hence, for all $t \in [0,T]$ we have

$$\|u(\cdot,t) - u_0\|_{L^\infty(\Omega)} \le C_3(R) t^{\frac{\theta_2}{2}}$$

and thus

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + C_3(R) t^{\frac{\theta_2}{2}}.$$

If we take $T = T_0 < (\frac{1}{C_3(R)})^{\frac{2}{\theta_2}}$, then we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le R \quad \text{for all} \quad t \in [0,T_0].$$

Therefore, $\Phi(\tilde{u}) = u \in X_T$ and Φ maps X_T into itself. By similar arguments of the above procedures, one can further deduce that Φ becomes a contraction on X_T if T is small enough. For such T, by the contraction mapping principle (cf. [7, Theorem 5.1]), we find a unique $u \in X_T$ such that $u = \Phi(u)$. With Lemma 2.1, and the standard L^p and Schauder theories of linear parabolic equation (cf. [21]), one can derive that the solution enjoys the regularity (2.3) by the bootstrap arguments. Since T is independent of $\|u_0\|_{L^\infty(\Omega)}$, the solution can be prolonged in the interval $[0,T_{\max})$ with either $T_{\max} = \infty$ or

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as} \quad t \nearrow T_{\max},$$

which proves (2.4). Finally, the strong maximum principle implies that $u,v > 0$ in $\Omega \times (0,T_{\max})$. □

Now we state the following basic mass-preserving property.

Lemma 2.3. *Suppose that the conditions in Theorem* 1.1 *hold. Then*

$$\|u(\cdot,t)\|_{L^1(\Omega)} = m|\Omega| \quad \textit{for all} \quad t \in (0,T_{\max}), \tag{2.8}$$

where m is defined by (1.8).

Proof. Integrating Eq. (1.2a) with respect to $x\in\Omega$, and using the boundary conditions in (1.3), we obtain (2.8) immediately. $\square$

We shall need the following property before deriving uniform-in-time L^∞-estimate of solutions.

Lemma 2.4 (cf. [20, Lemma 1]). *Let Ω be a bounded domain in $\mathbb{R}^n$, $n\geq 1$, with smooth boundary. Let $T\in(0,\infty]$ and suppose that $z\in C^0(\bar{\Omega}\times[0,T))\cap C^{2,1}(\bar{\Omega}\times(0,T))$ is a solution of*

$$\begin{cases} z_t=\Delta z-z+g, & x\in\Omega, \quad t\in(0,T),\\ \dfrac{\partial z}{\partial \nu}=0, & x\in\partial\Omega, \quad t\in(0,T), \end{cases}$$

where $g\in L^\infty((0,T);L^q(\Omega))$. Then there exists a constant $C>0$ such that

$$\|z(\cdot,t)\|_{W^{1,r}(\Omega)}\leq C$$

with

$$r\in\begin{cases} [1,\frac{nq}{n-q}), & \text{if} \quad q\leq n,\\ [1,\infty], & \text{if} \quad q>n. \end{cases}$$

We are in a position to derive the following result.

Lemma 2.5. *Suppose that the conditions in Theorem* 1.1 *hold. Then there exists a constant $C>0$ independent of χ and t such that*

$$\begin{aligned} &\|u\|_{L^\infty(\Omega)}+\|v\|_{W^{1,\infty}(\Omega)}+\|\mathbf{w}\|_{W^{1,\infty}(\Omega)}\\ &\leq C\left(\chi^{\frac{56+2n-n^2}{8(4-n)}}+1\right) \quad \text{for all} \quad t\in(0,T_{\max}). \end{aligned} \tag{2.9}$$

Proof. By (2.8) and Lemma 2.4, for $r_1\in[1,\frac{3}{2})$, we have

$$\|v(\cdot,t)\|_{W^{1,r_1}(\Omega)}\leq C \tag{2.10}$$

for all $t\in(0,T_{\max})$. Along with Lemma 2.1 this implies

$$\begin{aligned} \|\mathbf{w}\|_{W^{2,r_1}(\Omega)}&\leq C\|\chi\nabla v\|_{L^{r_1}(\Omega)}\leq C\chi\|v(\cdot,t)\|_{W^{1,r_1}(\Omega)}\\ &\leq C\chi \quad \text{for all} \quad t\in(0,T_{\max}). \end{aligned} \tag{2.11}$$

In view of the Sobolev embedding $W^{2,\frac{4}{3}}(\Omega)\hookrightarrow W^{1,2}(\Omega)$, we have from (2.11) with $r_1=\frac{4}{3}$ that

$$\|\mathbf{w}\|_{W^{1,2}(\Omega)}\leq C\|\mathbf{w}\|_{W^{2,\frac{4}{3}}(\Omega)}\leq C\chi \quad \text{for all} \quad t\in(0,T_{\max}). \tag{2.12}$$

Multiplying Eq. (1.2a) by qu^{q-1} for $q>1$, integrating the resulting equation by parts alongside the boundary conditions in (1.3), we have

$$\begin{aligned}&\frac{d}{dt}\|u\|^q_{L^q(\Omega)}+\frac{4(q-1)}{q}\|\nabla u^{\frac{q}{2}}\|^2_{L^2(\Omega)}\\ =&-(q-1)\int_\Omega u^q\nabla\cdot\mathbf{w}\quad\text{for all}\quad t\in(0,T_{\max}).\end{aligned}\tag{2.13}$$

Taking $q=2$ in (2.13), and using (2.12) and Hölder's inequality, we get

$$\begin{aligned}&\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)}+2\|\nabla u\|^2_{L^2(\Omega)}\\ \le&\|u\|^2_{L^2(\Omega)}+\|u^2\|_{L^2(\Omega)}\|\nabla\cdot\mathbf{w}\|_{L^2(\Omega)}\\ \le&C(\chi+1)\|u\|^2_{L^4(\Omega)}\quad\text{for all}\quad t\in(0,T_{\max}).\end{aligned}\tag{2.14}$$

By the Gagliardo-Nirenberg inequality, one has

$$\begin{aligned}C(\chi+1)\|u\|^2_{L^4(\Omega)}&\le C(\chi+1)\left(\|\nabla u\|^{\frac{6n}{4+2n}}_{L^2(\Omega)}\|u\|^{\frac{4-n}{2+n}}_{L^1(\Omega)}+\|u\|^2_{L^1(\Omega)}\right)\\ &\le C(\chi+1)\left(\|\nabla u\|^{\frac{6n}{4+2n}}_{L^2(\Omega)}+1\right)\\ &\le 2\|\nabla u\|^2_{L^2(\Omega)}+C\left(\chi^{\frac{2(2+n)}{4-n}}+1\right).\end{aligned}\tag{2.15}$$

Substituting (2.15) into (2.14), and using Grönwall's inequality, we get

$$\begin{aligned}\|u\|_{L^2(\Omega)}&\le\|u_0\|_{L^2(\Omega)}+C\left(\chi^{\frac{2(2+n)}{4-n}}+1\right)^{\frac{1}{2}}\\ &\le C\left(\chi^{\frac{2+n}{4-n}}+1\right)\quad\text{for all}\quad t\in(0,T_{\max}),\end{aligned}\tag{2.16}$$

which along with Lemma 2.4 implies that (2.10) and (2.11) actually hold for any $r_1\in[1,6)$. Then taking $r_1=4$ in (2.11) and using the Sobolev embedding $W^{2,4}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)\hookrightarrow L^\infty(\Omega)$, one has

$$\|\mathbf{w}\|_{W^{1,\infty}(\Omega)}+\|\mathbf{w}\|_{L^\infty(\Omega)}\le C\chi\quad\text{for all}\quad t\in(0,T_{\max}).\tag{2.17}$$

Now taking $q=4$ in (2.13), using (2.16), (2.17), the Gagliardo-Nirenberg inequality and Young's inequality, we arrive at

$$\frac{d}{dt}\|u\|^4_{L^4(\Omega)}+\|u\|^4_{L^4(\Omega)}+3\|\nabla u^2\|^2_{L^2(\Omega)}$$

$$
\begin{aligned}
&=\|u\|_{L^4(\Omega)}^4-3\int_\Omega u^4\nabla\cdot\mathbf{w}\le\big(1+3\|\nabla\cdot\mathbf{w}\|_{L^\infty(\Omega)}\big)\|u^2\|_{L^2(\Omega)}^2\\
&\le C(\chi+1)\left(\|\nabla u^2\|_{L^2(\Omega)}^{\frac{2n}{n+2}}\|u^2\|_{L^1(\Omega)}^{\frac{4}{n+2}}+\|u^2\|_{L^1(\Omega)}^2\right)\\
&\le\|\nabla u^2\|_{L^2(\Omega)}^2+C\|u^2\|_{L^1(\Omega)}^2\left(\chi^{\frac{n+2}{2}}+1\right)\\
&\le\|\nabla u^2\|_{L^2(\Omega)}^2+C\left(\chi^{\frac{2+n}{4-n}}+1\right)^4\left(\chi^{\frac{n+2}{2}}+1\right)
\end{aligned}
$$

for all $t\in(0,T_{\max})$. This implies

$$
\frac{d}{dt}\|u\|_{L^4(\Omega)}^4+\|u\|_{L^4(\Omega)}^4\le C\left(\chi^{\frac{2+n}{4-n}}+1\right)^4\left(\chi^{\frac{n+2}{2}}+1\right) \tag{2.18}
$$

for all $t\in(0,T_{\max})$. Applying Grönwall's inequality to (2.18), for all $t\in(0,T_{\max})$ we get

$$
\begin{aligned}
\|u\|_{L^4(\Omega)}&\le\|u_0\|_{L^4(\Omega)}+C\left(\chi^{\frac{2+n}{4-n}}+1\right)\left(\chi^{\frac{n+2}{2}}+1\right)^{\frac{1}{4}}\\
&\le C\big(\chi^{\alpha(n)}+1\big),
\end{aligned}
$$

where

$$
\alpha(n):=\frac{2+n}{4-n}+\frac{n+2}{8}
$$

is a positive constant. The above inequality alongside (2.17) and Young's inequality shows that

$$
\begin{aligned}
\|u\mathbf{w}\|_{L^4(\Omega)}&\le\|\mathbf{w}\|_{L^\infty(\Omega)}\|u\|_{L^4(\Omega)}\le C\chi\|u\|_{L^4(\Omega)}\\
&\le C\big(\chi^{\alpha(n)+1}+1\big),\quad t\in(0,T_{\max}).
\end{aligned}
$$

Then using the L^p-L^q estimates of the Neumann heat semigroup (cf. [42, Lemma 1.3]) and $\frac{1}{2}+\frac{n}{8}<1$ for $n\in\{1,2,3\}$, we deduce that

$$
\begin{aligned}
\|u(\cdot,t)\|_{L^\infty(\Omega)}&\le\big\|e^{t\Delta}u_0\big\|_{L^\infty(\Omega)}+\int_\Omega\big\|e^{(t-s)\Delta}\nabla\cdot(u\mathbf{w})\big\|_{L^\infty(\Omega)}ds\\
&\le C+C\int_0^t\big(1+(t-s)\big)^{-\frac{1}{2}-\frac{n}{8}}e^{-\lambda_1(t-s)}\|u\mathbf{w}\|_{L^4(\Omega)}ds\\
&\le C+C\big(\chi^{\alpha(n)+1}+1\big)\int_0^t\big(1+(t-s)\big)^{-\frac{1}{2}-\frac{n}{8}}e^{-\lambda_1(t-s)}ds\\
&\le C\big(\chi^{\alpha(n)+1}+1\big),\quad t\in(0,T_{\max}),
\end{aligned} \tag{2.19}
$$

where λ_1 denotes the first nonzero eigenvalue of $-\Delta$ in Ω under the homogeneous Neumann boundary condition. Then it follows from Lemma 2.4 and (2.19) immediately that

$$\|v\|_{W^{1,\infty}(\Omega)} \leq C\big(\chi^{\alpha(n)+1}+1\big) \quad \text{for all} \quad t\in(0,T_{\max}). \tag{2.20}$$

Applying Young's inequality to (2.17), we obtain

$$\|\mathbf{w}\|_{W^{1,\infty}(\Omega)} \leq C\big(\chi^{\alpha(n)+1}+1\big) \quad \text{for all} \quad t\in(0,T_{\max}). \tag{2.21}$$

In view of (2.19)-(2.21), (2.9) is proved by noting that

$$\alpha(n)+1=\frac{56+2n-n^2}{8(4-n)}$$

and the constant C appeared in this proof is independent of χ and t. □

Proof of Theorem 1.1. In view of (2.4) and (2.9), we get $T_{\max}=\infty$ and (1.7) immediately. □

3 Global stability

In this section, we shall use the Lyapunov functional method along with compactness arguments to investigate the large time behavior of solutions to the system (1.2)-(1.3). Throughout this section, we suppose that the conditions in Theorem 1.1 hold and $(u,v,\mathbf{w})$ is the global classical solution of (1.2)-(1.3) obtained in Theorem 1.1. To this end, we first derive the following higher-order estimates of solutions as time t is away from 0.

Lemma 3.1. *For any $\theta\in(0,1)$, there exists some constant $C(\theta)>0$ such that*

$$\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega}\times[1,\infty))}+\|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega}\times[1,\infty))}+\|\mathbf{w}\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega}\times[1,\infty))}\leq C(\theta). \tag{3.1}$$

Proof. With Lemma 2.1, (3.1) can be proved by the bootstrap argument based on the L^p estimate and the Schauder estimate (cf. [23]), see [40, Theorem 2.1] for instance. We omit the details here for brevity. □

In view of Lemma 2.5, we have the following result immediately.

Corollary 3.1. *For any $\chi \in (0,1)$, there exists some positive constant M independent of χ and t such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq M$$

for all $t>0$.

Now we are in a position to construct the Lyapunov functional.

Lemma 3.2. *Let C_P and M be given by* (2.1) *and Corollary* 3.1, *respectively. Define two positive constants*

$$\Gamma_1 := \frac{4\varepsilon}{M^2}, \quad \Gamma_2 := \frac{1}{2}\left(\chi^2 + \frac{4\varepsilon}{C_P M^2}\right). \tag{3.2}$$

If

$$\chi < \tilde{\chi} := \min\left\{1, \left(\frac{4\varepsilon}{C_P M^2}\right)^{\frac{1}{2}}\right\},$$

then the energy functional

$$\mathcal{E}(t) := \Gamma_1 \int_\Omega (u-u_*)^2 + \Gamma_2 \int_\Omega (v-v_*)^2$$

satisfies

$$\frac{d}{dt}\mathcal{E}(t) \leq -\theta \mathcal{E}(t), \quad t>0, \tag{3.3}$$

where $\theta>0$ is a positive constant.

Proof. Clearly, it has that $\chi^2 < \frac{4\varepsilon}{C_P M^2}$ from our assumptions. Then it follows from (3.2) that

$$\chi^2 < \Gamma_2 < \frac{\Gamma_1}{C_P}. \tag{3.4}$$

Using the integration by parts with the boundary condition $\nabla u \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0$, we obtain

$$\begin{aligned} \Gamma_1 \frac{d}{dt}\int_\Omega (u-u_*)^2 &= 2\Gamma_1 \int_\Omega (u-u_*)\left[\Delta u - \nabla\cdot(u\mathbf{w})\right] \\ &= -2\Gamma_1 \int_\Omega |\nabla u|^2 + 2\Gamma_1 \int_\Omega u\mathbf{w}\cdot\nabla u \quad \text{for all} \quad t>0. \end{aligned} \tag{3.5}$$

Since $\int_\Omega (u-u_*) = 0$, (2.1) implies that

$$\Gamma_2 C_P \int_\Omega |\nabla u|^2 - \Gamma_2 \int_\Omega (u-u_*)^2 \geq 0 \quad \text{for all} \quad t>0,$$

which together with (3.5) shows that

$$\begin{aligned}&\Gamma_1\frac{d}{dt}\int_\Omega(u-u_*)^2\\&\le -(2\Gamma_1-\Gamma_2C_P)\int_\Omega|\nabla u|^2-\Gamma_2\int_\Omega(u-u_*)^2+2\Gamma_1\int_\Omega u\mathbf{w}\cdot\nabla u\end{aligned}\tag{3.6}$$

for all $t>0$. Similarly, for all $t>0$, we have

$$\begin{aligned}\Gamma_2\frac{d}{dt}\int_\Omega(v-v_*)^2&=2\Gamma_2\int_\Omega(v-v_*)\left[\Delta v+(u-u_*)-(v-v_*)\right]\\&=-2\Gamma_2\int_\Omega|\nabla v|^2+2\Gamma_2\int_\Omega(u-u_*)(v-v_*)-2\Gamma_2\int_\Omega(v-v_*)^2.\end{aligned}\tag{3.7}$$

Multiplying Eq. (1.2c) by $\mathbf{w}$, integrating the resulting equation by parts along with the boundary conditions in (1.3), and using (1.4) and Young's inequality, we have

$$\begin{aligned}&\varepsilon\|\nabla\mathbf{w}\|^2_{L^2(\Omega)}+\|\mathbf{w}\|^2_{L^2(\Omega)}\\&=\chi\int_\Omega\nabla(v-v_*)\cdot\mathbf{w}\le\varepsilon\|\nabla\mathbf{w}\|^2_{L^2(\Omega)}+\frac{\chi^2}{4\varepsilon}\|v-v_*\|^2_{L^2(\Omega)},\end{aligned}\tag{3.8}$$

which yields

$$\chi^2\int_\Omega(v-v_*)^2-4\varepsilon\|\mathbf{w}\|^2_{L^2(\Omega)}\ge 0,\quad t>0.\tag{3.9}$$

With (3.6)-(3.9), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t)\le&-(2\Gamma_1-\Gamma_2C_P)\int_\Omega|\nabla u|^2-\Gamma_2\int_\Omega(u-u_*)^2+2\Gamma_1\int_\Omega u\mathbf{w}\cdot\nabla u-2\Gamma_2\int_\Omega|\nabla v|^2\\&+2\Gamma_2\int_\Omega(u-u_*)(v-v_*)-(2\Gamma_2-\chi^2)\int_\Omega(v-v_*)^2-4\varepsilon\|\mathbf{w}\|^2_{L^2(\Omega)}\\=:&-\int_\Omega XAX^T-\int_\Omega YBY^T,\quad t>0,\end{aligned}\tag{3.10}$$

where

$$X:=(u-u_*,v-v_*),\quad Y:=(\nabla u,\nabla v,\mathbf{w})$$

and A,B are matrices denoted by

$$A:=\begin{pmatrix}\Gamma_2 & -\Gamma_2\\ -\Gamma_2 & 2\Gamma_2-\chi^2\end{pmatrix},\quad B:=\begin{pmatrix}2\Gamma_1-\Gamma_2C_P & 0 & -\Gamma_1u\\ 0 & 2\Gamma_2 & 0\\ -\Gamma_1u & 0 & 4\varepsilon\end{pmatrix}.$$

Using (3.2) and (3.4), for all $t>0$, we have

$$
\begin{aligned}
&|A|=\begin{vmatrix} \Gamma_2 & -\Gamma_2 \\ -\Gamma_2 & 2\Gamma_2-\chi^2 \end{vmatrix}=\Gamma_2\left(\Gamma_2-\chi^2\right)>0,\\
&2\Gamma_1-\Gamma_2 C_P>0, \quad 2\Gamma_2(2\Gamma_1-\Gamma_2 C_P)>0,
\end{aligned}
$$

and the determinant of B which satisfies

$$|B|=-2\Gamma_2\left(\Gamma_1^2 u^2-8\varepsilon\Gamma_1+4\varepsilon\Gamma_2 C_P\right)>0$$

since

$$|B|\geq -2\Gamma_2\left(\Gamma_1^2 M^2-8\varepsilon\Gamma_1+4\varepsilon\Gamma_2 C_P\right)=8\varepsilon\Gamma_2(\Gamma_1-\Gamma_2 C_P)>0.$$

Based on the Sylvester's criterion, we know that the matrices A and B are positive definite. Hence we can find some constant $\alpha>0$ such that

$$XAX^T\geq\alpha|X|^2, \quad YBY^T\geq\alpha|Y|^2, \quad t>0. \tag{3.11}$$

In view of (3.10) and (3.11), we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}(t)&\leq-\alpha\left(\int_\Omega(u-u_*)^2+\int_\Omega(v-v_*)^2\right)\\
&\leq-\theta\left(\Gamma_1\int_\Omega(u-u_*)^2+\Gamma_2\int_\Omega(v-v_*)^2\right)\\
&=-\theta\mathcal{E}(t), \quad t>0,
\end{aligned}
$$

where

$$\theta:=\min\left\{\frac{\alpha}{\Gamma_1},\frac{\alpha}{\Gamma_2}\right\},$$

and hence (3.3) is proved. □

We next give the result on the convergence rate of solutions.

Lemma 3.3. *Let the conditions of Lemma* 3.2 *hold. There exist two constants $\sigma>0$ and $C>0$ independent of t such that*

$$\|u(\cdot,t)-m\|_{W^{1,\infty}(\Omega)}+\|v(\cdot,t)-m\|_{W^{1,\infty}(\Omega)}+\|\mathbf{w}\|_{W^{1,\infty}(\Omega)}\leq Ce^{-\sigma t} \quad \textit{for all} \quad t>1,$$

where m is given by (1.8).

Proof. It follows from (3.3) that

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-\theta t}, \quad t>0,$$

which along with (3.9) implies that

$$\|u(\cdot,t)-m\|_{L^2(\Omega)}+\|v(\cdot,t)-m\|_{L^2(\Omega)}+\|\mathbf{w}\|_{L^2(\Omega)} \leq Ce^{-\theta t}, \quad t>0. \tag{3.12}$$

The combination of Lemma 3.1, (3.12) and the Gagliardo-Nirenberg inequality

$$\|u-m\|_{W^{1,\infty}(\Omega)} \leq C\left(\|D^2u\|_{L^4}^{\frac{2n+4}{n+8}}\|u-m\|_{L^2}^{\frac{4-n}{n+8}}+\|u-m\|_{L^2}\right)$$

yields

$$\|u(\cdot,t)-m\|_{W^{1,\infty}(\Omega)} \leq Ce^{-\sigma t}, \quad t>1$$

by choosing C appropriately large and taking $\sigma:=\frac{(4-n)\theta}{n+8}$. Similar procedures give

$$\|v(\cdot,t)-m\|_{W^{1,\infty}(\Omega)}+\|\mathbf{w}\|_{W^{1,\infty}(\Omega)} \leq Ce^{-\sigma t}, \quad t>1,$$

which completes the proof. □

Proof of Theorem 1.2. Clearly, Theorem 1.2 is a direct consequence of Lemmas 3.2 and 3.3. □

4 Discussion and numerical simulations

In the classical chemotaxis model, the advective velocity of species is directly proportional to the chemical signal concentration gradient. In this paper, we consider a new type of chemotaxis model (1.2)-(1.3), where it is not the advective velocity but its acceleration that is proportional to the chemical signal concentration gradient. This mechanism entails that the species adjusts its migration velocity according to the spatial variation of the signal concentration, which gives more details of how the species responds to the presence of signals. Such a process has been used in mathematical models (cf. [1,35]) to successfully explain the spatio-temporal heterogeneity observed in the experiments like in [30, 32]). The main goal of this paper is to apply this modeling idea to model the chemotaxis process and surprisingly find that the resulting model will produce distinct outcomes from the classical chemotaxis model. In the classical chemotaxis model (1.1), the solution blows up in two dimensions with a critical mass and in three dimensions for any positive mass. Here we show that the new chemotaxis model

(1.2)-(1.3) by using the idea of acceleration has globally bounded solutions in two and three dimensions and further shows that when the chemotactic sensitivity parameter χ is small, the solution stabilizes into a constant and hence there is no pattern formation. Though we are only able to prove the case $\tau=0$ (i.e., the velocity field $\mathbf{w}$ is in its equilibrium state) in the paper, we conjecture that the similar results will hold true for $\tau=1$. Then a natural question is whether the considered system (1.2)-(1.3) can generate aggregation patterns as χ is suitably large, since the typical consequence of chemotaxis is the spatial aggregation. In this section, we shall explore this question to demonstrate that (1.2)-(1.3) indeed can produce the aggregation patterns as desired.

We proceed with two steps. In step 1, we perform linear stability analysis to find the possible pattern formation parameter regime. Then in step 2, we use numerical simulations to illustrate the patterns generated from the system (1.2)-(1.3).

4.1 Linear instability analysis

We start with the corresponding ODE system of (1.2)-(1.3)

$$\begin{cases} \mathbf{u}_t = G(\mathbf{u}), \\ \mathbf{w} \equiv \mathbf{0}, \end{cases} \tag{4.1}$$

where

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad G(\mathbf{u}) := \begin{pmatrix} 0 \\ u-v \end{pmatrix}.$$

It is obvious that the equilibrium $(m,m,\mathbf{0})$ of (4.1) is marginally stable since the two eigenvalues of

$$G_{\mathbf{u}}(u_*,v_*) := \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

are 0 and -1.

Next we restrict our analysis to the one-dimensional case ($\Omega = (0,l)$ with some $l>0$) for simplicity and linearize the system (1.2)-(1.3) at the equilibrium $(m,m,0)$ to get the linearized system

$$\begin{cases} u_t = u_{xx} - m w_x, & x \in (0,l), \quad t>0, \\ v_t = v_{xx} + u - v, & x \in (0,l), \quad t>0, \\ 0 = \varepsilon w_{xx} - w + \chi v_x, & x \in (0,l), \quad t>0, \\ u_x = v_x = 0, \ w = 0, & x = 0,l, \quad t>0, \end{cases} \tag{4.2}$$

which has solutions in the form of

$$\begin{cases} u(x,t)=\sum\limits_{k\geq 0} U_k e^{\lambda t}\cos(kx),\\ v(x,t)=\sum\limits_{k\geq 0} V_k e^{\lambda t}\cos(kx),\\ w(x,t)=\sum\limits_{k\geq 0} W_k e^{\lambda t}\sin(kx),\end{cases} \tag{4.3}$$

where the constants U_k, V_k and W_k are determined by the Fourier expansion of the initial value, λ (depending on k) is the temporal growth rate and $k=\frac{N\pi}{l}$ is the wave number with the mode $N=0,1,2,\dots$. Substituting (4.3) into (4.2), we obtain

$$\begin{cases} \sum\limits_{k\geq 0}\left(\lambda U_k+k^2U_k+mkW_k\right)e^{\lambda t}\cos(kx)=0, & \text{(4.4a)}\\ \sum\limits_{k\geq 0}\left(\lambda V_k+k^2V_k-U_k+V_k\right)e^{\lambda t}\cos(kx)=0, & \text{(4.4b)}\\ \sum\limits_{k\geq 0}\left(\varepsilon k^2W_k+W_k+\chi kV_k\right)e^{\lambda t}\sin(kx)=0. & \text{(4.4c)}\end{cases}$$

Multiplying Eqs. (4.4a) and (4.4b) by $\cos\frac{N\pi}{l}x$, integrating the results with respect to x over $(0,l)$, using $e^{\lambda t}\neq 0$ and recall the basic fact

$$\int_0^l \cos\left(\frac{N\pi}{l}x\right)\cos\left(\frac{M\pi}{l}x\right)dx=\begin{cases} l, & M=N=0,\\ \dfrac{l}{2}, & M=N>0,\\ 0, & M\neq N,\end{cases}$$

we obtain

$$\begin{cases} \lambda U_k+k^2U_k+mkW_k=0,\\ \lambda V_k+k^2V_k-U_k+V_k=0.\end{cases} \tag{4.5}$$

Similarly, using Eq. (4.4c) and

$$\int_0^l \sin\left(\frac{N\pi}{l}x\right)\sin\left(\frac{M\pi}{l}x\right)dx=\begin{cases} \dfrac{l}{2}, & M=N>0,\\ 0, & M=N=0 \quad \text{or} \quad M\neq N,\end{cases}$$

one has

$$\varepsilon k^2W_k+W_k+\chi kV_k=0,\quad k=\frac{N\pi}{l}\neq 0,\quad N=1,2,3,\dots. \tag{4.6}$$

The stability of the mode $N=0$ ($k=0$) is corresponding to the ODE system (4.1) which has been discussed above.

Next we consider the case $k=\frac{N\pi}{l}\neq 0$. By (4.6) we have

$$W_k=-\frac{\chi k}{1+\varepsilon k^2}V_k,$$

which together with (4.5) shows that

$$\lambda\begin{pmatrix}U_k\\V_k\end{pmatrix}=\mathscr{A}\begin{pmatrix}U_k\\V_k\end{pmatrix}\quad\text{with}\quad \mathscr{A}=\begin{pmatrix}-k^2 & \frac{m\chi k^2}{1+\varepsilon k^2}\\ 1 & -k^2-1\end{pmatrix}.$$

This implies that λ is the eigenvalue of the matrix $\mathscr{A}$. By calculation, we obtain the eigenvalue $\lambda(k^2)$ depending on the wavenumber k as the roots of

$$P(\lambda):=\lambda^2+a_1(k^2)\lambda+a_0(k^2)=0,$$

where

$$a_1(k^2)=2k^2+1>0,\quad a_0(k^2)=k^4+k^2-\frac{m\chi k^2}{\varepsilon k^2+1}.$$

Denote the two zeros of $P(\lambda)$ by

$$\lambda_1 \text{ and } \lambda_2 \text{ with } Re(\lambda_1)\leq Re(\lambda_2).$$

Then for each $k\neq 0$, it is clear that

$$\begin{cases}\lambda_1 \text{ and } \lambda_2 \text{ are complex with } Re(\lambda_1)\leq Re(\lambda_2)<0, & \text{if} \quad a_0(k^2)>0,\\ \lambda_1 \text{ and } \lambda_2 \text{ are real with } \lambda_1<\lambda_2=0, & \text{if} \quad a_0(k^2)=0,\\ \lambda_1 \text{ and } \lambda_2 \text{ are real with } \lambda_1<0<\lambda_2, & \text{if} \quad a_0(k^2)<0.\end{cases} \tag{4.7}$$

The three cases in (4.7) are corresponding to $\chi<\chi_0(k^2)$, $\chi=\chi_0(k^2)$ and $\chi>\chi_0(k^2)$, respectively, where

$$\chi_0(k^2):=\frac{(k^2+1)(\varepsilon k^2+1)}{m}\quad\text{is strictly increasing with } k.$$

We denote

$$\chi_*:=\inf_{k=N\pi/l}\chi_0(k^2)=\chi_0\left(\frac{\pi^2}{l^2}\right)=\frac{(l^2+\pi^2)(l^2+\pi^2\varepsilon)}{ml^4}. \tag{4.8}$$

Therefore, if

$$\chi>\chi_*,$$

then there exists at least one mode (e.g. $N=1$ with $k=\frac{\pi}{l}$), such that $\lambda_2>0$, and hence the equilibrium $(m,m,0)$ of the system (1.2)-(1.3) is linearly unstable, which may enforce patterns bifurcating from $(m,m,0)$.

Remark 4.1. It can be seen from the above analysis that $Re(\lambda_1)$ is always negative for all $k\neq 0$, which means that there will be no periodic patterns since $P(\lambda)$ cannot have a pair of conjugate purely imaginary roots.

4.2 Spatio-temporal patterns

In this subsection, we shall use numerical simulations to illustrate the patterns generated by the system (1.2)-(1.3) in one and two dimensions. The unique equilibrium of (1.2)-(1.3) is $(m,m,0)$, and we shall set the initial value (u_0,v_0,w_0) as a small perturbation of $(m,m,0)$. We first look at the one-dimensional case say $\Omega=(0,10)$, where the boundary condition for w in (1.3) simply becomes the zero Dirichlet boundary condition. By (4.8), we find the critical value of χ is

$$\chi_*=\frac{(100+\pi^2)^2}{10000}\approx 1.20713.$$

We set the initial value (u_0,v_0,w_0) as a small perturbation of the equilibrium $(m,m,0)$ with 1% deviation:

$$(u_0,v_0,w_0)=\big(m+0.01\cdot\cos(\pi x),m+0.01\cdot\cos(\pi x),0.01\cdot\sin(\pi x)\big). \tag{4.9}$$

Then it is expected that patterns will arise from the equilibrium $(1,1,0)$ for any $\chi>\chi_*$. From Remark 4.1, we know that the periodic pattern bifurcating from $(m,m,0)$ is impossible. Without loss of generality, we shall assume the species mass is one, namely $m=1$. Then we perform the numerical simulations in $(0,10)$ by the Matlab PDE solver and the pattern formation generated by (1.2)-(1.3) with $\chi=5$ are plotted in Fig. 1 where we observe stable aggregation patterns developing from the equilibrium $(1,1,0)$. Then we proceed to perform the numerical

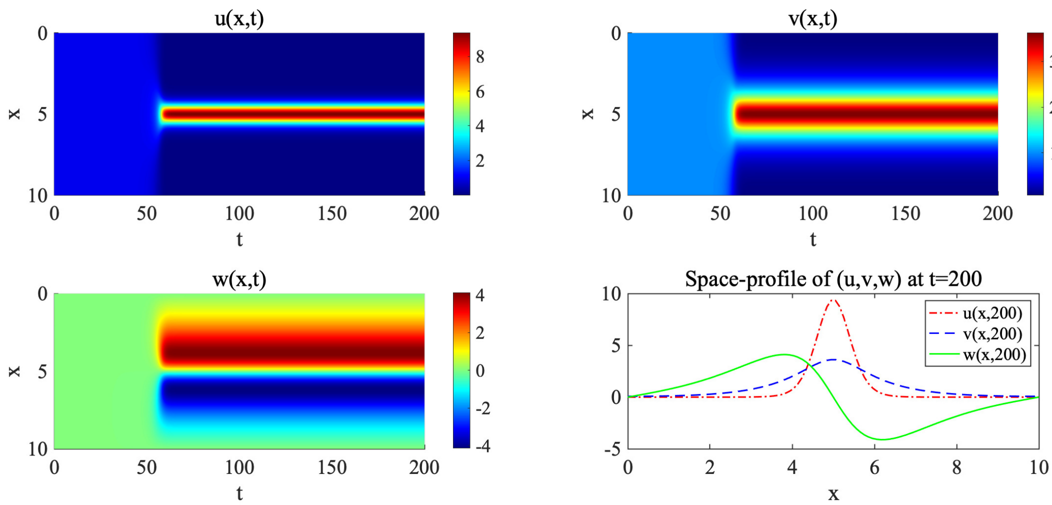

Figure 1: Spatio-temporal patterns generated by the system (1.2)-(1.3) in the interval $\Omega=[0,10]$ with $\chi=5,\varepsilon=1$, where the initial value (u_0,v_0,w_0) is given by (4.9) with $m=1$.

simulations in two dimensions. In this case, the boundary condition for **w** is complicated and hard to specify for general domain Ω. For simplicity, we do the numerical simulations in a disk and assume that the solution is radially symmetric. Then the system (1.2)-(1.3) is equivalent to the system (1.6) which is much easier to handle numerically. Then we turn to simulate the system (1.6) in a disk $B_5(0)$ with radius 5 centered at the origin using Comsol. The initial value (u_0,v_0,w_0) is again set as a small perturbation of $(1,1,0)$

$$u_0=v_0=1+0.01\cdot\cos(\pi x)\cdot\cos(\pi y), \quad w_0=0.$$

The numerical patterns for u and v at different time steps are plotted in Fig. 2 and Fig. 3, respectively, where we see that aggregation patterns arise from the homogeneous steady state $(1,1,0)$ and stabilize into two boundary aggregates eventually. This not only confirms our analytical results that the solutions will not blow up, but also shows that the chemotaxis model (1.2)-(1.3) we propose can generate aggregation patterns as desired.

There are many interesting questions left to pursue further. For example, the global well-posedness problem of (1.2)-(1.3) with $\tau=1$ remains unsolved in a general domain Ω due to the unavailability of parabolic regularity with the boundary conditions in (1.3). If Ω is a disk, the system (1.2)-(1.3) is equivalent to (1.6) and its global well-posedness can be asserted for $\tau=1$ as discussed in Remark 1.2. Another interesting question is the stationary problem of (1.2)-(1.3). The numerical simulations have demonstrated that the reduced system (1.6) admits stationary

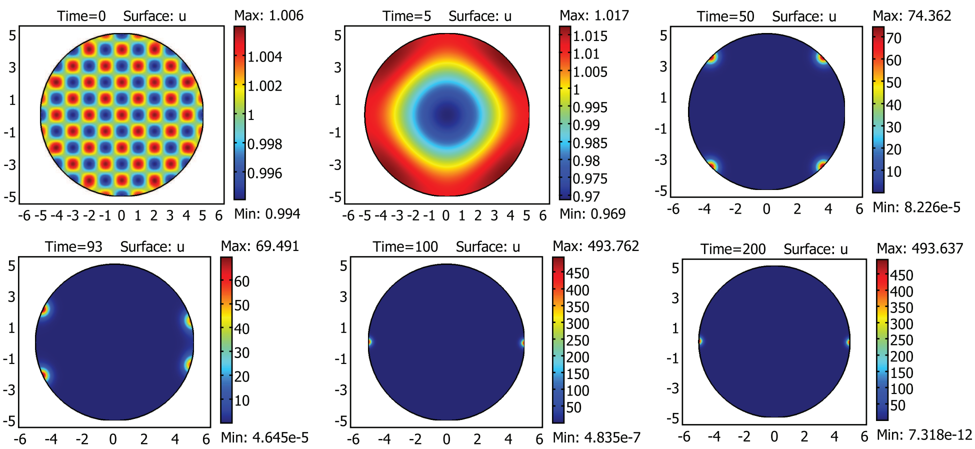

Figure 2: Time evolution patterns of $u(x,t)$ generated by the system (1.6) in a disk with radius 5 centered at the origin, where $\chi=5$, $\lambda=1$ and $\varepsilon=1$. The initial value (u_0,v_0,w_0) is chosen as $u_0=v_0=1+0.01\cdot\cos(\pi x)\cdot\cos(\pi y)$, $w_0=0$.

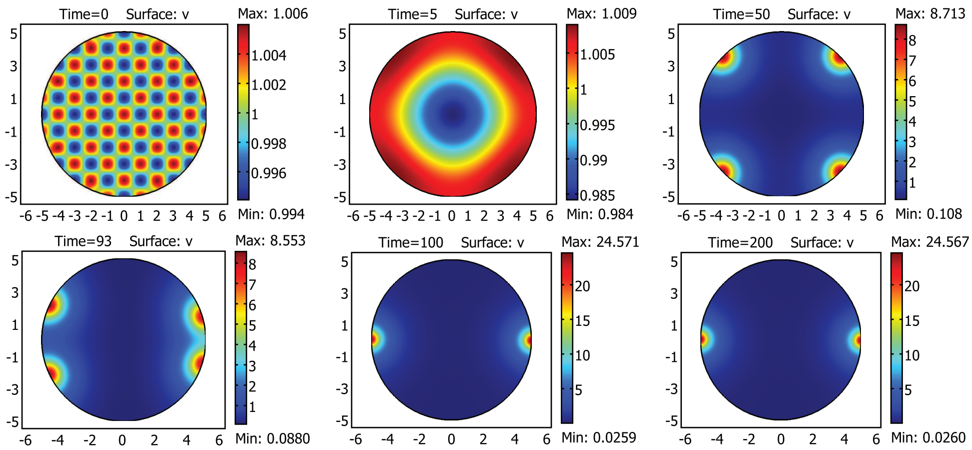

Figure 3: Time evolution patterns of $v(x,t)$ generated by the system (1.6) in a disk with radius 5 centered at the origin, where $\chi=5,\lambda=1$ and $\varepsilon=1$. The initial value (u_0,v_0,w_0) is chosen as $u_0=v_0=1+0.01\cdot\cos(\pi x)\cdot\cos(\pi y), w_0=0$.

aggregation patterns, which suggests that (1.6) has non-constant steady states. This, however, has not been justified in this paper. In a more general domain Ω, we conjecture that the problem (1.2)-(1.3) admits non-constant steady states as confirmed numerically in one dimension in Fig. 1. Furthermore as $\chi>0$ is large, the profile of non-constant steady states will be of spikes. All these questions are interesting and have to be left open for further research.

Acknowledgments

The research of W.R. Tao is partially supported by the fellowship of China Postdoctoral Science Foundation (No. 2020M683232), by the Fundamental Research Funds for the Central University (No. 2021CDJQY-047), PolyU Postdoc Matching Fund Scheme Project ID P0030816/B-Q75G and 1-W15F, and Chongqing Key Laboratory of Analytic Mathematics and Applications. The research of Z.A. Wang is partially supported by Hong Kong RGC GRF (Grant No. PolyU 15303019) and Postdoc Matching Fund Scheme Project ID P0034904 (Primary Work Programme W15F).

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