Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **15**, No. 6, pp. 1631-1654

DOI: 10.4208/aamm.OA-2022-0173
December 2023

Spectral Galerkin Approximation of Fractional Optimal Control Problems with Fractional Laplacian

Jiaqi Zhang[1], Yin Yang[2,*] and Zhaojie Zhou[3]

[1] *School of Mathematics and Computational Science, Xiangtan University, Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Key Laboratory of Intelligent Computing and Information Processing of Ministry of Education, Xiangtan 411105, Hunan, China*
[2] *School of Mathematics and Computational Science, Xiangtan University, National Center for Applied Mathematics in Hunan, Hunan International Scientific and Technological Innovation Cooperation Base of Computational Science, Xiangtan 411105, Hunan, China*
[3] *School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, Shandong, China*

Received 28 June 2022; Accepted (in revised version) 29 December 2022

Abstract. In this paper spectral Galerkin approximation of optimal control problem governed by fractional elliptic equation is investigated. To deal with the nonlocality of fractional Laplacian operator the Caffarelli-Silvestre extension is utilized. The first order optimality condition of the extended optimal control problem is derived. A spectral Galerkin discrete scheme for the extended problem based on weighted Laguerre polynomials is developed. A priori error estimates for the spectral Galerkin discrete scheme is proved. Numerical experiments are presented to show the effectiveness of our methods and to verify the theoretical findings.

AMS subject classifications: 35Q93, 49M25, 49M41

Key words: Fractional Laplacian, optimal control problem, Caffarelli-Silvestre extension, weighted Laguerre polynomials.

1 Introduction

The goal of this paper is to investigate spectral Galerkin approximation of optimal control problem governed by fractional elliptic equation with fractional Laplacian operator defined by spectral expansion. Let Ω be an open, bounded and connected domain in

*Corresponding author.
Emails: zjqss1992@163.com (J. Zhang), yangyinxtu@xtu.edu.cn (Y. Yang), zhouzhaojie@sdnu.edu.cn (Z. Zhou)

$\mathbb{R}^n$, with Lipschitz boundary $\partial\Omega$. We consider the following fractional optimal control problem:

$$\min_{z\in Z_{ad}} J(u,z) := \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \frac{\mu}{2}\|z\|^2_{L^2(\Omega)} \tag{1.1}$$

subject to

$$\begin{cases} (-\Delta)^s u(x) = f+z, & x\in\Omega, \\ u(x)=0, & x\in\partial\Omega. \end{cases} \tag{1.2}$$

Here the constraint set of control variable z is defined by

$$Z_{ad} = \left\{ z\in L^2(\Omega): \int_\Omega z(x)dx \geq 0 \right\}.$$

$\mu>0$ is the regularization parameter, and u_d is the desired state. The operator $(-\Delta)^s$, $(s\in(0,1))$ is the fractional power of Laplacian operator, which will be defined later.

In recent years, optimal control problem [15,28,32,34] has developed into a hot subject across computational mathematics, applied mathematics and systems science. It has a very wide range of applications in engineering control, medical imaging, aerospace and many other fields. The solution of the optimal control problem is to find a way to achieve the optimal performance index of the control system under the constraint conditions. In various fields of human activities, many problems can be described by the optimal control problem with a partial differential equation as the state equation.

Compared with integer order equations, fractional order differential equations can more accurately describe materials and physical processes with memory and heredity, such as viscoelastic materials, diffusion and heat conduction in porous media, etc. Therefore, more and more scholars pay attention to the discussion and analysis of fractional order problems [16, 23, 25–27, 29, 33]. Although optimal control theory has been developed for many years, fractional optimal control theory is a new field in mathematics. In recent years, many numerical methods and algorithms have been developed to solve fractional order optimal control problems. In [30], Ye and Xu proposed a space-time spectral method to solve the time fractional optimal control problems. In [31], they used the space-time spectral method to solve the optimal control problem of time fractional diffusion equation with integral constraints on state variable. In [15], Li and Zhou use spectral collocation method to solve the optimal control problem of space fractional diffusion equation. In [28], Yang, Zhang, Liu, et al proposed the Jacobi spectral collocation method to solve the time fractional optimal control problem. In [24, 32], the authors discussed the spectral Galerkin approximation of optimal control problem governed by fractional differential equation with control integral constraint. Unlike aforementioned works the weighted Jacobi polynomials are used to approximate the state equation. In finite element method aspects the authors discussed finite element approximation [35] of time fractional optimal control problem with pointwise control constraint. A priori error estimate for the semi-discrete scheme is derived. Regularity of time fractional optimal control problem and a fully discrete error estimate for $L1$ and backward euler convolution quadrature scheme were presented in [14]. In [12], Gunzburger and Wang propose a time discrete fully discrete finite element method based on convolution quadrature to solve the time fractional optimal control problem. In [36], Zhou and Tan proposed a fast Primal-dual Active set algorithm for optimal control problem governed by space fractional diffusion equation with control constraints based on finite element approximation.

To our best knowledge the numerical methods or algorithms developed for optimal control problems with fractional Laplacian are not much, and mainly focus on the finite element method. In [8], D'Elia, Glusa and Otárola proposed semi-discrete and fully discrete methods to solve a linear quadratic optimal control problem including integral fractional Laplace operator. For optimal control problems with fractional Laplacian in spectral definition a serial of works, for examples, see [1,19,20], are developed in recent years based on Caffarelli-Silvestre extension and finite element discretization. Since the finite element method is local, a truncated problem is introduced and the approximate properties of its solution are obtained. The Caffarelli-Silvestre extension can overcome the nonlocality. However, the solution of the extended equation is weakly singular at $y=0$ due to the degenerate/ singular weight y^{α}. Therefore, the accuracy of the finite element method is limited. In [3], Chen and Shen developed a spectral method to solve the extended problem and achieved high-order convergence rate in the extended y-direction despite the weak singularity at $y=0$. In [13], Gu et al. expressed the d-dimensional spectral fractional equation as a d+1-dimensional regular partial differential equation by using the Caffarelli-Silvestre extension, and estimates on the error made by the deep Ritz method.

Inspired by the above work, in this paper we use the spectral method in the extended y-direction to solve the optimal control problem with fractional Laplacian in spectral definition. Due to the low regularity in y-direction seriously deteriorates the convergence rate of the usual numerical method. To overcome this, we use the enriched spectral method to improve the numerical method and enhance its convergence rate. The first order optimality condition of the extended optimal control problem is derived. A spectral Galerkin discrete scheme for the extended problem based on weighted Laguerre polynomials is developed. Due to the global nature of the spectral method, we do not need to introduce the truncated problem in the extended y-direction. A priori error estimates for the spectral discrete scheme is proved. Numerical experiments are carried out to verify the theoretical findings.

The rest of the paper is organized as follows. In Section 2, we will introduce fractional operator, Caffarelli-Silvestre extension and generalized Laguerre function. In Section 3, the first-order optimality conditions for the extended problem and the original optimal control problem are given. In Section 4, we use the spectral Galerkin method to discretize the optimal control problem and derive the error estimates. In Section 5, the enriched spectral Galerkin discrete scheme is presented and the error estimates are derived.

2 Preliminaries

2.1 Properties of the fractional operators

In this section the definition of fractional Laplacian is based on spectral theory [5]. Let $\{\lambda_n,\varphi_n\}$ be the eigenvalues and orthonormal eigenfunctions of the Laplacian with homogeneous Direchlet boundary condition, i.e.,

$$-\Delta\varphi_n=\lambda_n\varphi_n \quad \text{in } \Omega, \qquad \varphi_n=0 \quad \text{on } \partial\Omega, \qquad (\varphi_n,\varphi_n)=1.$$

It is well-known that $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n\to+\infty$ and $\{\varphi_n\}$ forms an orthonormal basis of $L^2(\Omega)$ [9]. Then the fractional Laplacian in spectral form is defined by

$$(-\Delta)^s v:=\sum_{n=1}^{\infty}\lambda_n^s\hat{v}_n\varphi_n, \quad v\in C_0^\infty(\Omega), \quad s\in(0,1),$$

where $\hat{v}_n=\int_\Omega v\varphi_n dx$. We also define the Hilbert space associated with the spectrum of the Laplacian

$$\mathbb{H}^r(\Omega)=\left\{v=\sum_{n=1}^{\infty}\hat{v}_n\varphi_n\in L^2(\Omega):\ |v|^2_{\mathbb{H}^r(\Omega)}=\sum_{n=1}^{\infty}(\lambda_n)^r|\hat{v}_n|^2<\infty\right\}.$$

For any $s<r$, there exists by Sobolev imbedding theorem

$$|v|_{\mathbb{H}^s(\Omega)}\le c|v|_{\mathbb{H}^r(\Omega)}.$$

2.2 The Caffarelli-Silvestre extension

Set $\Lambda:=(0,\infty)$. We define the semi-infinite cylinder in R^{n+1} and its lateral boundary, respectively, by

$$\mathcal{C}:=\Omega\times\Lambda, \quad \partial_L\mathcal{C}=\partial\Omega\times\bar{\Lambda}.$$

Thus, we can use the Caffarelli-Silvestre extension [4] to rewrite the state equation (1.2) as the following mixed boundary value problem

$$\begin{cases} -\mathrm{div}\big(y^\alpha\nabla\mathcal{U}(x,y)\big)=0 & \text{in } \mathcal{C}=\Omega\times\Lambda,\\ \mathcal{N}\mathcal{U}:=-\lim\limits_{y\to0}y^\alpha\mathcal{U}_y=d_s(f+z) & \text{on } \Omega\times\{0\},\\ \mathcal{U}=0 & \text{on } \partial_L\mathcal{C}=\Omega\times\bar{\Lambda}. \end{cases} \tag{2.1}$$

Here $\alpha=1-2s\in(-1,1)$, $d_s=2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}$. We call y the extended variable.

Let $\mathcal{Z}$ be either Ω, Λ or $\mathcal{C}$, and w be a positive weight function. We denote

$$\begin{aligned} &(p,q)_{w,\mathcal{Z}}:=\int_{\mathcal{Z}}p(t)q(t)w(t)dt, \qquad \|p\|^2_{w,\mathcal{Z}}=(p,p)_{w,\mathcal{Z}},\\ &H^1_w(\mathcal{Z}):=\big\{v\in L^2_w(\mathcal{Z}):\nabla v\in L^2_w(\mathcal{Z})\big\}, \end{aligned}$$

equipped with norm

$$\|v\|_{w,\mathcal{Z}}^2 := (v,v)_{w,\mathcal{Z}}, \qquad \|v\|_{1,w,\mathcal{Z}} := \left(\|v\|_{w,\mathcal{Z}}^2 + \|\nabla v\|_{w,\mathcal{Z}}^2\right)^{\frac{1}{2}}.$$

We will omit the weight from the notation when $\omega \equiv 1$. In order to study the extended problem (2.1) we define

$$\mathcal{H}_{y^\alpha}^1(\mathcal{C}) := \left\{ \nabla v \in L_{y^\alpha}^2(\mathcal{C}) : \lim_{y\to\infty} v(x,y) = 0,\ v(x,y)|_{\partial_L \mathcal{C}} = 0 \right\}$$

equipped with norm

$$\|v\|_{\mathcal{H}_{y^\alpha}^1(\mathcal{C})} = \|\nabla v\|_{y^\alpha,\mathcal{C}}. \tag{2.2}$$

Moreover, we define the trace of function $v \in \mathcal{H}_{y^\alpha}^1(\mathcal{C})$ by

$$\boldsymbol{tr}\{v\}(x) := v(x,0). \tag{2.3}$$

Lemma 2.1 ([17]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\alpha = 1-2s$. The trace operator $\boldsymbol{tr}$ satisfies $\boldsymbol{tr}\mathcal{H}_{y^\alpha}^1(\mathcal{C}) = \mathbb{H}^s(\Omega)$ and*

$$\|\boldsymbol{tr}\{v\}\|_{\mathbb{H}^s(\Omega)} \le c\|v\|_{\mathcal{H}_{y^\alpha}^1(\mathcal{C})}, \quad \forall v \in \mathcal{H}_{y^\alpha}^1(\mathcal{C}). \tag{2.4}$$

Then for given $f+z \in \mathbb{H}^{-s}(\Omega)$ the weak formulation of (2.1) is to find $\mathcal{U} \in \mathcal{H}_{y^\alpha}^1(\mathcal{C})$ such that

$$(y^\alpha \nabla \mathcal{U}, \nabla V)_{\mathcal{C}} = d_s(f+z, \boldsymbol{tr}\{V\})_\Omega, \quad \forall V \in \mathcal{H}_{y^\alpha}^1(\mathcal{C}). \tag{2.5}$$

Here

$$(y^\alpha \nabla \mathcal{U}, \nabla V)_{\mathcal{C}} := \frac{1}{d_s}\int_{\mathcal{C}} y^\alpha \nabla \mathcal{U}(x,y) \cdot \nabla V(x,y).$$

The wellposedness of the above weak formulation is a direct consequence of Lax-Milagram lemma and Lemma 2.1.

2.3 Generalized Laguerre functions

Since (2.5) involves a singular weight function y^α, it is natural to use the generalized Laguerre functions $\{\widehat{\mathscr{L}}_n^\alpha(y)\}$, which are orthogonal with respect to weight y^α. We start by reviewing some basic properties of the generalized Laguerre functions

$$\widehat{\mathscr{L}}_n^\alpha(y) := e^{-\frac{y}{2}} \mathscr{L}_n^\alpha(y),$$

where $\mathscr{L}_n^\alpha(y)$ is the generalized Laguerre polynomial [22]. It is clear that $\{\widehat{\mathscr{L}_n^\alpha}(y)\}$ forms a complete basis in $L^2_{y^\alpha}(\Lambda)$ [11], and they are mutually orthogonal with respect to the weight y^α:

$$\int_0^\infty \widehat{\mathscr{L}_n^\alpha}(y)\widehat{\mathscr{L}_m^\alpha}(y)y^\alpha dy=\gamma_n^\alpha\delta_{mn},$$

where δ_{mn} is Dirac Delta function, and $\gamma_n^\alpha=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}$.

The generalized Laguerre functions can be efficiently and stably computed by the three-term recurrence formula

$$\begin{aligned}
&\widehat{\mathscr{L}_{-1}^\alpha}(y)\equiv 0, \qquad \widehat{\mathscr{L}_0^\alpha}(y)=e^{-\frac{y}{2}},\\
&\widehat{\mathscr{L}_{n+1}^\alpha}(y)=\frac{2n+\alpha+1-y}{n+1}\widehat{\mathscr{L}_n^\alpha}(y)-\frac{n+\alpha}{n+1}\widehat{\mathscr{L}_{n-1}^\alpha}(y).
\end{aligned}$$

Denote $\widehat{\mathcal{P}}_N^y=span\{\widehat{\mathscr{L}_n^\alpha}(y),\, 0\le n\le N\}$. For any $u\in L^2_{y^\alpha}(\Lambda)$, we define

$$(\pi_N^y u-u,v)_{y^\alpha}=0, \quad \forall v\in\widehat{\mathcal{P}}_N^y. \tag{2.6}$$

Next, we define a generalized derivative by $\widehat{\partial}_y=\partial_y+\frac{1}{2}$ and the corresponding non-uniformly weighted Sobolev space

$$\widehat{B}_\alpha^m(\Lambda):=\big\{v:\widehat{\partial}_y^l v\in L^2_{y^{\alpha+l}}(\Lambda),\, 0\le l\le m\big\}, \quad \alpha>-1, \quad m\in\mathbb{N}.$$

According to [22] we have the following result.

Lemma 2.2. *For any $u\in\widehat{B}_\alpha^m(\Lambda)$ and $0\le m\le N+1$, the following estimate holds*

$$\left\|\widehat{\partial}_y^l(u-\pi_N^y u)\right\|_{y^{\alpha+l},\Lambda}\le\sqrt{\frac{(N-m+1)!}{(N-l+1)!}}\|\widehat{\partial}_y^m u\|_{y^{\alpha+m},\Lambda}, \quad 0\le l\le m. \tag{2.7}$$

3 First order necessary conditions for the extended control problem

Using the Caffarelli-Silvestre extension we can define the extended optimal control problem as follows

$$\min_{z\in Z_{ad}} J(\boldsymbol{tr}\{\mathcal{U}\},z) \tag{3.1}$$

subject to

$$\frac{1}{d_s}(y^\alpha\nabla\mathcal{U},\nabla V)_{\mathcal{C}}=(f+z,\boldsymbol{tr}\{V\})_\Omega, \quad \forall V\in\mathcal{H}^1_{y^\alpha}(\mathcal{C}). \tag{3.2}$$

Theorem 3.1. *Suppose* $(\mathcal{U},z)$ *is the solution of the extended control problems* (3.1)-(3.2), *then the following first order optimal conditions hold*

$$\begin{cases} \dfrac{1}{d_s}(y^\alpha \nabla \mathcal{U}, \nabla V)_{\mathcal{C}} = (f+z, \boldsymbol{tr} V)_\Omega, & \forall V \in \mathcal{H}^1_{y^\alpha}(\mathcal{C}), \\ \dfrac{1}{d_s}(y^\alpha \nabla \mathcal{Q}, \nabla P)_{\mathcal{C}} = \big(tr\{\mathcal{U}\} - u_d, \boldsymbol{tr}\{P\}\big)_\Omega, & \forall P \in \mathcal{H}^1_{y^\alpha}(\mathcal{C}), \\ \displaystyle\int_\Omega (\mu z + \boldsymbol{tr}\{\mathcal{Q}\})(v-z) \geq 0, & \forall v \in Z_{ad}. \end{cases} \tag{3.3}$$

Proof. Let $\hat{J}(z) := J(\boldsymbol{tr}\{\mathcal{U}(z)\}, z)$. Then we can rewrite the optimal control problem as the following reduced optimization problem

$$\min_{z \in Z_{ad}} \hat{J}(z).$$

Then the first order necessary optimality condition takes the form

$$\hat{J}'(z)(v-z) \geq 0, \quad \forall v \in Z_{ad}.$$

By simple calculation we obtain

$$\begin{aligned} \hat{J}'(z)(v-z) &= \lim_{h \to 0^+} \frac{\hat{J}\big(z+h(v-z)\big) - \hat{J}(z)}{h} \\ &= \int_\Omega (\boldsymbol{tr}\{\mathcal{U}\} - u_d)[\boldsymbol{tr}\{\mathcal{U}'(z)(v-z)\}]dx + \mu \int_\Omega z(v-z)dx. \end{aligned}$$

By (3.2) we have

$$\frac{1}{d_s}\Big(y^\alpha \nabla \big(\boldsymbol{tr}\{\mathcal{U}'(z)(v-z)\}\big), \nabla V\Big)_{\mathcal{C}} = \Big(v-z, \boldsymbol{tr}\{V\}\Big)_\Omega.$$

To simplify above optimality condition we introduce the adjoint state equation

$$\frac{1}{d_s}(y^\alpha \nabla \mathcal{Q}, \nabla P)_{\mathcal{C}} = (\boldsymbol{tr}\{\mathcal{U}\} - u_d, \boldsymbol{tr}\{P\})_\Omega.$$

Then we have

$$\begin{aligned} \Big(v-z, \boldsymbol{tr}\{\mathcal{Q}\}\Big)_\Omega &= \frac{1}{d_s}\Big(y^\alpha \nabla \big(\boldsymbol{tr}\{\mathcal{U}'(z)(v-z)\}\big), \nabla \mathcal{Q}\Big)_{\mathcal{C}} \\ &= \frac{1}{d_s}(y^\alpha \nabla \mathcal{Q}, \nabla \big(\boldsymbol{tr}\{\mathcal{U}'(z)(v-z)\}\big))_{\mathcal{C}} \\ &= \big(\boldsymbol{tr}\{\mathcal{U}\} - u_d, \boldsymbol{tr}\{\mathcal{U}'(z)(v-z)\}\big)_\Omega. \end{aligned}$$

This leads to

$$\hat{J}'(z)(v-z) = \int_\Omega (\mu z + \boldsymbol{tr}\{\mathcal{Q}\})(v-z)dx \geq 0, \quad \forall v \in Z_{ad}.$$

Thus, we complete the proof. □

Finally we present the first order necessary condition of the original control problem according to [1].

Theorem 3.2. *Suppose* (u,z) *is the solution of the optimal control problems* (1.1)-(1.2)*, then the following first order optimal conditions hold*

$$\begin{cases} (-\Delta)^s u = f+z \quad in\ \Omega, & u(x)=0 \quad in\ \partial\Omega, \\ (-\Delta)^s q = u-u_d \quad in\ \Omega, & q(x)=0 \quad in\ \partial\Omega, \\ \int_\Omega (\mu z+q)(v-z)\geq 0, & \forall v\in Z_{ad}. \end{cases} \tag{3.4}$$

4 Galerkin approximation

4.1 Galerkin discrete scheme

Since the domain $\mathcal{C}$ is a tensor-product domain, it is natural to use a tensor-product approximation. Let X_h be a suitable approximation space in the x-direction,

$$\begin{aligned} &X_h = span\{\phi_m^x(x): 1\leq m\leq M\}, \\ &Y_N = \{\phi_n^y(y) = \widehat{\mathscr{L}}_{n-2}^\alpha(y) - \widehat{\mathscr{L}}_{n-1}^\alpha(y): 1\leq n\leq N\}. \end{aligned}$$

Then, the Galerkin method for (2.5) is to find $\mathcal{U}_N^h(z)\in X_h\times Y_N$ such that

$$\frac{1}{d_s}(y^\alpha \nabla \mathcal{U}_N^h(z), \nabla V_N^h)_{\mathcal{C}} = (f+z, \boldsymbol{tr}\{V_N^h\})_\Omega, \quad \forall V_N^h\in X_h\times Y_N. \tag{4.1}$$

The corresponding discrete scheme of problem (1.1)-(1.2) can be expressed as

$$\min_{z_N\in Z_{ad}} J(\mathcal{U}_N^h, z_N) := \frac{1}{2}\|\boldsymbol{tr}\{\mathcal{U}_N^h\} - u_d\|_{L^2(\Omega)}^2 + \frac{\mu}{2}\|z_N\|_{L^2(\Omega)}^2, \tag{4.2}$$

such that

$$\frac{1}{d_s}(y^\alpha \nabla \mathcal{U}_N^h, \nabla V_N^h)_{\mathcal{C}} = (f+z_N, \boldsymbol{tr}\{V_N^h\})_\Omega, \quad \forall V_N^h\in X_h\times Y_N. \tag{4.3}$$

Similar to the continuous case we can derive the discrete first order optimality conditions

$$\begin{cases} \frac{1}{d_s}(y^\alpha \nabla \mathcal{U}_N^h, \nabla V_N^h)_{\mathcal{C}} = (f+z_N, \boldsymbol{tr}\{V_N^h\})_\Omega, & \forall V_N^h\in X_h\times Y_N, \\ \frac{1}{d_s}(y^\alpha \nabla \mathcal{Q}_N^h, \nabla P_N^h)_{\mathcal{C}} = (\boldsymbol{tr}\{\mathcal{U}_N^h\} - u_d, \boldsymbol{tr}\{P_N^h\})_\Omega, & \forall P_N^h\in X_h\times Y_N, \\ \int_\Omega (\mu z_N + \boldsymbol{tr}\{\mathcal{Q}_N^h\})(v-z_N)\geq 0, & \forall v\in Z_{ad}. \end{cases} \tag{4.4}$$

To derive a priori error analysis we need to introduce the following auxiliary problems

$$\begin{cases} \dfrac{1}{d_s}(y^\alpha \nabla \mathcal{U}_N^h(z), \nabla V_N^h)_{\mathcal{C}} = (f+z, \boldsymbol{tr}\{V_N^h\})_\Omega, & \forall V_N^h \in X_h \times Y_N, \\ \dfrac{1}{d_s}(y^\alpha \nabla \mathcal{Q}_N^h(z), \nabla P_N^h)_{\mathcal{C}} = (\boldsymbol{tr}\{\mathcal{U}_N^h(z)\} - u_d, \boldsymbol{tr}\{P_N^h\})_\Omega, & \forall P_N^h \in X_h \times Y_N, \\ \dfrac{1}{d_s}(y^\alpha \nabla \mathcal{Q}_N^h(u), \nabla P_N^h)_{\mathcal{C}} = (\boldsymbol{tr}\{\mathcal{U}\} - u_d, \boldsymbol{tr}\{P_N^h\})_\Omega, & \forall P_N^h \in X_h \times Y_N. \end{cases} \tag{4.5}$$

Combining (4.4) and (4.5) we obtain

$$\begin{cases} \dfrac{1}{d_s}\big(y^\alpha(\nabla \mathcal{U}_N^h(z) - \nabla \mathcal{U}_N^h)), \nabla V_N^h\big)_{\mathcal{C}} = \big(z - z_N, \boldsymbol{tr}\{V_N^h\}\big)_\Omega, \\ \dfrac{1}{d_s}\big(y^\alpha(\nabla \mathcal{Q}_N^h(z) - \nabla \mathcal{Q}_N^h), \nabla P_N^h\big)_{\mathcal{C}} = \big(\boldsymbol{tr}\{\mathcal{U}_N^h(z) - \mathcal{U}_N^h\}, \boldsymbol{tr}\{P_N^h\}\big)_\Omega, \\ \dfrac{1}{d_s}\big(y^\alpha(\nabla \mathcal{Q}_N^h(u) - \nabla \mathcal{Q}_N^h(z)), \nabla P_N^h\big)_{\mathcal{C}} = \big(\boldsymbol{tr}\{\mathcal{U} - \mathcal{U}_N^h(z)\}, \boldsymbol{tr}\{P_N^h\}\big)_\Omega, \\ \dfrac{1}{d_s}\big(y^\alpha(\nabla \mathcal{Q}_N^h - \nabla \mathcal{Q}_N^h(z)), \nabla P_N^h\big)_{\mathcal{C}} = \big(\boldsymbol{tr}\{\mathcal{U}_N^h - \mathcal{U}_N^h(z)\}, \boldsymbol{tr}\{P_N^h\}\big)_\Omega. \end{cases} \tag{4.6}$$

4.2 Error estimate

4.2.1 Error estimate of the extended control problem

To better describe the error, we introduce the weighted Hilbert space

$$H^1_{y^\alpha}(\Lambda) = \Big\{ v \in L^2_{y^\alpha}(\Lambda) : \partial_y v \in L^2_{y^\alpha}(\Lambda) \Big\}, \quad \alpha > -1.$$

The projection errors in the $H^1_{y^\alpha}(\Lambda)$ norm are given below.

Lemma 4.1 ([3]). *For any $u \in H^1_{y^\alpha}(\Lambda) \cap \widehat{B}^m_\alpha(\Lambda)$ and $\partial_y u \in H^1_{y^\alpha}(\Lambda) \cap \widehat{B}^{m-1}_\alpha(\Lambda)$, $m \geq 2$, we have*

$$\|\partial_y(u - \pi_N^y u)\|_{y^\alpha,\Lambda} \leq C N^{\frac{2-m}{2}} \|\widehat{\partial}_y^m u\|_{y^{\alpha+m-1},\Lambda}. \tag{4.7}$$

Lemma 4.2 ([3]). *For any $u \in H^1_{y^\alpha}(\Lambda) \cap \widehat{B}^m_\alpha(\Lambda)$ and $\partial_y u \in H^1_{y^\alpha}(\Lambda) \cap \widehat{B}^{m-1}_\alpha(\Lambda)$, $2 \leq m \leq N+1$, we have*

$$\|\pi_N^y u - u\|_{1,y^\alpha,\Lambda} \leq C N^{-\frac{m}{2}} \big(\|\widehat{\partial}_y^m u\|_{y^{\alpha+m},\Lambda} + N \|\widehat{\partial}_y^m u\|_{y^{\alpha+m-1},\Lambda} \big). \tag{4.8}$$

To be clear for reader, we define

$$\begin{aligned} \widehat{U} = & \|\nabla_x(\pi_h^x - \mathcal{I})\mathcal{U}\|_{y^\alpha,\mathcal{C}} + \|(\pi_h^x - \mathcal{I})\widehat{\partial}_y^2 \mathcal{U}\|_{y^{\alpha+1},\mathcal{C}} \\ & + N^{-\frac{m}{2}} (\|(\nabla_x(\widehat{\partial}_y^m \mathcal{U})\|_{y^{\alpha+m},\mathcal{C}} + N\|\widehat{\partial}_y^m \mathcal{U}\|_{y^{\alpha+m-1},\mathcal{C}}), \\ \widehat{Q} = & \|\nabla_x(\pi_h^x - \mathcal{I})\mathcal{Q}\|_{y^\alpha,\mathcal{C}} + \|(\pi_h^x - \mathcal{I})\widehat{\partial}_y^2 \mathcal{Q}\|_{y^{\alpha+1},\mathcal{C}} \\ & + N^{-\frac{m}{2}} (\|(\nabla_x(\widehat{\partial}_y^m \mathcal{Q})\|_{y^{\alpha+m},\mathcal{C}} + N\|\widehat{\partial}_y^m \mathcal{Q}\|_{y^{\alpha+m-1},\mathcal{C}}). \end{aligned}$$

Here π_h^x denotes a projection operator satisfying

$$\|u-\pi_h^x u\|_{H^1(\Omega)} \lesssim \inf_{u_h\in X_h}\|u-u_h\|_{H^1(\Omega)}.$$

For example, we can choose π_h^x to be the Ritz projection if X_h is a finite element space.

Theorem 4.1 ([3]). *Let $\mathcal{U}$ and $\mathcal{U}_N^h(z)$ be the solutions to problem* (2.5) *and* (4.1), *respectively. Assume that $\mathcal{U}(x,\cdot)\in H^1_{y^\alpha}(\Lambda)\cap\widehat{B}^m_\alpha(\Lambda)$ and $\partial_y\mathcal{U}(x,\cdot)\in H^1_{y^\alpha}(\Lambda)\cap\widehat{B}^{m-1}_\alpha(\Lambda)$, $2\le m\le N+1$. Then the following estimate holds*

$$\|\mathcal{U}-\mathcal{U}_N^h(z)\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\le C\widehat{U}. \tag{4.9}$$

Theorem 4.2. *Let $\mathcal{Q}$ and $\mathcal{Q}_N^h(u)$ be the solutions to problem* (3.3) *and* (4.4), *respectively. Suppose that $\mathcal{Q}(x,\cdot)\in H^1_{y^\alpha}(\Lambda)\cap\widehat{B}^m_\alpha(\Lambda)$ and $\partial_y\mathcal{Q}(x,\cdot)\in H^1_{y^\alpha}(\Lambda)\cap\widehat{B}^{m-1}_\alpha(\Lambda)$, $2\le m\le N+1$. Then we have*

$$\|\mathcal{Q}-\mathcal{Q}_N^h(u)\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\le C\widehat{Q}. \tag{4.10}$$

Proof. The main idea follows [3]. Here we just sketch the poof. From (3.3) and (4.4), we find

$$\Big(y^\alpha\nabla\big(\mathcal{Q}-\mathcal{Q}_N^h(u)\big),\nabla P_N^h\Big)_{\mathcal{C}}=0,$$

which implies that

$$\begin{aligned}\big\|\mathcal{Q}-\mathcal{Q}_N^h(u)\big\|^2_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}&=\Big(y^\alpha\nabla\big(\mathcal{Q}-\mathcal{Q}_N^h(u)\big),\nabla\big(\mathcal{Q}-P_N^h\big)\Big)_{\mathcal{C}}\\&\le\big\|\mathcal{Q}-\mathcal{Q}_N^h(u)\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\big\|\mathcal{Q}-P_N^h\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})},\end{aligned}$$

namely,

$$\big\|\mathcal{Q}-\mathcal{Q}_N^h(u)\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\le\inf_{P_N^h\in X_h\times Y_N}\big\|\mathcal{Q}-P_N^h\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}. \tag{4.11}$$

Substituting $P_N^h=\pi_N^y\pi_h^x\mathcal{Q}$ in (4.11) results in

$$\big\|\mathcal{Q}-P_N^h\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\le\big\|\nabla(\pi_N^y\pi_h^x\mathcal{Q}-\mathcal{Q})\big\|_{y^\alpha,\mathcal{C}}.$$

Let $\mathcal{I}$ be the identity operator, then we further have

$$\big\|\nabla(\pi_N^y\circ\pi_h^x\mathcal{Q}-\mathcal{Q})\big\|_{y^\alpha,\mathcal{C}}\le\big\|\nabla(\pi_h^x-\mathcal{I})\circ\pi_N^y\mathcal{Q})\big\|_{y^\alpha,\mathcal{C}}+\big\|\nabla(\pi_N^y-\mathcal{I})\circ\mathcal{Q}\big\|_{y^\alpha,\mathcal{C}}.$$

By Lemmas 2.2 and 4.1 we derive

$$\begin{aligned}\big\|\nabla(\pi_h^x-\mathcal{I})\circ\pi_N^y\mathcal{Q})\big\|_{y^\alpha,\mathcal{C}}&\le\big\|\nabla_x(\pi_h^x-\mathcal{I})\circ\pi_N^y\mathcal{Q})\big\|_{y^\alpha,\mathcal{C}}+\big\|\partial_y(\pi_h^x-\mathcal{I})\circ\pi_N^y\mathcal{Q})\big\|_{y^\alpha,\mathcal{C}}\\&\le C\big\|\nabla_x(\pi_h^x-\mathcal{I})\mathcal{Q}\big\|_{y^\alpha,\mathcal{C}}+C\big\|(\pi_h^x-\mathcal{I})\widehat{\partial}_y^2\mathcal{Q})\big\|_{y^{\alpha+1},\mathcal{C}}\end{aligned}$$

and

$$\begin{aligned}\left\|\nabla(\pi_N^y-\mathcal{I})\mathcal{Q}\right\|_{y^\alpha,\mathcal{C}} \leq& \left\|\nabla_x(\pi_N^y-\mathcal{I})\mathcal{Q}\right\|_{y^\alpha,\mathcal{C}}+\left\|\partial_y(\pi_N^y-\mathcal{I})\mathcal{Q}\right\|_{y^\alpha,\mathcal{C}}\\ \leq& CN^{-\frac{m}{2}}\Big(\left\|\nabla_x(\widehat{\partial}_y^m\mathcal{Q})\right\|_{y^{\alpha+m},\mathcal{C}}+N\left\|\widehat{\partial}_y^m\mathcal{Q}\right\|_{y^{\alpha+m-1},\mathcal{C}}\Big).\end{aligned}$$

Combining above estimates yields the theorem result. □

Theorem 4.3. *Let* $(\mathcal{U},\mathcal{Q},z)$ *and* $(\mathcal{U}_N^h,\mathcal{Q}_N^h,z_N)$ *be the solutions of* (3.3) *and* (4.4). *Suppose that* $\mathcal{U}(x,\cdot),\mathcal{Q}(x,\cdot)\in H^1_{y^\alpha}(\Lambda)\cap\widehat{B}_\alpha^m(\Lambda)$ *and* $\partial_y\mathcal{U}(x,\cdot),\partial_y\mathcal{Q}(x,\cdot)\in H^1_{y^\alpha}(\Lambda)\cap\widehat{B}_\alpha^{m-1}(\Lambda)$, $2\leq m\leq N+1$. *Then we have*

$$\|\mathcal{U}-\mathcal{U}_N^h\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}+\|\mathcal{Q}-\mathcal{Q}_N^h\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}+\|z-z_N\|_{L^2(\Omega)}\leq C(\widehat{U}+\widehat{Q}). \tag{4.12}$$

Proof. We decompose $\mathcal{U}-\mathcal{U}_N^h$ and $\mathcal{Q}-\mathcal{Q}_N^h$ into

$$\begin{aligned}\mathcal{U}-\mathcal{U}_N^h&=\mathcal{U}-\mathcal{U}_N^h(z)+\mathcal{U}_N^h(z)-\mathcal{U}_N^h,\\ \mathcal{Q}-\mathcal{Q}_N^h&=\mathcal{Q}-\mathcal{Q}_N^h(u)+\mathcal{Q}_N^h(u)-\mathcal{Q}_N^h.\end{aligned}$$

Then we obtain

$$\begin{aligned}\|\mathcal{U}-\mathcal{U}_N^h\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}&\leq\|\mathcal{U}-\mathcal{U}_N^h(z)\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}+\|\mathcal{U}_N^h(z)-\mathcal{U}_N^h\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})},\\ \|\mathcal{Q}-\mathcal{Q}_N^h\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}&\leq\|\mathcal{Q}-\mathcal{Q}_N^h(u)\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})},+\|\mathcal{Q}_N^h(u)-\mathcal{Q}_N^h\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}.\end{aligned}$$

Setting

$$\begin{aligned}&V_N^h=\mathcal{U}_N^h(z)-\mathcal{U}_N^h, &&P_N^h=\mathcal{Q}_N^h-\mathcal{Q}_N^h(z),\\ &P_N^h=\mathcal{Q}_N^h(u)-\mathcal{Q}_N^h(z), &&P_N^h=\mathcal{Q}_N^h(z)-\mathcal{Q}_N^h,\end{aligned}$$

in (4.6), respectively, we derive

$$\begin{cases}\Big(y^\alpha\nabla(\mathcal{U}_N^h(z)-\mathcal{U}_N^h),\nabla(\mathcal{U}_N^h(z)-\mathcal{U}_N^h)\Big)_{\mathcal{C}}=d_s\Big(z-z_N,tr\{\mathcal{U}_N^h(z)-\mathcal{U}_N^h\}\Big)_{\Omega},\\ \Big(y^\alpha\nabla(\mathcal{Q}_N^h(u)-\mathcal{Q}_N^h(z)),\nabla(\mathcal{Q}_N^h(u)-\mathcal{Q}_N^h(z))\Big)_{\mathcal{C}}=d_s\Big(\boldsymbol{tr}\{\mathcal{U}-\mathcal{U}_N^h(z)\},\boldsymbol{tr}\{\mathcal{Q}_N^h(u)-\mathcal{Q}_N^h(z)\}\Big)_{\Omega},\\ \Big(y^\alpha\nabla(\mathcal{Q}_N^h(z)-\nabla\mathcal{Q}_N^h),\nabla(\mathcal{Q}_N^h(z)-\mathcal{Q}_N^h)\Big)_{\mathcal{C}}=d_s\Big(\boldsymbol{tr}\{\mathcal{U}_N^h-\mathcal{U}_N^h(z)\},\boldsymbol{tr}\{\mathcal{Q}_N^h(z)-\mathcal{Q}_N^h\}\Big)_{\Omega}.\end{cases}$$

By (2.4) we have

$$\begin{cases}\|\mathcal{U}_N^h(z)-\mathcal{U}_N^h\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\leq C\|z-z_N\|_{L^2(\Omega)},\\ \|\mathcal{Q}_N^h(u)-\mathcal{Q}_N^h(z)\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\leq C\|\mathcal{U}-\mathcal{U}_N^h(z)\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\leq C\widehat{U},\\ \|\mathcal{Q}_N^h(z)-\mathcal{Q}_N^h\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\leq C\|\mathcal{U}_N^h(z)-\mathcal{U}_N^h\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\leq C\|z-z_N\|_{L^2(\Omega)}.\end{cases} \tag{4.13}$$

Choosing $v=z_N$ in (3.3) and $v=z$ in (4.4) leads to

$$\mu\|z-z_N\|^2_{L^2(\Omega)}\le\Big(\boldsymbol{tr}\{\mathcal{Q}-\mathcal{Q}^h_N\},z_N-z\Big). \tag{4.14}$$

Setting

$$V^h_N=\mathcal{Q}^h_N(z)-\mathcal{Q}^h_N \quad \text{and} \quad P^h_N=\mathcal{U}^h_N(z)-\mathcal{U}^h_N$$

in (4.6) gives

$$\begin{aligned}&\Big(y^\alpha\nabla\big(\mathcal{U}^h_N(z)-\mathcal{U}^h_N\big),\nabla\big(\mathcal{Q}^h_N(z)-\mathcal{Q}^h_N\big)\Big)_{\mathcal{C}}=-d_s\Big(\boldsymbol{tr}\{\mathcal{Q}^h_N(z)-\mathcal{Q}^h_N\},z_N-z\Big)_\Omega,\\&\Big(y^\alpha\nabla\big(\mathcal{Q}^h_N(z)-\mathcal{Q}^h_N\big),\nabla\big(\mathcal{U}^h_N(z)-\mathcal{U}^h_N\big)\Big)_{\mathcal{C}}=d_s\Big(\boldsymbol{tr}\{\mathcal{U}^h_N(z)-\mathcal{U}^h_N\},\boldsymbol{tr}\{\mathcal{U}^h_N(z)-\mathcal{U}^h_N\}\Big)_\Omega.\end{aligned}$$

Then we have

$$\Big(\boldsymbol{tr}\{\mathcal{Q}^h_N(z)-\mathcal{Q}^h_N\},z_N-z\Big)_\Omega=-\big\|\boldsymbol{tr}\{\mathcal{U}^h_N(z)-\mathcal{U}^h_N\}\big\|^2_{L^2(\Omega)}\le 0.$$

Using above estimate we further obtain

$$\begin{aligned}\mu\|z-z_N\|^2_{L^2(\Omega)}&\le\Big(\boldsymbol{tr}\{\mathcal{Q}-\mathcal{Q}^h_N(z)\},z_N-z\Big)\\&\le\big\|\boldsymbol{tr}\{\mathcal{Q}-\mathcal{Q}^h_N(z)\}\big\|_{\mathbb{H}^s(\Omega)}\|z_N-z\|_{L^2(\Omega)}.\end{aligned}$$

By Theorem 4.1, Theorem 4.2 and (4.13), we derive

$$\|z-z_N\|_{L^2(\Omega)}\le C\big\|\boldsymbol{tr}\{\mathcal{Q}-\mathcal{Q}^h_N(z)\}\big\|_{\mathbb{H}^s(\Omega)}\le C(\widehat{U}+\widehat{Q}). \tag{4.15}$$

Combining (4.13) and (4.15), we arrive at

$$\big\|\mathcal{U}-\mathcal{U}^h_N\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}+\big\|\mathcal{Q}-\mathcal{Q}^h_N\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}+\|z-z_N\|_{L^2(\Omega)}\le C(\widehat{U}+\widehat{Q}).$$

Thus, we complete the proof. □

4.2.2 Error estimate of the original optimal control problem

Theorem 4.4. *Let (u,q,z) and (u_N,q_N,z_N) be the solutions of* (3.4) *and* (4.4)*. If $\mathcal{U}(x,\cdot),\mathcal{Q}(x,\cdot)\in H^1_{y^\alpha}(\Lambda)\cap\widehat{B}^m_\alpha(\Lambda)$ and $\partial_y\mathcal{U}(x,\cdot),\partial_y\mathcal{Q}(x,\cdot)\in H^1_{y^\alpha}(\Lambda)\cap\widehat{B}^{m-1}_\alpha(\Lambda)$, $2\le m\le N+1$, then we have*

$$\|u-u_N\|_{\mathbb{H}^s}+\|q-q_N\|_{\mathbb{H}^s}+\|z-z_N\|_{L^2(\Omega)}\le C(\widehat{U}+\widehat{Q}).$$

Proof. According to the definition of trace, i.e., (2.3), we can get

$$\begin{aligned}&\|u-u_N\|_{\mathbb{H}^s(\Omega)}=\|\boldsymbol{tr}\{\mathcal{U}-\mathcal{U}^h_N\}\|_{\mathbb{H}^s(\Omega)}\le\big\|\mathcal{U}-\mathcal{U}^h_N\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\le C(\widehat{U}+\widehat{Q}),\\&\|q-q_N\|_{\mathbb{H}^s(\Omega)}=\|\boldsymbol{tr}\{\mathcal{Q}-\mathcal{Q}^h_N\}\|_{\mathbb{H}^s(\Omega)}\le\big\|\mathcal{Q}-\mathcal{Q}^h_N\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\le C(\widehat{U}+\widehat{Q}).\end{aligned}$$

This completes the proof. □

4.3 Numerical example

In this section we present a numerical example to support the theoretical result.

Example 4.1. We consider the following one-dimension fractional optimal control problem

$$\min_{z\in Z_{ad}} J(u,z) := \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|z\|^2_{L^2(\Omega)}$$

subject to

$$\begin{cases} (-\Delta)^s u(x) = f+z, & x\in\Omega=(-1,1), \\ u(-1)=u(1)=0. \end{cases}$$

For given

$$f(x)=\pi^{2s}\sin(\pi x)+\frac{1}{2}\sin(\pi x) \quad \text{and} \quad u_d(x)=\left(1-\frac{1}{2}\pi^{2s}\right)\sin(\pi x)$$

the exact solutions are defined by

$$z(x)=-\frac{1}{2}\sin(\pi x), \quad q(x)=\frac{1}{2}\sin(\pi x), \quad u(x)=\sin(\pi x).$$

In this example we choose the generalized Jacobi polynomials [10]

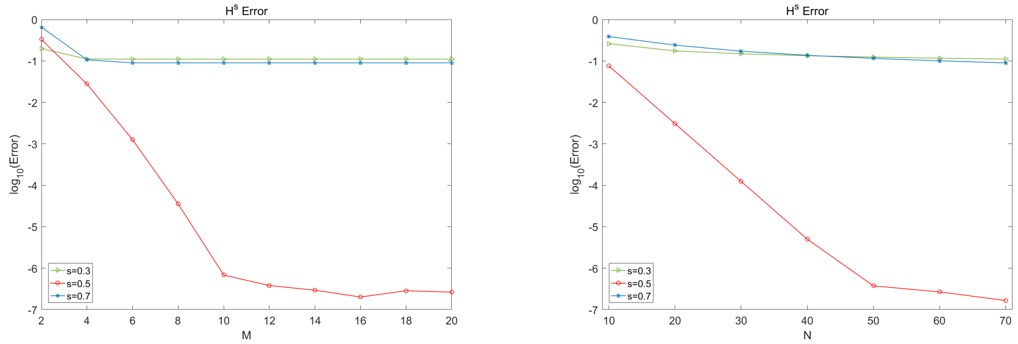
$$\phi^x_m(x)=Q^{-1,-1}_{m+1}(x):=-\frac{1}{4}P^{-1,-1}_{m+1}=-\frac{1}{4}(1-x^2)J^{1,1}_{m-1},$$

and

$$\phi^y_n(y)=\widehat{\mathscr{L}}^{\alpha}_{n-2}(y)-\widehat{\mathscr{L}}^{\alpha}_{n-1}(y),$$

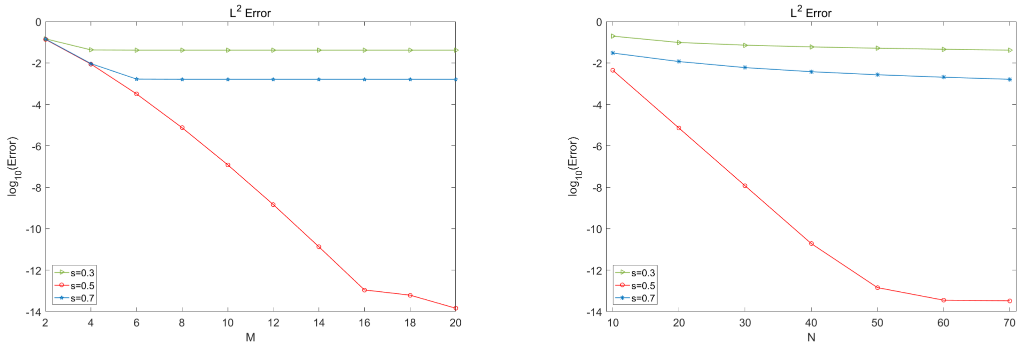
as the basis in x and y direction to derive the corresponding stiffness matrix and mass matrix.

In the following figures we show the H^s error of state variable and adjoint variable and the L^2 error of control variable for $s=0.3$, $s=0.5$ and $s=0.7$. Note that the solutions are smooth. From Figs. 1-3, we can observe the high accuracy and high efficiency (exponential decay) of the spectral scheme for $s=0.5$, while for $s\neq 0.5$, the convergence rate is very low. According to [3] we know that the solution of the extended state equation has low regularity in y direction, which would seriously deteriorate the convergence rate of the usual numerical method. To overcome this, we will use the enriched spectral method (see [3]) to improve the numerical method and enhance its convergence rate.



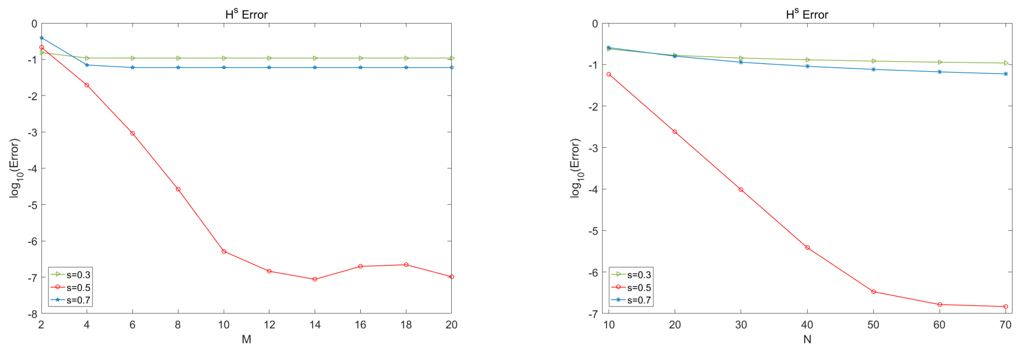
(a) (b)

Figure 1: The H^s errors of state $u(x)$: (a) $N=70$, (b) $M=30$.



(a) (b)

Figure 2: The L^2 errors of control $z(x)$: (a) $N=70$, (b) $M=30$.



(a) (b)

Figure 3: The H^s errors of adjoint $q(x)$: (a) $N=70$, (b) $M=30$.

5 Galerkin approximation with enriched space in extended direction

5.1 Enriched spectral discrete scheme

The solution of the problem (1.2) can be derived from the Caffarelli-Silvestre extension (3.1)-(3.2), i.e., $u(x) = \mathcal{U}(x,0)$. Indeed, owe to [2, Lemma 2.2] and [6, Proposition 2.1],

$$\mathcal{U}(x,y) = \sum_{n=1}^{\infty} \tilde{u}_n \phi_n(x) \psi_n(y),$$

where $\psi_n(y)$, $n = 1,2,\cdots$, solves

$$\begin{cases} -\psi''(y) - \dfrac{\alpha}{y} - \psi'(y) + \lambda\psi(y) = 0, & y \in \Lambda, \\ \psi(0) = 1, \quad \lim\limits_{y\to\infty} \psi(y) = 0. \end{cases}$$

Here $\lambda > 0$, $\alpha = 1-2s$, $s \in (0,1)$. Thanks to [6, Proposition 2.1], we have

$$\psi_n(y) = \begin{cases} e^{-\sqrt{\lambda_n} y}, & s = \dfrac{1}{2}, \\ \dfrac{2^{1-s}}{\Gamma(s)} (\sqrt{\lambda_n} y)^s K_s(\sqrt{\lambda_n} y), & s \in (0,1) \setminus \left\{ \dfrac{1}{2} \right\}, \end{cases}$$

where K_s is the modified Bessel function of second kind. That implies that there exists singularity in y direction for $s \in (0,1) \setminus \{\frac{1}{2}\}$, which affects the convergence rate. Therefore we need to apply enriched spectral method in y-axis to improve the accuracy of numerical approximation.

According to [3] we introduce

$$Y_N^k := Y_N \oplus \mathrm{span}\{\mathcal{S}_i\}_{i=1}^k, \qquad \mathcal{S}_i(y) := y^{i-\alpha} e^{-\frac{y}{2}}.$$

The enriched spectral approximation for (3.2) is to find $\mathcal{U}_{N,k}^h(z) \in X_h \times Y_N^k$ such that

$$\left(y^\alpha \nabla \mathcal{U}_{N,k}^h(z), \nabla V_{N,k}^h\right)_{\mathcal{C}} = d_s(f+z, \boldsymbol{tr}\{V_{N,k}^h\})_\Omega, \quad \forall V_{N,k}^h \in X_h \times Y_N^k. \tag{5.1}$$

Then the enriched spectral discrete scheme for problems (3.1)-(3.2) can be expressed as

$$\min_{z_{N,k} \in Z_{ad}} J(\mathcal{U}_{N,k}^h, z_{N,k}) := \frac{1}{2} \|\boldsymbol{tr}\{\mathcal{U}_{N,k}^h\} - u_d\|_{L^2(\Omega)}^2 + \frac{\mu}{2} \|z_{N,k}\|_{L^2(\Omega)}^2, \tag{5.2}$$

such that

$$\frac{1}{d_s} \left(y^\alpha \nabla \mathcal{U}_{N,k}^h, \nabla V_{N,k}^h\right)_{\mathcal{C}} = (f + z_{N,k}, \boldsymbol{tr}\{V_{N,k}^h\})_\Omega, \quad \forall V_{N,k}^h \in X_h \times Y_N^k. \tag{5.3}$$

In an analogous way to Section 4, we can obtain the following first order necessary conditions

$$\begin{cases} \dfrac{1}{d_s}\big(y^\alpha \nabla \mathcal{U}^h_{N,k}, \nabla V^h_{N,k}\big)_{\mathcal{C}} = \big(f + z_{N,k}, \boldsymbol{tr}\{V^h_{N,k}\}\big)_\Omega, & \forall V^h_{N,k} \in X_h \times Y^k_N, \\ \dfrac{1}{d_s}\big(y^\alpha \nabla \mathcal{Q}^h_{N,k}, \nabla P^h_{N,k}\big)_{\mathcal{C}} = \big(\boldsymbol{tr}\{\mathcal{U}^h_{N,k}\} - u_d, \boldsymbol{tr}\{P^h_{N,k}\}\big)_\Omega, & \forall P^h_{N,k} \in X_h \times Y^k_N, \\ \displaystyle\int_\Omega \big(\mu z_{N,k} + \boldsymbol{tr}\{\mathcal{Q}^h_{N,k}\}\big)(v - z_{N,k}) \geq 0, & \forall v \in Z_{ad}. \end{cases} \tag{5.4}$$

To derive error estimate we introduce the corresponding auxiliary problems

$$\begin{cases} \dfrac{1}{d_s}\big(y^\alpha \nabla \mathcal{U}^h_{N,k}(z), \nabla V^h_{N,k}\big)_{\mathcal{C}} = \big(f + z, \boldsymbol{tr}\{V^h_{N,k}\}\big)_\Omega, & \forall V^h_{N,k} \in X_h \times Y^k_N, \\ \dfrac{1}{d_s}\big(y^\alpha \nabla \mathcal{Q}^h_{N,k}(z), \nabla P^h_{N,k}\big)_{\mathcal{C}} = \big(\boldsymbol{tr}\{\mathcal{U}^h_{N,k}(z)\} - u_d, \boldsymbol{tr}\{P^h_{N,k}\}\big)_\Omega, & \forall P^h_{N,k} \in X_h \times Y^k_N, \\ \dfrac{1}{d_s}\big(y^\alpha \nabla \mathcal{Q}^h_{N,k}(u), \nabla P^h_{N,k}\big)_{\mathcal{C}} = \big(\boldsymbol{tr}\{\mathcal{U}\} - u_d, \boldsymbol{tr}\{P^h_{N,k}\}\big)_\Omega, & \forall P^h_{N,k} \in X_h \times Y^k_N. \end{cases} \tag{5.5}$$

Combining (5.4) and (5.5), we can obtain

$$\begin{cases} \dfrac{1}{d_s}\big(y^\alpha (\nabla \mathcal{U}^h_{N,k}(z) - \nabla \mathcal{U}^h_{N,k}), \nabla V^h_{N,k}\big)_{\mathcal{C}} = \big(z - z_{N,k}, \boldsymbol{tr}\{V^h_{N,k}\}\big)_\Omega, \\ \dfrac{1}{d_s}\big(y^\alpha (\nabla \mathcal{Q}^h_{N,k}(z) - \nabla \mathcal{Q}^h_{N,k}), \nabla P^h_{N,k}\big)_{\mathcal{C}} = \big(\boldsymbol{tr}\{\mathcal{U}^h_{N,k}(z) - \mathcal{U}^h_{N,k}\}, \boldsymbol{tr}\{P^h_{N,k}\}\big)_\Omega, \\ \dfrac{1}{d_s}\big(y^\alpha (\nabla \mathcal{Q}^h_{N,k}(u) - \nabla \mathcal{Q}^h_{N,k}(z)), \nabla P^h_{N,k}\big)_{\mathcal{C}} = \big(\boldsymbol{tr}\{\mathcal{U} - \mathcal{U}^h_{N,k}(z)\}, \boldsymbol{tr}\{P^h_{N,k}\}\big)_\Omega. \end{cases} \tag{5.6}$$

5.2 Error estimate

5.2.1 Error estimate of the extended optimal control problem

The error analysis for the enriched spectral approximation of the extended control problem is analogous to the standard spectral discretization presented in previous section. Therefore we just sketch the main results for the sake of brevity.

For convenience, we define

$$\begin{aligned} \widetilde{F} = {} & N^{-\frac{m}{2}} |f + z|_{\mathbb{H}^{\frac{m}{2}-s}(\Omega)} + N^{1-\frac{m}{2}} |f + z|_{\mathbb{H}^{\frac{m-1}{2}-s}(\Omega)} \\ & + h^r |f + z|_{\mathbb{H}^{r+\frac{1}{2}-s}(\Omega)} + h^{r-1} |f + z|_{\mathbb{H}^{r-1-s}(\Omega)}, \\ \widetilde{U} = {} & N^{-\frac{m}{2}} |u - u_d|_{\mathbb{H}^{\frac{m}{2}-s}(\Omega)} + N^{1-\frac{m}{2}} |u - u_d|_{\mathbb{H}^{\frac{m-1}{2}-s}(\Omega)} \\ & + h^r |u - u_d|_{\mathbb{H}^{r+\frac{1}{2}-s}(\Omega)} + h^{r-1} |u - u_d|_{\mathbb{H}^{r-1-s}(\Omega)}. \end{aligned}$$

Note that $\mathcal{U}^h_{N,k}(z)$ can be viewed as the enriched spectral approximation of $\mathcal{U}$. Then we have the following error estimate from [3].

Theorem 5.1 ([3]). *Let $\mathcal{U}$ and $\mathcal{U}_{N,k}^h(z)$ be the solutions to problem* (2.5) *and* (5.1)*, respectively. If $f+z\in\mathbb{H}^{l-s}(\Omega)$, $l=\max\{\frac{m}{2},r+\frac{1}{2}\}$, then it holds*

$$\|\mathcal{U}-\mathcal{U}_{N,k}^h(z)\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\leq\widetilde{F}, \tag{5.7}$$

where $m=2k+1+[2s]$.

Combing variational formulation (3.3) and numerical scheme (5.5) leads to

$$\Big(y^\alpha\nabla\big(\mathcal{Q}-\mathcal{Q}_{N,k}^h(u)\big),\nabla P_{N,k}^h\Big)_{\mathcal{C}}=0,\quad \forall P_{N,k}^h\in X_h\times Y_N^k.$$

Then, by Cauchy-Schwarz inequality, we have that for any $\Phi\in X_h\times Y_N^k$

$$\begin{aligned}\big\|\mathcal{Q}-\mathcal{Q}_{N,k}^h(u)\big\|^2_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}&=\Big(y^\alpha\nabla\big(\mathcal{Q}-\mathcal{Q}_{N,k}^h(u)\big),\nabla\big(\mathcal{Q}-\Phi\big)\Big)_{\mathcal{C}}\\&\leq\big\|\mathcal{Q}-\mathcal{Q}_{N,k}^h(u)\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\big\|\mathcal{Q}-\Phi\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})},\end{aligned}$$

namely,

$$\big\|\mathcal{Q}-\mathcal{Q}_{N,k}^h(u)\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\leq\inf_{\Phi\in X_h\times Y_N^k}\big\|\mathcal{Q}-\Phi\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}. \tag{5.8}$$

Similar to the state equation we have the following estimate according to [3].

Theorem 5.2. *Let $\mathcal{Q}$ and $\mathcal{Q}_{N,k}^h(u)$ be the solutions to problem* (3.3) *and* (5.4)*, respectively. If $u-u_d\in\mathbb{H}^{l-s}(\Omega)$, $l=\max\{\frac{m}{2},r+\frac{1}{2}\}$, then it holds*

$$\|\mathcal{Q}-\mathcal{Q}_{N,k}^h(u)\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}\leq C\widetilde{U}, \tag{5.9}$$

where $m=2k+1+[2s]$.

Theorem 5.3. *Let $(\mathcal{U},\mathcal{Q},z)$ and $(\mathcal{U}_{N,k}^h,\mathcal{Q}_N^{h,k},z_N)$ be the solution of* (3.3) *and* (5.4)*, respectively. Suppose that $f+z,u-u_d\in\mathbb{H}^{l-s}(\Omega)$, $l=\max\{\frac{m}{2},r+\frac{1}{2}\}$, $m=2k+1+[2s]$. Then we have*

$$\|\mathcal{U}-\mathcal{U}_{N,k}^h\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}+\big\|\mathcal{Q}-\mathcal{Q}_{N,k}^h\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}+\|z-z_{N,k}\|_{L^2(\Omega)}\leq C(\widetilde{F}+\widetilde{U}). \tag{5.10}$$

Proof. First we can decompose $\mathcal{U}-\mathcal{U}_{N,k}^h$ and $\mathcal{Q}-\mathcal{Q}_{N,k}^h$ into

$$\begin{aligned}\mathcal{U}-\mathcal{U}_{N,k}^h&=\mathcal{U}-\mathcal{U}_{N,k}^h(z)+\mathcal{U}_{N,k}^h(z)-\mathcal{U}_{N,k}^h,\\\mathcal{Q}-\mathcal{Q}_{N,k}^h&=\mathcal{Q}-\mathcal{Q}_{N,k}^h(u)+\mathcal{Q}_{N,k}^h(u)-\mathcal{Q}_{N,k}^h.\end{aligned}$$

Then we have

$$\begin{aligned}\|\mathcal{U}-\mathcal{U}_{N,k}^h\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}&\leq\|\mathcal{U}-\mathcal{U}_{N,k}^h(z)\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}+\|\mathcal{U}_{N,k}^h(z)-\mathcal{U}_{N,k}^h\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})},\\\big\|\mathcal{Q}-\mathcal{Q}_{N,k}^h\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}&\leq\big\|\mathcal{Q}-\mathcal{Q}_{N,k}^h(u)\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}+\big\|\mathcal{Q}_{N,k}^h(u)-\mathcal{Q}_{N,k}^h\big\|_{\mathcal{H}^1_{y^\alpha}(\mathcal{C})}.\end{aligned}$$

Next, setting

$$V_{N,k}^h = \mathcal{U}_{N,k}^h(z) - \mathcal{U}_{N,k}^h, \qquad P_{N,k}^h = \mathcal{Q}_{N,k}^h - \mathcal{Q}_{N,k}^h(z),$$
$$P_{N,k}^h = \mathcal{Q}_{N,k}^h(u) - \mathcal{Q}_{N,k}^h(z), \qquad P_{N,k}^h = \mathcal{Q}_{N,k}^h(z) - \mathcal{Q}_{N,k}^h,$$

in (5.6), and using (2.4) we obtain

$$\|\mathcal{U}_{N,k}^h(z) - \mathcal{U}_{N,k}^h\|_{\mathcal{H}_{y^\alpha}^1(\mathcal{C})} \le C\|z - z_{N,k}\|_{L^2(\Omega)}, \tag{5.11a}$$
$$\|\mathcal{Q}_{N,k}^h(u) - \mathcal{Q}_{N,k}^h(z)\|_{\mathcal{H}_{y^\alpha}^1(\mathcal{C})} \le C\|\mathcal{U} - \mathcal{U}_{N,k}^h(z)\|_{\mathcal{H}_{y^\alpha}^1(\mathcal{C})} \le C\widetilde{F}, \tag{5.11b}$$
$$\|\mathcal{Q}_{N,k}^h(z) - \mathcal{Q}_{N,k}^h\|_{\mathcal{H}_{y^\alpha}^1(\mathcal{C})} \le C\|\mathcal{U}_{N,k}^h(z) - \mathcal{U}_{N}^h\|_{\mathcal{H}_{y^\alpha}^1(\mathcal{C})}. \tag{5.11c}$$

Then, choosing

$$V_{N,k}^h = \mathcal{Q}_{N,k}^h(z) - \mathcal{Q}_{N,k}^h \quad \text{and} \quad P_{N,k}^h = \mathcal{U}_{N,k}^h(z) - \mathcal{U}_{N,k}^h$$

in (5.6) yields

$$\Big(\boldsymbol{tr}\{\mathcal{Q}_{N,k}^h(z) - \mathcal{Q}_{N,k}^h\}, z_{N,k} - z\Big)_\Omega = -\|\boldsymbol{tr}\{\mathcal{U}_{N,k}^h(z) - \mathcal{U}_{N,k}^h\}\|_{L^2(\Omega)}^2 \le 0.$$

Setting $v = z_{N,k}$ in (3.3), and $v = z$ in (5.4), we can obtain

$$\begin{aligned}\mu\|z - z_{N,k}\|_{L^2(\Omega)}^2 &\le \Big(\boldsymbol{tr}\{\mathcal{Q} - \mathcal{Q}_{N,k}^h(z)\}, z_{N,k} - z\Big) \\ &\le \|\boldsymbol{tr}\{\mathcal{Q} - \mathcal{Q}_{N,k}^h(z)\}\|_{\mathbb{H}^s(\Omega)}\|z_{N,k} - z\|_{L^2(\Omega)}.\end{aligned}$$

By Theorem 5.1 and Theorem 5.2, we have

$$\|z - z_{N,k}\|_{L^2(\Omega)} \le C\|\mathcal{Q} - \mathcal{Q}_{N,k}^h(z)\|_{\mathcal{H}_{y^\alpha}^1(\mathcal{C})} \le C(\widetilde{F} + \widetilde{U}). \tag{5.12}$$

Combining (5.11a)-(5.11c) and (5.12), we obtain

$$\|\mathcal{U} - \mathcal{U}_{N,k}^h\|_{\mathcal{H}_{y^\alpha}^1(\mathcal{C})} + \|\mathcal{Q} - \mathcal{Q}_{N,k}^h\|_{\mathcal{H}_{y^\alpha}^1(\mathcal{C})} + \|z - z_{N,k}\|_{L^2(\Omega)} \le C(\widetilde{F} + \widetilde{U}).$$

So, we complete the proof of the theorem. □

5.2.2 Error estimate of original optimal control problems

Theorem 5.4. *Let* (u,q,z) *and* $(u_{N,k},q_{N,k},z_{N,k})$ *be the solution of* (3.4) *and* (5.4), *respectively. Assume that* $f+z, u-u_d \in \mathbb{H}^{l-s}(\Omega), l = \max\{\frac{m}{2}, r+\frac{1}{2}\}$, $m = 2k+1+[2s]$. *Then we have*

$$\|u - u_{N,k}\|_{\mathbb{H}^s} + \|q - q_{N,k}^h\|_{\mathbb{H}^s} + \|z - z_N\|_{L^2(\Omega)} \le C(\widetilde{F} + \widetilde{U}). \tag{5.13}$$

Proof. According to the definition of trace, i.e., (2.3), we can get

$$\|u - u_{N,k}\|_{\mathbb{H}^s} = \|\boldsymbol{tr}\{\mathcal{U}\} - \boldsymbol{tr}\{\mathcal{U}_{N,k}^h\}\|_{\mathbb{H}^s} \le \|\mathcal{U} - \mathcal{U}_{N,k}^h\|_{\mathcal{H}_{y^\alpha}^1(\mathcal{C})} \le C(\widetilde{F} + \widetilde{U}),$$
$$\|q - q_{N,k}\|_{\mathbb{H}^s} = \|\boldsymbol{tr}\{\mathcal{Q}\} - \boldsymbol{tr}\{\mathcal{Q}_{N,k}^h\}\|_{\mathbb{H}^s} \le \|\mathcal{Q} - \mathcal{Q}_{N,k}^h\|_{\mathcal{H}_{y^\alpha}^1(\mathcal{C})} \le C(\widetilde{F} + \widetilde{U}).$$

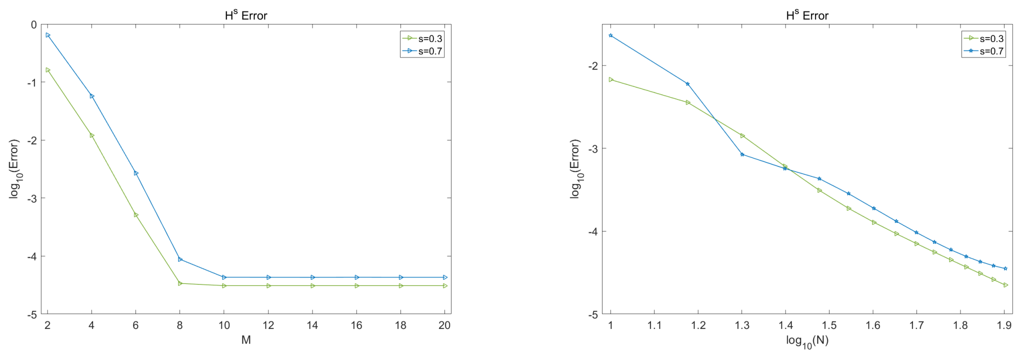
This complete the proof. □

5.3 Numerical example

In this section we choose the same numerical example of Section 4, and calculate the H^s error of $u(x)$ and $q(x)$ as well as the L^2 error of $z(x)$ for $s \neq 0.5$ and $k=2$. We also compare the numerical results with the method in Section 4.

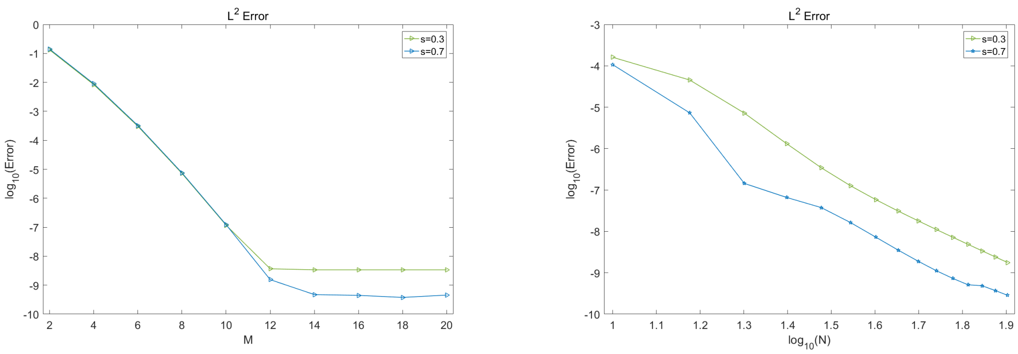
The convergence rates for H^s error of $u(x)$ and $q(x)$ as well as L^2 error of $z(x)$ for $s=0.3$ and 0.7 are displayed in Figs. 4-6, where we fix $M=30$, $k=2$ and $N=70$, $k=2$ respectively. We can observe that the enriched spectral scheme have a high accuracy and high efficiency (exponential decay), which verify our theoretical findings.

Moreover, we also compare the convergence rates for different k and $s=0.3$ and $s=0.7$. From Figs. 7-12, we can see that the usual spectral method $(k=0)$ hardly converges. However, the enriched spectral method significantly improves the convergence rates and numerical accuracy.



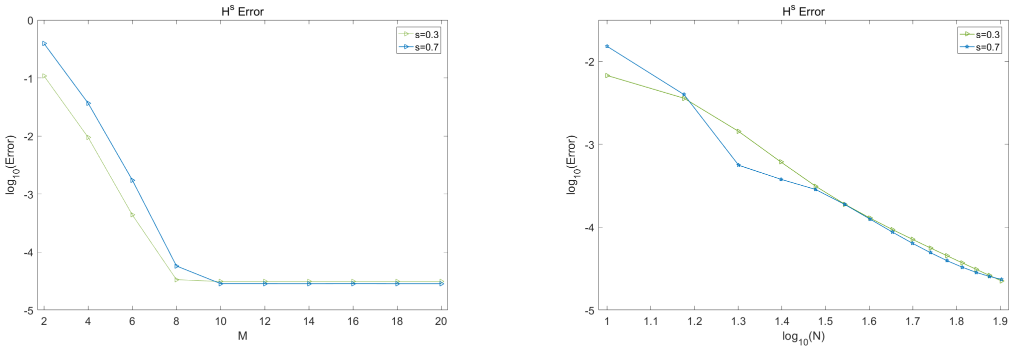
(a) (b)

Figure 4: The H^s errors of state $u(x)$: (a) $N=70$, $k=2$, (b) $M=30$, $k=2$.



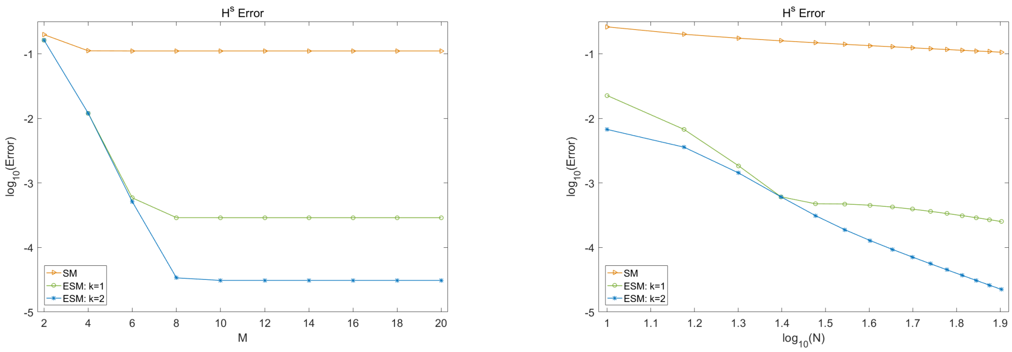
(a) (b)

Figure 5: The L^2 errors of control $z(x)$: (a) $N=70$, $k=2$, (b) $M=30$, $k=2$.



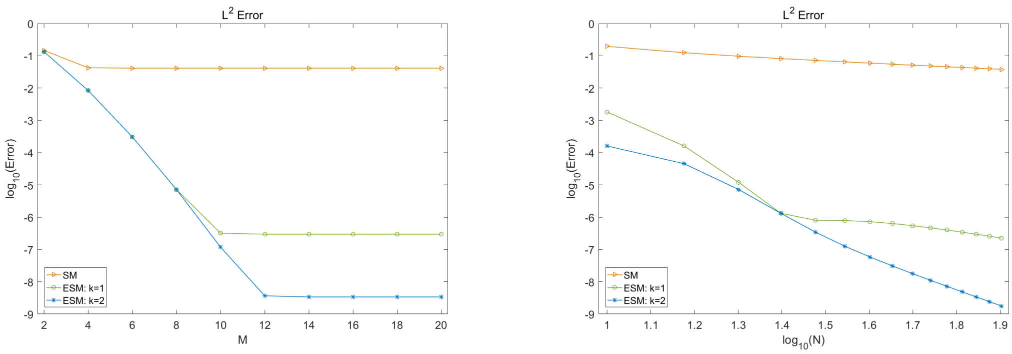
(a) (b)

Figure 6: The H^s errors of adjoint $q(x)$: (a) $N=70$, $k=2$, (b) $M=30$, $k=2$.



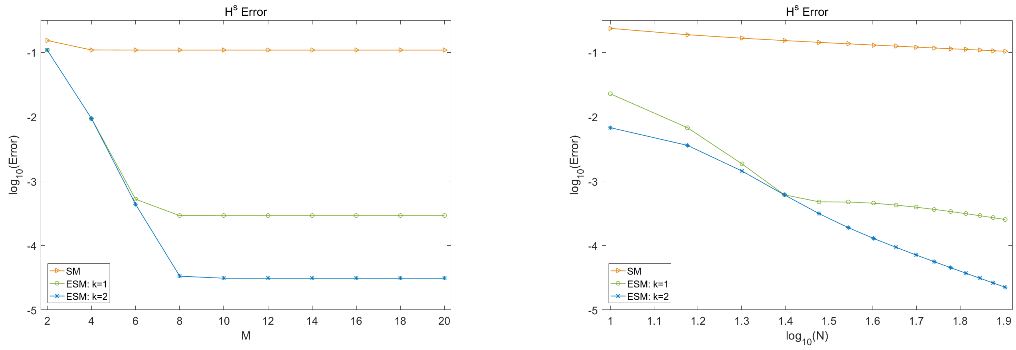
(a) (b)

Figure 7: The H^s errors of state $u(x)$: (a) $N=70$, $s=0.3$, (b) $M=30$, $s=0.3$.



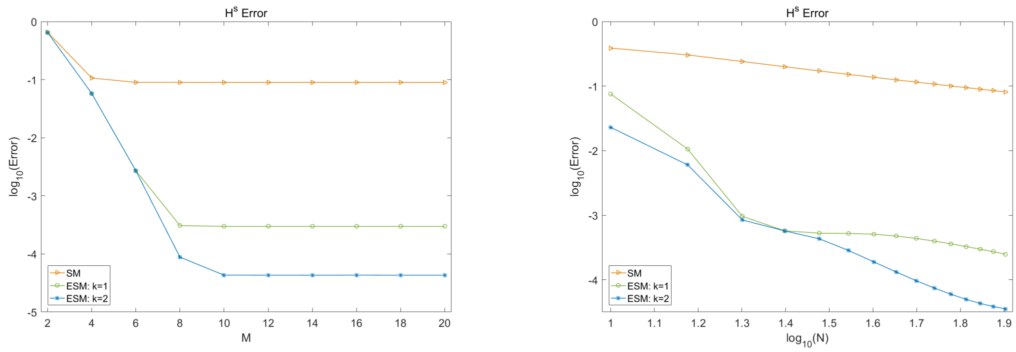
(a) (b)

Figure 8: The L^2 errors of control $z(x)$: (a) $N=70$, $s=0.3$, (b) $M=30$, $s=0.3$.



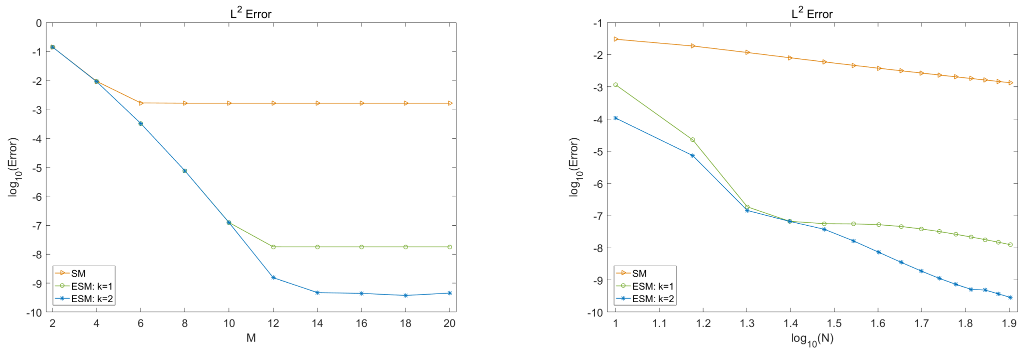
(a) (b)

Figure 9: The H^s errors of adjoint $q(x)$: (a) $N=70$, $s=0.3$, (b) $M=30$, $s=0.3$.



(a) (b)

Figure 10: The H^s errors of state $u(x)$: (a) $N=70$, $s=0.7$, (b) $M=30$, $s=0.7$.



(a) (b)

Figure 11: The L^2 errors of control $z(x)$: (a) $N=70$, $s=0.7$, (b) $M=30$, $s=0.7$.

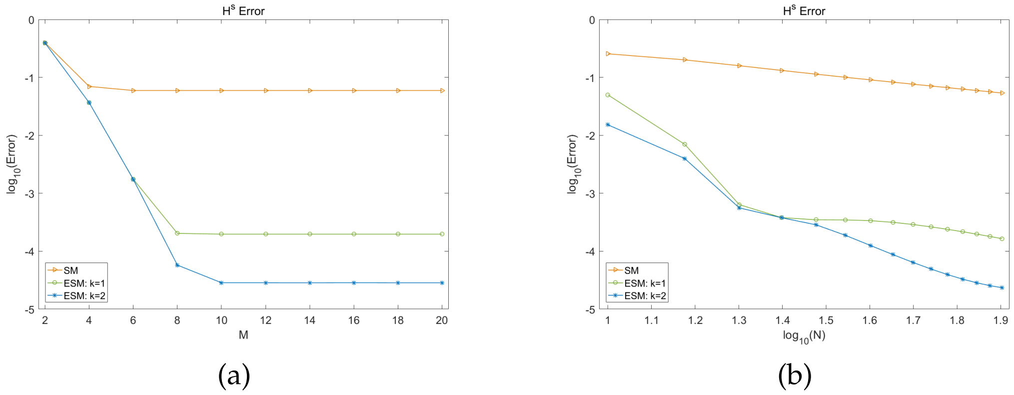


Figure 12: The H^s errors of adjoint $q(x)$: (a) $N=70$, $s=0.7$, (b) $M=30$, $s=0.7$.

Acknowledgements

This work was supported by the National Natural Science Foundation of China Project (Nos. 12071402, 11931003, 12261131501, and 11971276), the Project of Scientific Research Fund of the Hunan Provincial Science and Technology Department (No. 2022RC3022).

References

[1] H. ANTIL, AND E. OTÁROLA, *A FEM for an optimal control problem of fractional powers of elliptic operators,* SIAM J. Control. Optim., 53(6) (2015), pp. 3432–3456.

[2] C. BRÄNDLE, E. COLORADO, A. DE PABLO, AND U. SÁNCHEZ, *A concave convex elliptic problem involving the fractional Laplacian,* Proc. R. Soc. Edinb., 143(01) (2013), pp. 39–71.

[3] S. CHEN, AND J. SHEN, *An efficient and accurate numerical method for the spectral fractional Laplacian equation,* J. Sci. Comput., 82(1) (2020), pp. 1–25.

[4] L. CAFFARELLI, AND L. SILVESTRE, *An extension problem related to the fractional Laplacian,* Commun. Partial Differential Equations, 32 (2007), pp. 1245–1260.

[5] X. CABRÉ, AND J. TAN, *Positive solutions of nonlinear problems involving the square root of the Laplacian,* Adv. Math., 224 (2010), pp. 2052–2093.

[6] A. CAPELLA, J. DÁVILA, L. DUPAIGNE, AND Y. SIRE, *Regularity of radial extremal solutions for some non-local semilinear equations,* Commun. Partial Differential Equations, 36(8) (2011), pp. 1353–1384.

[7] S. CHEN, AND J. SHEN, *Enriched spectral methods and applications to problems with weakly singular solutions,* J. Sci. Comput., 77(3) (2018), pp. 1468–1489.

[8] M. D'ELIA, C. GLUSA, AND E. OTÁROLA, *A priori error estimates for the optimal control of the integral fractional Laplacian,* SIAM J. Control. Optim., 57(4) (2019), pp. 2775–2798.

[9] L. EVANS, Partial Differential Equations, vol. 19, 2nd edition, American Mathematical Society, Providence, 2010.

[10] B. GUO, L. WANG, AND Z. WANG, *Generalized Laguerre interpolation and pseudospectral method for unbounded domains,* SIAM J. Numer. Anal., 43(6) (2006), pp. 2567–2589.

[11] B. GUO, AND X. ZHANG, *Spectral method for differential equations of degenerate type on unbounded domains by using generalized Laguerre functions,* Appl. Numer. Math., 57(4) (2007), pp. 455–471.

[12] M. GUNZBURGER, AND J. WANG, *Error analysis of fully discrete finite element approximations to an optimal control problem governed by a time-fractional PDE,* SIAM J. Control Optim., 57(1) (2019), pp. 241–263.

[13] Y. GU, AND M. K. NG, *Deep Ritz Method for the Spectral Fractional Laplacian Equation Using the Caffarelli–Silvestre Extension,* SIAM J. Sci. Comput., 44(4) (2022), pp. A2018–A2036.

[14] B. JIN, B. LI, AND Z. ZHOU, *Pointwise-in-time error estimates for an optimal control problem with subdiffusion constraint,* IMA J.Numer. Anal., 40(1) (2020), pp. 377–404.

[15] S. LI, AND Z. ZHOU, *Fractional spectral collocation method for optimal control problem governed by space fractional diffusion equation,* Appl. Math Comput., 350 (2019), pp. 331–347.

[16] Y. LIN, AND C. XU, *Finite difference/spectral approximations for the time-fractional diffusion equation,* J. Comput. Phys., 225(2) (2007), pp. 1533–1552.

[17] R. NOCHETTO, E. OTÁROLA, AND A. SALGADO, *A PDE approach to fractional diffusion in general domains: A priori error analysis,* Found. Comput. Math., 15(3) (2015), pp. 733–791.

[18] R. NOCHETTO, E. OTÁROLA, AND A. SALGADO, *A PDE approach to space-time fractional parabolic problems,* SIAM J. Number. Anal., 54 (2016), pp. 848–873.

[19] E. OTÁROLA, *A piecewise linear FEM for an optimal control problem of fractional operators: error analysis on curved domains,* ESAIM: Math. Model. Numer. Anal., 51(4) (2017), pp. 1473–1500.

[20] E. OTÁROLA, AND A. J. SALGADO, *Sparse optimal control for fractional diffusion,* Comput. Meth. Appl. Mat., 18(1) (2018), pp. 95–110.

[21] J. SHEN, *Efficient spectral-Galerkin method I. direct solvers of second-and fourth-order equations using Legendre polynomials,* SIAM J. Sci. Comput., 15(6) (1994), pp. 1489–1505.

[22] J. SHEN, T. TANG, AND L. WANG, Spectral Methods: Algorithms, Analysis and Applications, Springer, Berlin, 2011.

[23] J. SHEN, AND C. T. SHENG, *An efficient space-time method for time fractional diffusion equation,* J. Sci. Comput., 81(2) (2019), pp. 1088–1110.

[24] F. WANG, Z. ZHANG, AND Z. ZHOU, *A spectral Galerkin approximation of optimal control problem governed by fractional advection diffusion reaction equations,* J. Comput. Appl. Math., 386 (2021), p. 113233.

[25] Y. YANG, Y. CHEN, Y. HUANG, AND H. WEI, *Spectral collocation method for the time-fractional diffusion-wave equation and convergence analysis,* Comput. Math. Appl., 73(6) (2017), pp. 1218–1232.

[26] Y. YANG, Y. HUANG, AND Y. ZHOU, *Numerical solutions for solving time fractional Fokker-Planck equations based on spectral collocation methods,* J. Comput. Appl. Math., 339 (2018), pp. 389–404.

[27] Y. YANG, Y. HUANG, AND Y. ZHOU, *Numerical simulation of time fractional Cable equations and convergence analysis,* Numer. Meth. Partial Differential Equations, 34 (2018), pp. 1556–1576.

[28] Y. YANG, J. ZHANG, AND H. LIU, ET AL., *An indirect convergent Jacobi spectral collocation method for fractional optimal control problems,* Math. Method. Appl. Sci., 44(4) (2021), pp. 2806–2824.

[29] Y. YANG, J. WANG, Y. CHEN, AND H. LIAO, *Compatible L^2 norm convergence of variable-step L1 scheme for the time-fractional MBE model with slope selection,* J. Comput. Phys., 467 (2022), p. 111467.

[30] X. YE, AND C. XU, *Spectral optimization methods for the time fractional diffusion inverse problem,* Numer. Math. Theor. Methods Appl., 6 (2013), pp. 499–519.

[31] X. YE, AND C. XU, *A spectral method for optimal control problem governed by the abnormal diffusion equation with integral constraint on the state,* Sci. Sin. Math., 46 (2016), pp. 1053–1070.

[32] L. ZHANG, AND Z. ZHOU, *Spectral Galerkin approximation of optimal control problem governed by Riesz fractional differential equation,* Appl. Numer. Math., 143 (2019), pp. 247–262.

[33] J. ZHANG, T. S. ALEROEV, Y. TANG, AND J. HUANG, *Numerical schemes for time-space fractional Vibration equations,* Adv. Appl. Math. Mech., 13 (2021), pp. 806–826.

[34] J. ZHANG, Y. CHEN, Y. HUANG, AND F. HUANG, *A posteriori error estimates for hp spectral element approximation of elliptic control problems with integral control and state constraints,* Adv. Appl. Math. Mech., 14 (2022), pp. 469–493.

[35] Z. ZHOU AND W. GONG, *Finite element approximation of optimal control problems governed by time fractional diffusion equation,* Comput. Math. Appl., 71 (2016), pp. 301–318.

[36] Z. ZHOU, AND Z. TAN, *Finite element approximation of optimal control problem governed by space fractional equation,* J. Sci. Comput., 78 (2019), pp. 1840–1861.