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# The Fractional Ginzburg-Landau Equation with Initial Data in Morrey Spaces $\phi$

Jingna Li[1,*], Xiaoting Yang[1] and Li Xia[2]

[1] *Department of Mathematics, Jinan University, Guangzhou 510632, China.*
[2] *Department of Mathematics, Guangdong University of Finance & Economics, Guangzhou 510320, China.*

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**Abstract.** The paper is concerned with fractional Ginzburg-Landau equation. Existence and uniqueness of local and global mild solution with initial data in Morrey spaces are obtained by contraction mapping principle and carefully choosing the working space, further regularity of mild solution is also discussed.

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**Key words**: Fractional Ginzburg-Landau equation, well-posedness, Morrey space, regularity.

## 1 Introduction

The effects of singularities, fractal support and long-range interactions of the system are involved in numerous applications such as chaotic dynamics [18], material science [11], physical kinetics [19], and among other (see, e.g., [8, 9, 15]). Fractional dynamics equations are just the right tool to describe these phenomena because they are nonlocal, which means they depend on the value of the whole space from the mathematical point of view, see Metzler and Klafter [6].

The fractional generalization of the Ginzburg-Landau equation was first proposed by Tarasov and Zaslavsky [13]. Its rescaled form is

$$\frac{\partial u}{\partial t} = Au-(a+\nu i)\Lambda^{2\alpha}u-(b+\mu i)u|u|^{2\sigma}, \tag{1.1}$$

*Corresponding author. *Email address:* `jingna8005@hotmail.com` (J. Li)

$$u(x,0)=u_0(x), \tag{1.2}$$

where $\Lambda=(-\triangle)^{\frac{1}{2}}$, and $\sigma>0, A\geq 0, a>0, b>0, \alpha\in(0,1], \nu, \mu$ are real constants. Actually, this equation can be used to describe dynamic processes in medium with fractional mass dimension or a continuum with fractional dispersion [12]. It is indicated by asymptotic analysis that an implication of the complex fractional Ginzburg-Landau equation was the renormalization of the transition state owing to the non-locality of competition [7]. In [10], the Psi-series solution of the one-dimensional fractional Ginzburg-Landau equation was proposed and the dominant order behavior and its structure of arbitrary singular solutions are discussed. In [5], we obtained local well-posedness result for the whole space case with initial data in $L^p(\mathbb{R}^n)$, $1\leq p\leq\infty$, $C(\mathbb{R}^n)$ and global well-posedness result for the periodic case. In [4], we proved that the initial-value problem of (1.1)-(1.2) with $0<\sigma\leq 1$ is locally well-posed with initial data in $\dot{W}^{r,p}(\mathbb{R}^n)$ and $\dot{W}^{r,p}(\mathbb{T}^n)$, $\mathbb{T}^n=\mathbb{R}^n/Z^n$ if $r$ and $p$ satisfy

$$1<p<\infty, \quad \frac{\alpha}{3}<\frac{n}{p}\leq\alpha, \quad r=\frac{n}{p}-\alpha\leq 0$$

by contraction mapping principle.

As we learned from [7, 12–14, 17], very singular initial data such as certain measures concentrated on smooth surfaces are of real physical interest for fractional Ginzburg-Landau equation, which motivates us to reconsider the problem of (1.1)-(1.2) containing initial data in the Morrey space. In this paper, we prove that if $0<\alpha\leq 1$ and $u_0\in M_{p,\lambda}(\mathbb{R}^n)$ with

$$1\leq p<\infty, \quad 0\leq\lambda<\infty, \quad \frac{n-\lambda}{p}<\frac{\alpha}{\sigma},$$

then the problem of (1.1)-(1.2) is locally well-posed for some $T>0$ and for sufficiently small initial data the solution is global. Moreover, we prove that the solution is actually smooth for $0<\sigma<1$. The precise statement of the results is presented in Theorem 3.1 of Section 3. For initial data $u_0(x)\in M_{p,\lambda}(\mathbb{R}^n)$, we prove that Eq. (1.1)-(1.2) admits a solution $u\in BC([0,T);L_{-k/q,q})$, and for the global solution $u$ we have decay rate

$$\|u\|_{L^\infty}=\mathcal{O}(t^z),$$

where

$$z=-\left[\frac{n-\lambda}{p\alpha}\left(\sigma+\frac{1}{2}\right)-1\right].$$

The results reduce to those in $L^p(\mathbb{R}^n)$ theory by taking $\lambda=0$.

The rest of this article is organized as follows. In Section 2, the definition and some properties of Morrey space, as well as the solution operator of semi-linear equations and their properties are given. In Section 3, we demonstrate that the solution to the initial value problem (1.1)-(1.2) exists globally with small initial data. And further regularity has been showed in Section 4.

# 2 The linear equation

In this section, we will introduce some of the basic properties of the Morrey space and then study the corresponding linear equation on the Morrey spaces. First we give the basic definition.

**Definition 2.1.** *For $1\le p<\infty$, $0\le\lambda<n$, the Morrey space $M_{p,\lambda}$ is defined as*

$$M_{p,\lambda}=M_{p,\lambda}(\mathbb{R}^n):=\Big\{f\in L^p_{loc}(\mathbb{R}^n)\,|\,\|f\|_{M_{p,\lambda}}<\infty\Big\},$$

*where the norm is given by*

$$\|f\|_{M_{p,\lambda}}=\sup_{\{x\in\mathbb{R}^n,R>0\}}R^{-\frac{\lambda}{p}}\left(\int_{|y-x|\le R}|f|^p(y)dy\right)^{\frac{1}{p}}.$$

*$\ddot{M}_{p,\lambda}$ is the following subspace of $M_{p,\lambda}$:*

$$\ddot{M}_{p,\lambda}:=\Big\{f\in M_{p,\lambda}\,|\,\|f(\cdot-y)-f(\cdot)\|_{M_{p,\lambda}}\to 0 \text{ as } y\to 0\Big\}.$$

We note that $M_{p,\lambda}$ is a Banach space and $\ddot{M}_{p,\lambda}$ is a closed subspace of $M_{p,\lambda}$. For $p>1$, $M_{p,0}=L^p$ and $M_{1,0}=M$, where $M$ is the space of finite measures. The index in the symbol $M_{p,\lambda}$ will be restricted to $1\le p<\infty$, $0\le\lambda<n$, when they are not specified. Sometimes we consider $p=\infty$ and then $M_{\infty,\lambda}$ simply means $L^\infty$. Some general properties of Morrey spaces [1] will be used in our article. More properties of Morrey spaces in another form were presented in [2]. For the reader's convenience, they are listed in the following lemma.

**Lemma 2.1.** *For $1\le p,q,r\le\infty$, we have*

*(i) Inclusion relations*

$$M_{p,\lambda}\subset M_{q,\mu},\quad \text{if}\quad \frac{n-\lambda}{p}=\frac{n-\mu}{q},\quad q\le p. \tag{2.1}$$

(ii) *The Hölder inequality*

$$\|fg\|_{M_{p,\lambda}} \leq \|f\|_{M_{q,\mu}} \|g\|_{M_{r,\nu}}, \tag{2.2}$$

*where*

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}, \quad \frac{\lambda}{p} = \frac{\mu}{q} + \frac{\nu}{r}.$$

(iii) *Continuous embedding in weighted Lebesgue space*

$$M_{p,\lambda} \hookrightarrow L_{-\frac{\mu}{p},p} \quad for \quad p>1, \quad \mu>\lambda, \tag{2.3}$$

*where $L_{s,p}$ is the weighted Lebesgue space composed of functions $f$ such that*

$$\left(1+|x|^2\right)^{\frac{s}{2}} f \in L^p$$

*with the norm*

$$\|f\|_{L_{s,p}} = \left\| \left(1+|x|^2\right)^{\frac{s}{2}} f \right\|_{L^p}.$$

Now let us consider the linear equation

$$\partial_t u - Au + (a+\nu i)\Lambda^{2\alpha} u = f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,+\infty), \tag{2.4}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n. \tag{2.5}$$

The exact solution of (2.4)-(2.5) is

$$\begin{aligned} u &= e^{At-(a+i\nu)t\Lambda^{2\alpha}} u_0 + \int_0^t e^{A(t-\tau)-(a+i\nu)(t-\tau)\Lambda^{2\alpha}} f(\tau) d\tau \\ &= e^{At} g_\alpha(\cdot,t) * u_0(\cdot) + \int_0^t e^{A(t-\tau)} g_\alpha(\cdot,t-\tau) * f(\cdot,\tau) d\tau \\ &\stackrel{\triangle}{=} Uu_0 + Gf, \end{aligned} \tag{2.6}$$

where $e^{-(a+i\nu)t\Lambda^{2\alpha}}$ is a convolution operator and the kernel $g_\alpha(x,t)$ is defined by Fourier transform

$$\widehat{g_\alpha}(\xi,t) = e^{-(a+i\nu)|\xi|^{2\alpha} t}.$$

Note that the Poisson kernel and the heat kernel are special cases of $g_\alpha(x,t)$. For $\alpha \in (0,1]$, and $t>0$, $|g_\alpha(x,t)|$ is a nonnegative and non-increasing radial function, and satisfies the dilation relation

$$g_\alpha(x,t) = t^{-\frac{n}{2\alpha}} g_\alpha\left(xt^{-\frac{1}{2\alpha}},1\right), \quad g_\alpha(x,1) \in L^p(\mathbb{R}^n), \quad 1 \leq p < \infty. \tag{2.7}$$

We now establish estimates for the operators $U$ and $G$ on the Morrey spaces. We first introduce the space of weighted continuous functions in time. These spaces was first defined as solving initial value problem for Navier-Stokes equations by Kato and collaborators [3].

**Definition 2.2.** *Let $0<T<\infty$. For a given Banach space $X$ and a real number $\beta\geq 0$, we denote by $C_\beta((0,T);X)$ the space of $X$-valued continuous functions $f$ on $(0,T)$ with the norm*

$$\|f\|_{C_\beta((0,T);X)}=\sup_{0<t<T} t^\beta \|f(\cdot,t)\|_X<\infty.$$

Particularly, $C_0((0,T);X)=BC((0,T);X)$ is the space of bounded continuous functions (note that $C_0((0,T);X)\neq C((0,T);X)$, the space of continuous functions). $\dot{C}_\beta((0,T);X)$ denotes a subspace of $C_\beta((0,T);X)$ consisting of all functions $f$ with $\lim\limits_{t\to 0} t^\beta\|f\|_X=0$.

In the following, the Morrey spaces will play the role of $X$ and the norm in $C_\beta((0,T);M_{p,\lambda})$ will be abbreviated as $\|\cdot\|_{\beta,p,\lambda}$.

For the linear operators $U$ and $G$, we have the following results.

**Proposition 2.1** ([16]). *Let $1\leq q_1\leq q_2<\infty$ and $0\leq\lambda_1=\lambda_2<n$. For any $t>0$, the operators $U(t)$, $W(t)=\nabla U(t)$ and $\partial_t U(t)$ are bounded operators from $M_{q_1,\lambda_1}$ to $M_{q_2,\lambda_2}$ and depend on $t$ continuously, where $\nabla\cdot$ denotes the space derivative. In addition, we have for $f\in M_{q_1,\lambda_1}$*

$$t^{\frac{1}{2\alpha}(\gamma_1-\gamma_2)}\|U(t)f\|_{M_{q_2,\lambda_2}}\leq Ce^{At}\|f\|_{M_{q_1,\lambda_1}}, \tag{2.8}$$

$$t^{\frac{1}{2\alpha}+\frac{1}{2\alpha}(\gamma_1-\gamma_2)}\|W(t)f\|_{M_{q_2,\lambda_2}}\leq Ce^{At}\|f\|_{M_{q_1,\lambda_1}}, \tag{2.9}$$

$$t^{1+\frac{1}{2\alpha}(\gamma_1-\gamma_2)}\|\partial_t U(t)f\|_{M_{q_2,\lambda_2}}\leq Ce^{At}\|f\|_{M_{q_1,\lambda_1}}, \tag{2.10}$$

*where*

$$\gamma_i=\frac{n-\lambda_i}{q_i},\quad i=1,2$$

*and constants $C$ depend on $\alpha,q_1,q_2,\lambda_1,\lambda_2$.*

**Remark 2.1.** Although for technical reasons we need $\lambda_1=\lambda_2$ in the proof of Proposition 2.1, we can see that it also works for $\lambda_1\neq\lambda_2$ satisfying

$$\frac{n-\lambda_2}{q_2}\leq\frac{n-\lambda_1}{q_1}$$

because of the embedding relations (2.1).

Generally, $U(t)$ is not a $C_0$-group on $M_{p,\lambda}$. In fact, $\ddot{M}_{p,\lambda}$ is the maximal closed subspace of $M_{p,\lambda}$ on which the $U(t)$ is a $C_0$-group [2]. Therefore, we need an estimate of the operator $G$ that acts between the weighted continuous function spaces introduced at the beginning.

**Lemma 2.2.** *Let* $u\in C_h((0,T);M_{q_1,\lambda_1})$ *and* $v\in C_l((0,T);M_{q_2,\lambda_2})$ *with*

$$1\le q_1\le\infty,\quad 1\le q_2\le\infty,\quad \frac{2\sigma}{q_1}+\frac{1}{q_2}\le 1,\quad l+2\sigma h<1.$$

*Assume that* $q$ *and* $\lambda$ *satisfy* $1\le q\le\infty$,

$$\frac{1}{q}\le\frac{2\sigma}{q_1}+\frac{1}{q_2},\quad 0\le\epsilon=\frac{2\sigma(n-\lambda_1)}{q_1}+\frac{n-\lambda_2}{q_2}-\frac{n-\lambda}{q}<2\alpha.$$

*Then*

$$G\big((b+i\mu)|u|^{2\sigma}v\big)\in C_m\big((0,T);M_{q,\lambda}\big)$$

*and*

$$\begin{aligned}&\big\|G\big((b+i\mu)|u|^{2\sigma}v\big)\big\|_{C_m((0,T);M_{q,\lambda})}\\ \le& Ce^{At}\|u\|^{2\sigma}_{C_h((0,T);M_{q_1,\lambda_1})}\|v\|_{C_l((0,T);M_{q_2,\lambda_2})},\end{aligned}$$

*where*

$$m=l+2\sigma h+\frac{\epsilon}{2\alpha}-1$$

*and* $C$ *is a constant.*

*Proof.* To prove this lemma, we use Proposition 2.1 and Hölder's inequality in Lemma 2.1

$$\begin{aligned}&\big\|G\big((b+i\mu)|u|^{2\sigma}v\big)\big\|_{M_{q,\lambda}}\\ \le& C\int_0^t\big\|e^{A(t-\tau)}g_\alpha(\cdot,t-\tau)*|u|^{2\sigma}v(\cdot,\tau)\big\|_{M_{q,\lambda}}d\tau\\ \le& Ce^{At}\int_0^t(t-\tau)^{-\frac{\epsilon}{2\alpha}}\big\||u|^{2\sigma}v(\tau)\big\|_{M_{\frac{q_1q_2}{2\sigma q_2+q_1},\frac{2\sigma\lambda_1q_2+\lambda_2q_1}{2\sigma q_2+q_1}}}d\tau.\end{aligned}\tag{2.11}$$

Therefore,

$$\begin{aligned}&\big\|G\big((b+i\mu)|u|^{2\sigma}v\big)\big\|_{M_{q,\lambda}}\\ \le& Ce^{At}\|u\|^{2\sigma}_{C_h((0,T);M_{q_1,\lambda_1})}\|v\|_{C_l((0,T);M_{q_2,\lambda_2})}t^{-m}\int_0^1(1-s)^{-\frac{\epsilon}{2\alpha}}s^{-l-2\sigma h}ds\end{aligned}$$

for

$$m=l+2\sigma h+\frac{\epsilon}{2\alpha}-1.$$

Note that

$$\big\||u|^{2\sigma}\big\|_{M_{\frac{p}{2\sigma},\lambda}} = \|u\|^{2\sigma}_{M_{p,\lambda}}.$$

The integral in the last inequality can be written as the Beta function

$$B\Big(1-\frac{\epsilon}{2\alpha}, 1-l-2\sigma h\Big).$$

Using the fact that the Beta function $B(a,b)$ is finite if $a>0$ and $b>0$, we have

$$\begin{aligned}&\big\|G\big((b+i\mu)|u|^{2\sigma}v\big)\big\|_{C_m((0,T);M_{q,\lambda})}\\ \le\, & Ce^{At}\|u\|^{2\sigma}_{C_h((0,T);M_{q_1,\lambda_1})}\|v\|_{C_l((0,T);M_{q_2,\lambda_2})}.\end{aligned} \tag{2.12}$$

The proof is complete. □

# 3 Main results and proofs

We will state the main theorem in this section.

**Theorem 3.1.** *Suppose that* $0<\alpha<1$ *and* $u_0\in M_{p,\lambda}$ *with*

$$1\le p<\infty, \quad 0\le\lambda<\infty, \quad \frac{n-\lambda}{p}<\frac{\alpha}{\sigma}. \tag{3.1}$$

*Then there is a* $\delta>0$ *such that if* $\|u_0\|_{M_{p,\lambda}}<\delta$, *the initial value problem of* (1.1)-(1.2) *admits a solution* $u(x,t)$ *on* $(0,T)$ *for some* $T>0$ *satisfying*

$$u\in C_\beta\big((0,T);M_{(2\sigma+1)p,\lambda}\big), \quad \beta=\frac{\sigma(n-\lambda)}{p\alpha(2\sigma+1)}. \tag{3.2}$$

*Further assume that* $A=0$, *then*

$$u\in\bigcap_{q<p}\bigcap_{k>n-\frac{q}{p}(n-\lambda)} BC\big([0,T);L_{-\frac{k}{q},q}\big), \tag{3.3}$$

*and*

$$u\in C_{\frac{\sigma(n-\lambda)}{p\alpha}-1}\big((0,T);M_{p,\lambda}\big). \tag{3.4}$$

*Moreover, for any* $p<p'\le\infty$, $A=0$ *and t is large,*

$$u\in\mathbb{Y}\equiv C_{m'}\big((t,T);\ddot{M}_{p',\lambda}\big), \quad m'=\frac{(n-\lambda)\sigma}{p\alpha}+\frac{(n-\lambda)}{2\alpha}\left(\frac{1}{p}-\frac{1}{p'}\right)-1, \tag{3.5}$$

*and $u$ is the unique solution in the class of functions satisfying* (3.5) *with small norm $\|u\|_{M_{p',\lambda}}$ for $p<p'\le\infty$. Moreover, the mapping*

$$\mathcal{O}: V\to\mathbb{Y},$$

*is Lipschitz and $V$ is a neighborhood of $u_0$.*

**Remark 3.1.** For $A=0$, it can be seen from the proof of the theorem that the solution is actually global if the norm $\|u_0\|_{M_{p,\lambda}}$ is small enough.

**Remark 3.2.** Note that (3.3) implies that for $1<q\le p$, $u(x,t)\to u_0(x)$ in $L_{-k/q,q}$ as $t\to 0$. Generally speaking, we do not anticipate $u(x,t)\to u_0(x)$ in $M_{p,\lambda}$ for any $u_0\in M_{p,\lambda}$, since $U(t)$ is not a $C_0$ semigroup on $M_{p,\lambda}$.

**Remark 3.3.** For $\|u_0\|_{M_{p,\lambda}}<\delta$, (3.5) also tells us the decay rate of $u$ for large $t$, namely

$$\|u\|_{L^\infty}=\mathcal{O}(t^z),$$

where

$$z=-\left[\frac{n-\lambda}{p\alpha}\left(\sigma+\frac{1}{2}\right)-1\right].$$

We prove this theorem by integral equations and contraction mapping. According to standard practice, we write the fractional Ginzburg-Landau equation (1.1) as an integral form

$$\begin{aligned} u(x,t)&=U(t)u_0-G\big((b+i\mu)|u|^{2\sigma}u\big)\\ &=U(t)u_0-\int_0^t U(t-\tau)(b+i\mu)|u|^{2\sigma}u(\tau)d\tau. \end{aligned} \tag{3.6}$$

*Proof.* Let $X$ denote the Banach space

$$X=C_\beta\big((0,T);M_{(2\sigma+1)p,\lambda}\big)$$

and $X_R$ represent the complete metric space of the closed ball in $X$ centred at 0 and of radius $R$, where $T$ and $R$ will be determined later on. Consider the nonlinear map $\mathcal{A}$ on $X_R$ defined by

$$\mathcal{A}(u)(t)=U(t)u_0-G\big((b+i\mu)|u|^{2\sigma}u\big)(t),\quad t\in(0,T).$$

First we show that $\mathcal{A}$ maps $X_R$ to itself and is a contraction. Applying Proposition 2.1 with

$$q_1=p,\quad q_2=(2\sigma+1)p,\quad \lambda_1=\lambda_2=\lambda,$$

we obtain

$$\|U(t)u_0\|_X = \|U(t)u_0\|_{C_\beta((0,T);M_{(2\sigma+1)p,\lambda})} \le C_0\|u_0\|_{M_{p,\lambda}}. \tag{3.7}$$

Using Proposition 2.1 and (2.2) leads to

$$\begin{aligned}
&\big\|G\big((b+i\mu)|u|^{2\sigma}u\big)\big\|_X \\
=& t^\beta\big\|G\big((b+i\mu)|u|^{2\sigma}u\big)\big\|_{M_{(2\sigma+1)p,\lambda}} \\
\le& t^\beta\int_0^t \big\|U(t-\tau)(b+i\mu)|u|^{2\sigma}u\big\|_{M_{(2\sigma+1)p,\lambda}}d\tau \\
\le& t^\beta Ce^{At}\int_0^t (t-\tau)^{-\beta}\big\||u|^{2\sigma}u(\tau)\big\|_{M_{p,\lambda}}d\tau \\
\le& t^\beta Ce^{At}\int_0^t (t-\tau)^{-\beta}\|u(\tau)\|^{2\sigma+1}_{M_{(2\sigma+1)p,\lambda}}d\tau,
\end{aligned}$$

where

$$\beta = \frac{(n-\lambda)\sigma}{p\alpha(2\sigma+1)}.$$

Therefore

$$\begin{aligned}
&\big\|G\big((b+i\mu)|u|^{2\sigma}u\big)\big\|_X \\
\le& Ct^\beta e^{At}\Big(\sup_{\tau\in(0,t)}\tau^\beta\|u\|_{M_{(2\sigma+1)p,\lambda}}\Big)^{2\sigma+1}\int_0^t(t-\tau)^{-\beta}\tau^{-(2\sigma+1)\beta}d\tau \\
\le& Ct^{1-\beta(2\sigma+1)}e^{At}\|u\|_X^{2\sigma+1}\int_0^1(1-s)^{-\beta}s^{-(2\sigma+1)\beta}ds.
\end{aligned}$$

The integral in the last inequality can be written as the Beta function $B(1-\beta,1-(2\sigma+1)\beta)$. Using the fact that the Beta function $B(a,b)$ is finite if $a>0$ and $b>0$, we conclude that $\|G((b+i\mu)|u|^{2\sigma}u)\|_X$ is finite if $T$ and $R$ are properly chosen.

Furthermore, for any $\tilde{u}$ and $u\in X_R$, since

$$|u|^{2\sigma}u-|\tilde{u}|^{2\sigma}\tilde{u} = \int_0^1\Big[(\sigma+1)(u-\tilde{u})|Z|^{2\sigma}+\sigma(\overline{u}-\overline{\tilde{u}})Z^2|Z|^{2\sigma-2}\Big]d\lambda,$$

where $Z=\lambda u+(1-\lambda)\tilde{u}$, we have

$$\begin{aligned}
\|\mathcal{A}(u)-\mathcal{A}(\tilde{u})\|_X =& C\big\|G(|u|^{2\sigma}u)-G(|\tilde{u}|^{2\sigma}\tilde{u})\big\|_X \\
\le& Ct^{1-\beta(2\sigma+1)}e^{At}\Big(\|u\|_X^{2\sigma}+\|\tilde{u}\|_X^{2\sigma}\Big)\|u-\tilde{u}\|_X \\
\le& Ct^{1-\beta(2\sigma+1)}R^{2\sigma}e^{At}\|u-\tilde{u}\|_X.
\end{aligned}$$

Hence, $\mathcal{A}$ maps $X_R$ to itself for some $T$ and $R$ properly chosen and is a contraction, which means (1.1)-(1.2) admits a unique solution $u\in C_\beta((0,T);M_{(2\sigma+1)p,\lambda})$.

To show that $u$ satisfies (3.3), we first note that

$$u=\mathcal{A}u=U(t)u_0-G\big((b+i\mu)|u|^{2\sigma}u\big).$$

For $u_0\in M_{p,\lambda}$,

$$U(t)u_0\in BC\big((0,T);\ddot{M}_{p,\lambda}\big)$$

as implied by Proposition 2.1. Moreover, we have

$$U(t)u_0\in BC\big([0,T);L_{-\frac{k}{p},q}\big)$$

for

$$1<q\le p,\quad k>n-\frac{q}{p}(n-\lambda)$$

due to

$$M_{p,\lambda}\hookrightarrow M_{q,n-\frac{q}{p}(n-\lambda)}\hookrightarrow L_{-\frac{k}{q},q},$$

and the fact that $U(t)$ is a $C_0$ semigroup on $L_{-k/q,q}$.

For the nonlinear term $G((b+i\mu)|u|^{2\sigma}u)$, we apply Lemma 2.2 with

$$\begin{aligned}&A=0,\quad \lambda_1=\lambda_2=\lambda,\quad q_1=q_2=(2\sigma+1)p,\quad q=p,\\ &h=l=\frac{\sigma(n-\lambda)}{p\alpha(2\sigma+1)},\quad m=\frac{\sigma(n-\lambda)}{p\alpha}-1<0\end{aligned}$$

to show that $G((b+i\mu)|u|^{2\sigma}u)\in BC([0,T);\ddot{M}_{p,\lambda})$, which means $G((b+i\mu)|u|^{2\sigma}u)\to 0$ as $t\to 0$ in $M_{p,\lambda}$ and $L_{-k/p,p}$. The application of Lemma 2.2 with

$$\begin{aligned}&A=0,\quad \lambda_1=\lambda_2=\lambda,\quad q_1=q_2=(2\sigma+1)p,\quad q=\frac{p}{1+\eta},\\ &h=l=\frac{\sigma(n-\lambda)}{(2\sigma+1)p\alpha},\quad m=\frac{(n-\lambda)\sigma}{p\alpha}-\frac{(n-\lambda)\eta}{2p\alpha}-1<0\end{aligned}$$

shows that $G((b+i\mu)|u|^{2\sigma}u)\in C_m((0,T);M_{q,\lambda})$ for $q\le p$, which implies that

$$G\big((b+i\mu)|u|^{2\sigma}u\big)\to 0\quad \text{as}\quad t\to 0\quad \text{in}\quad M_{q,\lambda}.$$

We get

$$G\big((b+i\mu)|u|^{2\sigma}u\big)\to 0\quad \text{in}\quad L_{-\frac{k}{q},q}\quad \text{as}\quad t\to 0,$$

due to the embedding

$$M_{q,\lambda} \hookrightarrow L_{-\frac{k}{q},q} \quad \text{for} \quad k > n - \frac{q}{p}(n-\lambda).$$

We now prove that $u$ satisfies (3.5). The linear term $U(t)u_0$ satisfying (3.5) is an easy consequence of Proposition 2.1. We apply Lemma 2.2 to $G((b+i\mu)|u|^{2\sigma}u)$ with

$$\begin{aligned} &h=l=\frac{(n-\lambda)\sigma}{p\alpha(2\sigma+1)}, \quad q_1=q_2=(2\sigma+1)p, \\ &m'=\frac{(n-\lambda)\sigma}{p\alpha}+\frac{(n-\lambda)}{2\alpha}\left(\frac{1}{p}-\frac{1}{p'}\right)-1, \\ &q=p', \quad \lambda_1=\lambda_2=\lambda, \end{aligned}$$

and use the fact that $u \in C_{\frac{(n-\lambda)\sigma}{p\alpha(2\sigma+1)}}((0,T);M_{(2\sigma+1)p,\lambda})$ to show that $G((b+i\mu)|u|^{2\sigma}u)$ is in the class defined by (3.5). This conclusion, combined with the uniqueness of $u$ in $C_{\frac{(n-\lambda)\sigma}{p\alpha(2\sigma+1)}}((0,T);M_{(2\sigma+1)p,\lambda})$ indicates the uniqueness of $u$ in (3.5). The proof of the Lipschitz property is standard and is therefore omitted. □

# 4 Further regularity

In this section we prove that the solution $u$ in Theorem 3.1 can be of higher regularity, which is actually smooth.

**Theorem 4.1.** *Let $u$ be the solution in Theorem* 3.1*. Then any derivatives of $u$ can be in the same Morrey space as $u$, namely, for any $1<p\le q<\infty$ and $k,j=0,1,\dots$*

$$\partial_t^k \nabla^j u \in C\big((0,T);\ddot{M}_{q,\lambda}\big), \tag{4.1}$$

*where $\nabla\cdot$ represents the spatial derivative and $C((0,T);X)$ is the space of $X$-valued continuous function on $(0,T)$.*

*Proof.* The smoothness of $u$ can also be proved by contraction mapping argument and building the regularity index in to the working space. First we consider the case when $k=0$. For $j=0$, (4.1) can be seen from (3.2)-(3.5) in Theorem 3.1. We now prove that (4.1) still holds true for $k=0,j=1$. We take any $t_1>0$ and show the results for $t>t_1$.

First, let $p<q$ and $X$ be the space consisting of function $u$ satisfying

$$u\in C\big([t_1,T);\ddot{M}_{q,\lambda}\big), \quad \nabla u\in C_{\frac{1}{2\alpha}}\big((t_1,T);\ddot{M}_{q,\lambda}\big), \tag{4.2}$$

and $X_R$ be the closed ball of radius $R$ in $X$. The idea is to apply contraction mapping principle to $\mathcal{A}$ on $X_R$ with $T$ and $R$ to be determined later. First, we choose $R$ appropriately such that $Uu_1\in X_R$, in which $u_1=u(t_1)$ is the value of $u$ at $t_1$. Similar to the proof of Theorem 3.1, we apply Lemma 2.2 to show that for $u\in X_R$

$$G\big((b+i\mu)|u|^{2\sigma}u\big)\in C_\gamma\big((0,T);\ddot{M}_{q,\lambda}\big), \tag{4.3}$$

$$\nabla G((b+i\mu)|u|^{2\sigma}u)\in C_{\frac{1}{2\alpha}+\gamma}\big((0,T);\ddot{M}_{q,\lambda}\big), \tag{4.4}$$

where

$$\gamma=\frac{\sigma(n-\lambda)}{\alpha q}-1.$$

Notice that $\gamma$ is negative under the condition (3.1). The relations (4.3) and (4.4) imply not only $Uu_0\in X_R$, but also that $\|G((b+i\mu)|u|^{2\sigma}u)\|$ in $X_R$ has small factor $(T-t_1)^{-\gamma}$ if $T-t_1$ is small.

If $T-t_1$ is chosen small and $R$ is set as described above, then $\mathcal{A}$ maps $X_R$ to itself and is a contraction. So $\mathcal{A}$ admits a fixed point $u$ in $X_R$, which solves (2.6). The uniqueness result in Theorem 3.1 indicates that this $u$ is exactly the original $u$ obtained in Theorem 3.1. Therefore we have proved that $u\in C((t_1,T);\ddot{M}_{q,\lambda})$, which means that $u\in C((0,T);\ddot{M}_{q,\lambda})$ due to the randomness of $t_1$.

Continuing to implement the same argument for higher space derivatives of $u$, we can obtain the result $\nabla^j u\in C((0,T);\ddot{M}_{q,\lambda})$. This completes the proof for $k=0$.

We now prove (4.1) for $k=1$. It can be easily seen from the regularity result we have just obtained that

$$\nabla^j u, \quad \Lambda^{2\alpha}\nabla^j u, \quad \nabla^j\big(|u|^{2\sigma}u\big)\in C\big((0,T);\ddot{M}_{q,\lambda}\big),$$

where $j=0,1,\ldots$ for any $p\le q<\infty$. Turning to Eq. (1.1)

$$\frac{\partial u}{\partial t}=Au-(a+\nu i)\Lambda^{2\alpha}u-(b+\mu i)u|u|^{2\sigma},$$

and the Hölder inequality for the Morrey space (i.e. (ii) of Lemma 2.1), we obtain for $j=0,1,2,\ldots$

$$\partial_t\nabla^j\theta\in C\big((0,T);\ddot{M}_{q,\lambda}\big).$$

The result for general $k$ can be established by an inducting process. This completes the proof of Theorem 4.1. $\square$

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