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Decay Rate Toward the Traveling Wave for Scalar Viscous Conservation Law

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Abstract. The time-decay rate toward the viscous shock wave for scalar viscous conservation law

$$u_t+f(u)_x=\mu u_{xx}$$

is obtained in this paper through an L^p estimate and the area inequality in [1] provided that the initial perturbations are small, i.e., $\|\Phi_0\|_{H^2}\le\varepsilon$, where Φ_0 is the anti-derivative of the initial perturbation. It is noted that there is no additional weighted requirement on Φ_0, i.e., $\Phi_0(x)$ only belongs to $H^2(R)$.

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1 Introduction

In this paper, we are concerned with the Cauchy problem of the viscous conservation law, which reads as,

$$\begin{cases} u_t+f(u)_x=\mu u_{xx}, \\ u(0,x)=u_0(x)\to u_\pm \quad \text{as} \quad x\to\pm\infty, \end{cases} \tag{1.1}$$

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where $u(x,t)$ is the unknown function, $f(u)$ is the so-called flux function, which is a given smooth function, $u_0(x)$ is the initial data, $\mu\in\mathbb{R}^+$ is the viscosity coefficient and $u_\pm$ are the far field states.

It is well-known that the long time behavior of solutions for the Cauchy problem (1.1) is closely related to the corresponding Riemann solutions, denoted as $u^R(x)$, of the Riemann problem

$$\begin{cases} u_t+f(u)_x=0, \\ u(0,x)=u_0^R(x), \end{cases} \tag{1.2}$$

where $u_0^R(x)$ is the Riemann initial data given by

$$u_0^R(x)=\begin{cases} u_-, & x<0, \\ u_+, & x>0. \end{cases} \tag{1.3}$$

The Riemann solutions contain two kinds of basic wave patterns, i.e., shock and rarefaction waves. In this paper, we focus on the shock wave case. Due to the effect of viscosity in (1.1), the shock wave is smoothed as a smooth function, called by viscous shock wave (or traveling wave). The stability of viscous shock wave has been extensively studied and many important achievements were obtained. Indeed, Il'in-Oleinik proved in 1960's that if $f(u)$ is strictly convex, i.e., $f''(u)>0$, the solution of (1.1) time-asymptotically tends to the viscous shock wave. Kawashima-Matsumura [8] further obtained the convergence rate if the initial data belongs to a weighted Sobolev space, see also [15] for the case that $f(u)$ is not convex or concave. An interesting L^1 stability theorem was established in [3].

Considerable progress on the asymptotic stability of traveling waves has been further achieved for the systems of viscous conservation laws such as compressible Navier-Stokes system since the pioneer works of Goodman [2] and Matsumura-Nishihara [14], see [4,5,8,10–12,18–20] and the references therein. In particular, Liu-Zeng [12] obtained the pointwise estimates in the stability analysis of viscous shock wave through approximate Green function approach and pointwise estimates.

Nevertheless, it is also interesting to study the decay rates toward the viscous shock wave through the basic energy method. To the best of our knowledge, Kawashima-Matsumura [8] first obtained the decay rate for scalar viscous conservation law (1.1) by a weighted energy method. Since then, there have been several works on the decay properties toward the viscous shock, cf. [7,15,16], in which all of the decay rates in time depend on the decay rates of the initial data at the far fields, that is, the initial data belongs to a weighted Sobolev space, and weighted estimates are needed, cf. [8].

The main purpose of this paper is to get the decay rate in time toward the traveling wave for the viscous conservation law (1.1) without additional condition on the initial data as in [8, 15]. In other word, more initial perturbations are allowed in our initial data.

We now state our main result. Without loss of generality, we assume that $u_- < u_+$. It is known that under the assumption of the so-called Oleinik entropy condition, cf. [13, 15],

$$h(u) := f(u) - f(u_\pm) - s(u - u_\pm) > 0, \quad (u_- < u < u_+), \tag{1.4}$$

the Riemann solution to the Riemann problem (1.2) consists of a single shock wave, cf. [17],

$$u^s(x-st) := \begin{cases} u_-, & x < st, \\ u_+, & x > st, \end{cases} \tag{1.5}$$

where s is the shock speed, determined by the Rankine-Hugoniot condition

$$-s(u_+ - u_-) + \big[f(u_+) - f(u_-)\big] = 0. \tag{1.6}$$

From (1.4), it holds that

$$h'(u_-) = f'(u_-) - s \geq 0, \quad h'(u_+) = f'(u_+) - s \leq 0.$$

In this paper, we only consider the case that

$$h'(u_-) = f'(u_-) - s > 0, \quad h'(u_+) = f'(u_+) - s < 0. \tag{1.7}$$

The viscous version of shock wave (viscous shock wave) is a special solution of (1.1) with the form

$$u = U(\xi), \quad \xi = x - st, \quad \lim_{\xi \to \pm\infty} U(\xi) = u_\pm. \tag{1.8}$$

The traveling wave $U(\xi)$ satisfies

$$\begin{cases} \big(-sU + f(U) - \mu U'\big)' = 0, \\ U(\pm\infty) = u_\pm, \end{cases} \tag{1.9}$$

where $' := \frac{d}{d\xi}$. Integrating (1.9) on $(-\infty, \xi)$ or $(\xi, +\infty)$, we get

$$-sU + f(U) - \mu U' = -su_\pm + f(u_\pm), \quad \xi \in \mathbb{R}. \tag{1.10}$$

We have

Lemma 1.1 ([15]). *Assume that the Oleinik entropy condition* (1.4) *and Rankine-Hugoniot condition* (1.6) *hold, then Eq.* (1.1) *admits a traveling wave solution* $U(\xi)$, $\xi = x - st$, *which is unique up to a shift and satisfies* $U' > 0$.

As in [6,8], we will reformulate the original Eq. (1.1) to an integrated equation. Let

$$\phi(x,t) := u(x,t) - U(\xi). \tag{1.11}$$

We assume that the initial data satisfies

$$u_0(x) - U(x) \in H^1 \cap L^1. \tag{1.12}$$

Thus we have

$$\phi_t + \left[f'(U)\phi\right]_x - \mu\phi_{xx} = N_x(x,t) \tag{1.13}$$

with the initial data $\phi_0(x) = u_0(x) - U(x)$, where

$$N(x,t) = -\left[f(u) - f(U) - f'(U)\phi\right] = \mathcal{O}(1)\phi^2. \tag{1.14}$$

Denote

$$\Phi(x,t) := \int_{-\infty}^{x} u(y,t) - U(y - st)dy. \tag{1.15}$$

Without loss of generality, we assume that $\Phi(\pm,0) = 0$ (otherwise we can replace $U(\xi)$ by $U(\xi + a)$ with a shift a determined by the initial data $u_0(x)$). Integrate (1.13) on $(-\infty, x)$, we get the following integrated equation:

$$\Phi_t + f'(U)\Phi_x - \mu\Phi_{xx} = N(x,t) \tag{1.16}$$

with the initial data

$$\Phi_0(x) = \int_{-\infty}^{x} \left[u_0(y) - U(y)\right] dy. \tag{1.17}$$

The main result is the following theorem.

Theorem 1.1. *Under the conditions* (1.7) *and* (1.12), *and* $\Phi_0(x) \in H^2(R)$, *there exists a positive constant* ε_0 *such that if* $\|\Phi_0\|_{H^2} \le \varepsilon_0$, *the Cauchy problem* (1.1) *has a unique global in time solution* $u(x,t)$ *satisfying*

$$u - U \in C\left([0,\infty); H^1\right) \cap L^2\left([0,\infty); H^2\right). \tag{1.18}$$

Furthermore it holds that, for any $2 < p < \infty$,

$$\|\Phi\|_{L^p}(t) \le Cp^{\frac{1}{4}}\varepsilon_0(1+t)^{-\frac{p-2}{4p}}, \tag{1.19}$$

$$\|u - U\|_{L^2}(t) \le Cp^{\frac{1}{8}}\varepsilon_0(1+t)^{-\frac{p-2}{8p}}, \tag{1.20}$$

$$\|u - U\|_{L^\infty}(t) \le Cp^{\frac{1}{6}}\varepsilon_0(1+t)^{-\frac{(p-2)(2p+1)}{4p(3p+2)}}. \tag{1.21}$$

Remark 1.1. In [8,15], the initial data $\Phi_0(x)$ belongs to a weighted Sobolev space, i.e.,

$$\int_R (1+x^2)^{\frac{\nu}{2}} \Phi_0^2(x)dx < +\infty, \quad \nu > 0. \tag{1.22}$$

Moreover the decay rates obtained in [8,15] depend on ν. The additional condition (1.22) is removed in Theorem 1.1.

Remark 1.2. The decay rate of $\|u-U\|_{L^2}$ is close to $(1+t)^{-1/8}$ for sufficiently large p. Similarly, the decay rate of $\|u-U\|_{L^\infty}$ is close to $(1+t)^{-1/6}$ and it can be improved a little as the initial data is more regular, see Remark 3.2 below.

We outline the proof of Theorem 1.1. From [6,8], $\|\Phi\|_{L^2}(t)$ is uniformly bounded by the initial data. Although the L^2 norm $\|\Phi\|_{L^2}(t)$ may not tend to zero as $t \to \infty$, we observe that the L^p norm $(p > 2)$ decays to zero with a rate of (1.19) by a delicate L^p estimate. This rate can yield a differential inequality for the derivative norm $f = \|\Phi_x\|_{L^2}^2$, i.e.,

$$f_t \le C(1+t)^{-\alpha}, \quad f \in L^1(0,\infty) \tag{1.23}$$

for some $0 < \alpha < 1$. The area inequality established in [1] is then applied to derive the desired decay rate (1.20) and the rate (1.21) of $\|u-U\|_{L^\infty}$ is finally obtained by the Gagliardo-Nirenberg inequality.

Notations. We denote $\|u\|_{L^p}$ by the norm of Sobolev space $L^p(\mathbb{R})$, C by the generic positive constants. If $p=2$, we omit the subscript, i.e $\|\cdot\|$ is the L^2 norm.

2 Preliminaries

In this section, we give some preliminaries, which will be used in the proof of the main theorem. First we introduce the famous Gagliardo-Nirenberg (GN) inequality which reads as,

Lemma 2.1 (Gagliardo-Nirenberg inequality). *For any $1 \le p \le +\infty$ and integer $0 \le j < m$,*

$$\|\nabla_x^j u\|_{L^p(\mathbb{R}^n)} \le C \|\nabla_x^m u\|_{L^r(\mathbb{R}^n)}^{\theta} \|u\|_{L^q(\mathbb{R}^n)}^{1-\theta}, \tag{2.1}$$

where

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\theta + \frac{1}{q}(1-\theta), \quad \frac{j}{m} \le \theta \le 1$$

and C is a constant independent of u.

Based on the Gagliardo-Nirenberg inequality, we have

Lemma 2.2 (Interpolation inequality [9])**.**

$$\|u\|_{L^p} \le C(p) \|\nabla(|u|^{\frac{p}{2}})\|_{L^2}^{\frac{2\gamma}{1+\gamma p}} \|u\|_{L^2}^{\frac{1}{1+\gamma p}} \tag{2.2}$$

for $2 \le p < \infty$, *where* $\gamma = \frac{1}{2}(\frac{1}{2} - \frac{1}{p})$ *and* $C(p)$ *is a positive constant.*

We now introduce an area inequality.

Lemma 2.3 (Area Inequality, cf. [1])**.** *Assume that a Lipschitz continuous function* $f(t) \ge 0$ *satisfies*

$$f'(t) \le C_0(1+t)^{-\alpha}, \tag{2.3}$$

and

$$\int_0^t f(s)ds \le C_1(1+t)^{\beta} \ln^{\gamma}(1+t), \quad \gamma \ge 0 \tag{2.4}$$

for some constants C_0 *and* C_1, *where* $0 \le \beta < \alpha$. *Then it holds that if* $\alpha + \beta < 2$,

$$f(t) \le 2\sqrt{C_0 C_1}(1+t)^{\frac{\beta-\alpha}{2}} \ln^{\frac{\gamma}{2}}(1+t), \quad t >> 1. \tag{2.5}$$

Moreover, if $\beta = \gamma = 0$, *i.e.,* $f(t) \in L^1[0,\infty)$ *and* $0 < \alpha \le 2$, *then*

$$f(t) = o(t^{-\frac{\alpha}{2}}) \quad as \quad t >> 1, \tag{2.6}$$

and the index $\frac{\alpha}{2}$ *is optimal.*

Remark 2.1. The time-decay rate (2.5) is surprising for the case $0 < \alpha < 1, \beta = \gamma = 0$, where the condition (2.4) becomes

$$\int_0^{+\infty} f(t)dt \le C_1 < \infty. \tag{2.7}$$

To get the decay rate of $f(t)$, the usual way is to multiply (2.3) by $1+t$, then we have

$$\big[(1+t)f(t)\big]' \le f(t) + C_0(1+t)^{1-\alpha}. \tag{2.8}$$

Integrating (2.8) on $[0,t]$ implies that

$$f(t) \le C(1+t)^{1-\alpha} \quad \text{as} \quad t >> 1. \tag{2.9}$$

It is impossible from (2.9) to get the time-decay rate of $f(t)$ as $0 < \alpha < 1$. So the decay rate (2.6) is surprising in this sense.

Proof. The complete proof of Lemma 2.3 can be found in [1]. Here we give a simple proof of the decay rate (2.6) for the special case that $0<\alpha<1,\beta=\gamma=0$ under additional condition that $f(t)$ is uniformly bounded.

Note that for any large time t, the function $f(\tau)$ on $[0,t]$ satisfies the inequality (2.3) which is equivalent to

$$\frac{df(\tilde{\tau})}{d\tilde{\tau}}\geq -C_0(1+t-\tilde{\tau})^{-\alpha}, \quad f(\tilde{\tau})|_{\tilde{\tau}=0}=f(t), \quad \tilde{\tau}\geq 0, \tag{2.10}$$

where $\tilde{\tau}=t-\tau$. Then we construct a function $g(\tilde{\tau})$ satisfying

$$\frac{dg(\tilde{\tau})}{d\tilde{\tau}}=-C_0(1+t-\tilde{\tau})^{-\alpha}, \quad g(\tilde{\tau})|_{\tilde{\tau}=0}=f(t), \quad \tilde{\tau}\geq 0. \tag{2.11}$$

It is straightforward to check that $f(\tau)\geq g(\tau)$ for any $\tau\in[0,t]$, see Fig. 1.

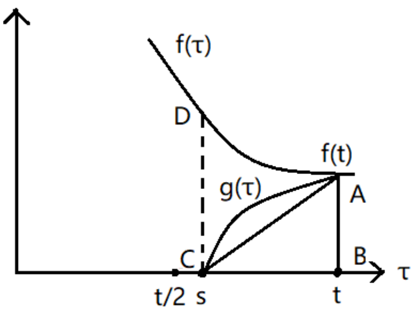

Figure 1: The structure of functions $f(\tau)$ and $g(\tau)$.

The ODE (2.11) is exactly a backward ordinary differential equation starting from time t, that is,

$$\begin{cases} \dfrac{dg(\tau)}{d\tau}=C_0(1+\tau)^{-\alpha}, & 0\leq\tau\leq t, \\ g(t)=f(t). \end{cases} \tag{2.12}$$

Without loss of generality, we assume that $f(t)>0$. A direct computation gives the formula of $g(\tau)$ for $0<\tau\leq t$,

$$\begin{aligned} g(\tau)&=f(t)-C_0\int_\tau^t(1+y)^{-\alpha}dy \\ &=f(t)-\frac{C_0}{1-\alpha}\left[(1+t)^{1-\alpha}-(1+\tau)^{1-\alpha}\right]. \end{aligned} \tag{2.13}$$

Taking $\tau=\frac{t}{2}$, it follows from $1-\alpha>0$ that if $t>>1$ then

$$g\left(\frac{t}{2}\right)=f(t)-\frac{C_0}{1-\alpha}\left[(1+t)^{1-\alpha}-\left(1+\frac{t}{2}\right)^{1-\alpha}\right]<0, \tag{2.14}$$

where we have used the additional condition that $f(t)$ is uniformly bounded. Since $g(\tau)$ is monotonically increasing, there exists a unique $s\in(\frac{t}{2},t)$ such that $g(s)=0$. Taking $\tau=s$ in (2.13), it follows from the mean value theorem that

$$\begin{aligned}
s=&\left[(1+t)^{1-\alpha}-\frac{1-\alpha}{C_0}f(t)\right]^{\frac{1}{1-\alpha}}-1\\
=&(1+t)\left[1-\frac{1-\alpha}{C_0}f(t)(1+t)^{\alpha-1}\right]^{\frac{1}{1-\alpha}}-1\\
=&(1+t)\left[1-\frac{1}{C_0}f(t)(1+t)^{\alpha-1}(1-\xi_t)^{\frac{\alpha}{1-\alpha}}\right]-1 \quad \text{for some} \quad \xi_t\in(0,1/2)\\
\le& t-\frac{1}{C_0}\left(\frac{1}{2}\right)^{\frac{\alpha}{1-\alpha}}f(t)(1+t)^{\alpha},
\end{aligned} \tag{2.15}$$

due to $f(t)(1+t)^{\alpha-1}=o(1)$ as $t>>1$. Thus it holds that

$$t-s\ge\frac{1}{C_0}2^{-\frac{\alpha}{1-\alpha}}f(t)(1+t)^{\alpha}. \tag{2.16}$$

Note that the curve $g(\tau)$ is concave due to the fact that the derivative of $(1+\tau)^{-\alpha}$ in (2.12) is negative. Thus the region S_{ABCD} surrounded by the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and the curve $\widetilde{DA}$ should cover the triangle △ABC, see Fig. 1.

That is, as $t>>1$,

$$\begin{aligned}
o(1)=&\int_s^t f(\tau)d\tau=\text{area of } S_{ABCD}\ge\text{area of } \Delta_{ABC}\\
=&\frac{1}{2}f(t)(t-s)\ge\frac{1}{2C_0}2^{-\frac{\alpha}{1-\alpha}}f^2(t)(1+t)^{\alpha},
\end{aligned} \tag{2.17}$$

which gives the desired convergence rate (2.6). □

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The global existence theorem was established in [15], i.e.,

Theorem 3.1 (Global existence, cf. [15]). *Under the conditions* (1.7) *and* (1.12), *and* $\Phi_0(x)\in H^2(R)$, *there exists a positive constant* ε_0 *such that if* $\|\Phi_0\|_{H^2}\le\varepsilon_0$, *the Cauchy problem* (1.16) *has a unique global in time solution* $\Phi(x,t)$ *satisfying*

$$\|\Phi\|^2_{H^2}(t)+\int_0^t\|\Phi\|^2_{H^3}(\tau)d\tau\le C\varepsilon_0^2. \tag{3.1}$$

Based on the global existence Theorem 3.1, we shall establish a L^p estimate for $\Phi(x,t)$. The L^p method was first used in [9] to study the decay properties of the solution toward the rarefaction wave for scalar viscous conservation law. Here we use the L^p method to study the decay rate for the anti-derivative $\Phi(x,t)$.

3.1 L^p estimate

Proposition 3.1. *Under the conditions of Theorem* 3.1, *it holds that, for* $2<p<\infty$,

$$\|\Phi\|_{L^p}\le Cp^{\frac{1}{4}}\varepsilon_0(1+t)^{-\frac{p-2}{4p}}. \tag{3.2}$$

Proof. For any $2<p<\infty$, multiply (1.16) by $w|\Phi|^{p-2}\Phi$, where $w=w(U)$ is a weight function which will be determined later, we have

$$\begin{aligned}&\frac{1}{p}w\frac{d}{dt}(|\Phi|^p)+f'(U)w|\Phi|^{p-2}\Phi\Phi_x-\mu w|\Phi|^{p-2}\Phi\Phi_{xx}\\&=N(x,t)|\Phi|^{p-2}\Phi w.\end{aligned} \tag{3.3}$$

Following the same line as in [15], see also [13], we arrive at

$$\begin{aligned}&\frac{d}{d\tau}\int\frac{1}{p}w|\Phi(t)|^p dx-\int\frac{1}{p}(hw)''|\Phi|^pU'dx\\&\quad+(p-1)\int\mu w|\Phi_x|^2|\Phi|^{p-2}dx\\&=\int w|\Phi|^{p-2}\Phi N dx,\end{aligned}$$

where $(hw)''$ means $\frac{d^2}{dU^2}(h(U)w(U))$. As in [15], we choose the weight function $w(u)$ as

$$w(u)=\begin{cases}-\dfrac{(u-u_-)(u-u_+)}{h(u)}, & (u_-<u<u_+),\\ -\dfrac{u_\pm-u_\mp}{f'(u_\pm)-s}, & (u=u_\pm).\end{cases} \tag{3.4}$$

By the condition (1.7), there exists a positive constant C such that

$$C^{-1}<w<C, \quad (hw)''=-2. \tag{3.5}$$

In view of Theorem 3.1, $\|\Phi\|_{L^\infty}\le C\varepsilon_0$ is small. Note that $N=\mathcal{O}(1)\Phi_x^2$ and $U'>0$ due to Lemma 1.1, one has

$$\frac{d}{d\tau}\int w|\Phi(t)|^p dx+\int \mu w\big\|\big(|\Phi|^{\frac{p}{2}}\big)_x\big\|^2 dx\le 0. \tag{3.6}$$

To get the decay rate (3.2), we multiply (3.6) by $(1+\tau)^\sigma$ and integrate the resulting equation on $(0,t)$, so that

$$\begin{aligned}&(1+t)^\sigma\|\Phi(t)\|_{L^p}^p+\int_0^t(1+\tau)^\sigma\big\|\partial_x\big(|\Phi|^{\frac{p}{2}}\big)\big\|^2 d\tau\\ \le&\|\Phi_0\|_{L^p}^p+\sigma\int_0^t(1+\tau)^{\sigma-1}\|\Phi(t)\|_{L^p}^p dt.\end{aligned} \tag{3.7}$$

We use the interpolation inequality (2.2) to estimate the RHS of (3.7). Since $p>2$ can be any number, we need to check the coefficient $C(p)$ carefully below. Thanks to the Sobolev inequality, we have

$$\|\Phi\|_{L^p}^p\le\|\Phi\|^2\|\Phi\|_{L^\infty}^{p-2}, \tag{3.8}$$

$$\|\Phi\|_{L^\infty}^p\le 2\|\Phi\|_{L^p}^{\frac{p}{2}}\big\|\big(|\Phi|^{\frac{p}{2}}\big)_x\big\|. \tag{3.9}$$

These inequalities give that

$$\|\Phi\|_{L^p}^p\le 2^{\frac{2(p-2)}{p+2}}\|\Phi\|^{\frac{4p}{p+2}}\big\|\big(|\Phi|^{\frac{p}{2}}\big)_x\big\|^{\frac{2(p-2)}{p+2}}, \tag{3.10}$$

from which, it follows from the Cauchy inequality

$$ab\le\frac{a^p}{p}+\frac{b^q}{q},\quad \frac{1}{p}+\frac{1}{q}=1,$$

that

$$\begin{aligned}&\sigma(1+\tau)^{\sigma-1}\|\Phi(t)\|_{L^p}^p\\ \le&\sigma(1+\tau)^{\sigma-1}2^{\frac{2(p-2)}{p+2}}\|\Phi\|^{\frac{4p}{p+2}}\big\|\big(|\Phi|^{\frac{p}{2}}\big)_x\big\|^{\frac{2(p-2)}{p+2}}\\ \le&\frac{p-2}{p+2}(1+t)^\sigma\|\partial_x\big(|\Phi|^{\frac{p}{2}}\big)\|^2+\frac{4}{p+2}2^{\frac{p-2}{2}}\sigma^{\frac{p+2}{4}}(1+t)^{\sigma-\frac{p+2}{4}}\|\Phi\|_{L^2}^p.\end{aligned} \tag{3.11}$$

Choosing $\sigma=\frac{p+2}{4}$, we get from Theorem 3.1 that

$$\|\Phi\|_{L^p}^p \le \|\Phi_0\|_{L^p}^p(1+t)^{-\frac{p+2}{4}} + \frac{4}{p+2}2^{\frac{p-2}{2}}\sigma^{\frac{p+2}{4}}(C\varepsilon_0)^p(1+t)^{-\frac{p-2}{4}}, \tag{3.12}$$

or

$$\|\Phi\|_{L^p} \le Cp^{\frac{1}{4}}\varepsilon_0(1+t)^{-\frac{p-2}{4p}}, \tag{3.13}$$

where C is independent of p. □

3.2 Decay of $\|\Phi_x\|=\|\phi\|$

Proposition 3.2. *Under the conditions of Theorem* 1.1, *it holds that*

$$\|\phi\|_{L^2} \le Cp^{\frac{1}{8}}\varepsilon_0(1+t)^{-\frac{p-2}{8p}}. \tag{3.14}$$

Proof. Multiplying (1.13) by ϕ and integrating the resulting equation on $\mathbb{R}$ with respect to x, we have

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2 + \int_{\mathbb{R}} [f'(U)\phi]_x \phi dx + \mu\|\phi_x\|^2 = \int_{\mathbb{R}} N_x(x,t)\phi dx. \tag{3.15}$$

Note that

$$N(x,t) = \int_0^1 f''(U+\theta\phi)\theta d\theta \phi^2 =: Q(U,\phi)\phi^2, \tag{3.16}$$

then we have

$$\begin{aligned}
\left|\int_{\mathbb{R}} N_x(x,t)\phi dx\right| &= \left|\int_{\mathbb{R}} Q(U,\phi)\left(\frac{2\phi^3}{3}\right)_x + Q(U,\phi)_x\phi^3 dx\right| \\
&= \left|\frac{1}{3}\int_{\mathbb{R}} \left[Q_U U' + Q_\phi \phi_x\right]\phi^3 dx\right| \\
&\le \int_{\mathbb{R}} U'\phi^2 dx + C\|\phi\|_{L^\infty}^2\|\phi_x\|\|\phi\| \\
&\le \int_{\mathbb{R}} U'\phi^2 dx + C\varepsilon_0^2\|\phi_x\|^2 \le C\|\phi\|_{L^\infty}^2 + C\varepsilon_0^2\|\phi_x\|^2.
\end{aligned} \tag{3.17}$$

On the other hand,

$$\left|\int_{\mathbb{R}} [f'(U)\phi]_x\phi dx\right| = \left|\int_{\mathbb{R}} f'(U)\left(\frac{\phi^2}{2}\right)_x dx\right| \le C\int_{\mathbb{R}} U'\phi^2 dx \le C\|\phi\|_{L^\infty}^2. \tag{3.18}$$

By the Garglilardo-Nirenberg inequality (2.1), we have

$$\|\phi\|_{L^\infty}^2=\|\Phi_x\|_{L^\infty}^2\le C\|\phi_x\|^{\frac{4(p+1)}{3p+2}}\|\Phi\|_{L^p}^{\frac{2p}{3p+2}}\le\frac{\mu}{4}\|\phi_x\|^2+C\|\Phi\|_{L^p}^2. \tag{3.19}$$

From (3.13) and (3.15)-(3.19), we get

$$\frac{d}{dt}\|\phi\|^2\le Cp^{\frac{1}{2}}\varepsilon_0^2(1+t)^{-\frac{p-2}{2p}}. \tag{3.20}$$

Thanks to

$$\int_0^\infty\|\phi\|^2(\tau)d\tau\le C\varepsilon_0^2$$

due to Theorem 3.1, we conclude from the area inequality in Lemma 2.3 that

$$\|\phi\|^2\le Cp^{\frac{1}{4}}\varepsilon_0^2(1+t)^{-\frac{p-2}{4p}}, \tag{3.21}$$

and finally obtain the rate (3.14). □

Remark 3.1. The Gagliardo-Nirenberg inequality

$$\|\phi\|_{L^2}\le\|\Phi\|_{L^p}^{1-\lambda}\|\phi_x\|^\lambda,\quad \lambda=\frac{p+2}{3p+2} \tag{3.22}$$

holds only for $p\le 2$, while $\|\Phi\|_{L^2}$ is known uniformly bounded. Thus it is difficult to obtain the decay rate of $\|\phi\|_{L^2}$ through the Gagliardo-Nirenberg inequality, and the area inequality can provide the rate (3.14).

3.3 Decay of $\|\phi_x\|$

Proposition 3.3. *Under the conditions of Theorem* 1.1*, it holds that,*

$$\|\phi_x\|_{L^2}\le Cp^{\frac{1}{8}}\varepsilon_0(1+t)^{-\frac{p-2}{8p}}. \tag{3.23}$$

Proof. Multiplying (1.13) by $-\phi_{xx}$ and integrating the resulting equation on $\mathbb{R}$ with respect to x, we have

$$\frac{1}{2}\frac{d}{dt}\|\phi_x\|^2+\mu\|\phi_{xx}\|^2=\int_{\mathbb{R}}[f'(U)\phi]_x\phi_{xx}dx-\int_{\mathbb{R}}N_x(x,t)\phi_{xx}dx. \tag{3.24}$$

We now estimate the right hand side of (3.24). A direct computation gives that

$$\left|\int_{\mathbb{R}}[f'(U)\phi]_x\phi_{xx}dx\right|\le\frac{1}{8}\mu\|\phi_{xx}\|^2+C\left(\|\phi\|_{L^\infty}^2+\|\phi_x\|_{L^\infty}^2\right)$$

$$\leq \frac{1}{4}\mu\|\phi_{xx}\|^2 + Cp^{\frac{1}{2}}\varepsilon_0^2(1+t)^{-\frac{p-2}{2p}}, \tag{3.25}$$

where we have used the fact that

$$\|\phi\|_{L^\infty}^2 \leq C\|\phi_{xx}\|^{\frac{4(p+1)}{5p+2}}\|\Phi\|_{L^p}^{\frac{6p}{5p+2}} \leq \frac{\mu}{16}\|\phi_{xx}\|^2 + Cp^{\frac{1}{2}}\varepsilon_0^2(1+t)^{-\frac{p-2}{2p}}, \tag{3.26}$$

$$\|\phi_x\|_{L^\infty}^2 \leq C\|\phi_{xx}\|^{\frac{4(2p+1)}{5p+2}}\|\Phi\|_{L^p}^{\frac{2p}{5p+2}} \leq \frac{\mu}{16}\|\phi_{xx}\|^2 + Cp^{\frac{1}{2}}\varepsilon_0^2(1+t)^{-\frac{p-2}{2p}}. \tag{3.27}$$

For the last term of the RHS of (3.24), the formula of N yields that

$$\begin{aligned}\left|\int_{\mathbb{R}} N_x(x,t)\phi_{xx}dx\right| &\leq \frac{\mu}{8}\|\phi_{xx}\|^2 + \int_{\mathbb{R}} N_x^2(x,t)dx \\ &\leq \frac{\mu}{8}\|\phi_{xx}\|^2 + C\int_{\mathbb{R}}\left(U_x^2\phi^4 + \phi^2\phi_x^2\right)dx \\ &\leq \frac{\mu}{4}\|\phi_{xx}\|^2 + Cp^{\frac{1}{2}}\varepsilon_0^2(1+t)^{-\frac{p-2}{2p}}. \end{aligned} \tag{3.28}$$

Thus we get

$$\frac{d}{dt}\|\phi_x\|^2 \leq Cp^{\frac{1}{2}}\varepsilon_0^2(1+t)^{-\frac{p-2}{2p}}, \tag{3.29}$$

and $\|\phi_x\|^2 \in L^1(0,\infty)$ due to Theorem 3.1. Again using the area inequality, one has

$$\|\phi_x\|^2 \leq Cp^{\frac{1}{4}}\varepsilon_0^2(1+t)^{-\frac{p-2}{4p}}, \tag{3.30}$$

and the estimate (3.23) is proved. □

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. It remains to show (1.21), which can be achieved from the Gagliardo-Nirenberg inequality and the decay rates (3.13), (3.23), i.e.,

$$\begin{aligned}\|\phi\|_{L^\infty} &\leq C\|\Phi\|_{L^p}^{\frac{p}{3p+2}}\|\phi_x\|^{\frac{2(p+1)}{3p+2}} \\ &\leq C\varepsilon_0 p^{\frac{2p+1}{4(3p+2)}}(1+t)^{-\frac{(p-2)(2p+1)}{4p(3p+2)}} \\ &\leq C\varepsilon_0 p^{\frac{1}{6}}(1+t)^{-\frac{(p-2)(2p+1)}{4p(3p+2)}}. \end{aligned} \tag{3.31}$$

This completes the proof. □

Remark 3.2. By the same idea as in the proof of Proposition 3.3, we can get a better decay rate for $\|\phi\|_{L^\infty}$ through higher order derivative estimates if the initial data $\Phi_0(x)$ is more regular, i.e., $\Phi_0 \in H^m(\mathbb{R}), m \geq 3$ with integer m.

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References

[1] Z. Dong, F. Huang and H. Su, *Large time behavior of strong solutions for stochastic Burgers equation with transport noise*, arXiv:2103.06608, 2021.

[2] J. Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Ration. Mech. Anal. 95(4) (1986), 325–344.

[3] H. Freistuhler and D. Serre, L^1 *stability of shock waves in scalar viscous conservation laws*, Comm. Pure Appl. Math. 51(3) (1998), 291–301.

[4] L. He and F. Huang, *Nonlinear stability of large amplitude viscous shock wave for general viscous gas*, J. Differential Equations, 269(2) (2020), 1226–1242.

[5] F. Huang and A. Matsumura, *Stability of a composite wave of two viscous shock waves for full compressible Navier-Stokes equation*, Commun. Math. Phys. 298 (2009), 841–861.

[6] A. M. Ilin and O. A. Olenik, *Asymptotic behavior of the solutions of the Cauchy problem for some quasi-linear equations for large values of the time*, Mat. Sb. 51 (1960), 191–216 (in Russian).

[7] C. Jones, R. Gardner and T. Kapitula, *Stability of travelling waves for non-convex scalar viscous conservation laws*, Commun. Pure Appl. Math. 46 (1993), 505–526.

[8] S. Kawashima and A. Matsumura, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion*, Commun. Math. Phys. 101 (1985), 97–127.

[9] S. Kawashima, S. Nishibata and M. Nishikawa, L^p *energy method for multi-dimensional viscous conservation laws and application to the stability of planar waves*, J. Hyperbolic Differ. Equations 01(03) (2004), 581–603.

[10] T. Liu, *Pointwise convergence to shock waves for viscous conservation laws*, Comm. Pure Appl. Math. 50(11) (1997), 1113–1182.

[11] T. Liu and Y. Zeng, *Time-asymptotic behavior of wave propagation around a viscous shock profile*, Comm. Math. Phys. 290(1) (2009), 23–82.

[12] T. Liu and Y. Zeng, *Shock Waves in Conservation Laws with Physical Viscosity*. Memoirs of the American Mathematical Society, Vol. 234(1105), AMS, 2015.

[13] A. Matsumura, *Waves in Compressible Fluids: Viscous Shock, Rarefaction, and Contact Waves. Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Y. Giga and A. Novotny (Eds.)*, Springer, 2016.

[14] A. Matsumura and K. Nishihara, *Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math. 3 (1986), 1–13.

[15] A. Matsumura and K. Nishihara, *Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity*, Comm. Math. Phys. 165 (1994), 83–96.

[16] K. Nishihara, *A note on the stability of traveling wave solutions of Burgers' equation*, Jpn. J. Appl. Math. 2 (1985), 27–35.

[17] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, 1983.

[18] A. Szepessy, Z. Xin, *Nonlinear stability of viscous shock waves*, Arch. Ration. Mech. Anal. 122 (1993), 53–103.

[19] N. Yoshida, *Decay properties of solutions toward a multiwave pattern for the scalar viscous conservation law with partially linearly degenerate flux*, Nonlinear Anal. 96 (2014), 189–210.

[20] K. Zumbrun, *Stability of large-amplitude shock waves of compressible Navier-Stokes equations*, in: Handbook of Mathematical Fluid Dynamics, Vol. III, 311–533, North-Holland, 2004.