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A MULTISCALE PROJECTION METHOD FOR SOLVING NONLINEAR INTEGRAL EQUATIONS UNDER THE LIPSCHITZ CONDITION*

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Abstract

We propose a multiscale projection method for the numerical solution of the irtatively regularized Gauss-Newton method of nonlinear integral equations. An a posteriori rule is suggested to choose the stopping index of iteration and the rates of convergence are also derived under the Lipschitz condition. Numerical results are presented to demonstrate the efficiency and accuracy of the proposed method.

Mathematics subject classification: 65J20, 65R20.
Key words: Nonlinear integral equations, Multiscale Galerkin method, parameter choice strategy, Gauss-Newton method.

1. Introduction

The aim of this paper is to propose a projection method of solving nonlinear integral equations of the type

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subset \mathbb{X} \to \mathbb{X}$ is a nonlinear Fredholm integral operater between the Hilert space $\mathbb{X}$ and defined by

$$F(x)(s) := \int_0^1 k(s,t,x(t))dt, \quad s \in [0,1],$$

where the kernel k is a continuous function on $[0,1] \times [0,1] \times R$. Eq. (1.1) is a typical example of an ill-posed problem, then the regularization technique has to be taken into account to yield the stable approximation [7, 17, 18].

Several regularization methods in the existing literature have been used to solve nonlinear integral equations. The regularization method [3, 16, 17], a two-step iterative process [15] have been considered to some extent and important results have already been obtained, but either lack of error analysis, or lack of an a posteriori rule.

Due to the faster convergence, the iteratively regularized Gauss-Newton method has received extensive attention in recent years [1,2,8,9]. Assume that the sequence $\{\alpha_k\}$ satisfy the following conditions:

$$\alpha_k > 0, \quad 1 \leq \frac{\alpha_k}{\alpha_{k+1}} \leq r, \quad \lim_{k\to\infty} \alpha_k = 0 \tag{1.2}$$

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with a constant $r > 1$, this method constructs the iterates $\{x_k^\delta\}$ by the following recursive algorithm:

$$x_{k+1}^\delta = x_k^\delta - (\alpha_k \mathcal{I} + F'(x_k^\delta)^* F'(x_k^\delta))^{-1}(F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta) + \alpha_k(x_k^\delta - x_*)) \tag{1.3}$$

from the initial guess $x_0 = x^* \in D(F)$, where y^δ is the only available approximation of y satisfying

$$\|y^\delta - y\| \leq \delta \tag{1.4}$$

with a given noise level $\delta > 0$.

We notice that most of the available results on the iteratively regularized Gauss-Newton are implemented in infinite dimensional space. In practical applications, we are more interested in considering this methods in a finite-dimensional setting.

There are many papers on the projection method to solve linear ill-posed equations, and many results have been obtained [5,10,11,14]. Therefore, we wonder if we can use the projection method to solve the nonlinear ill-posed problem? Motivated by this idea, this article attempts to solve the nonlinear ill-posed problem by using the projection method.

In this paper we propose a projection method for the iteratively regularized Gauss-Newton method and investigate the influence of the projection method. We focus on error analysis and and try to assert what conditions are appropriate for the discussions.

Throughout the paper it is assumed that F has continuous Fréchet derivative over $D(F)$. Assume that Eq. (1.1) has a solution $x^\dagger$ such that

$$B_\rho(x^\dagger) := \{x \in \mathbb{X}_n : \|x - x^\dagger\| \leq \rho\} \subset D(F) \tag{1.5}$$

with a positive number $\rho > 10r\|x^* - x^\dagger\|$.

We next describe the multiscale Galerkin method for solving Eq. (1.3). We denote $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. We suppose that $\{\mathbb{X}_n, n \in \mathbb{N}_0\}$ is a sequence of finite dimensional subspaces of $\mathbb{X}$ satisfying [4]

$$\mathbb{X}_n \subset \mathbb{X}_{n+1}, n \in \mathbb{N}_0, \quad \overline{\bigcup_{n\in\mathbb{N}_0} \mathbb{X}_n} = \mathbb{X}.$$

For each $i \in \mathbb{N}$, let $\mathbb{W}_i$ be the orthogonal complement of $\mathbb{X}_{i-1}$ in $\mathbb{X}_i$. For a fixed $n \in \mathbb{N}$, we have the decomposition

$$\mathbb{X}_n = \mathbb{X}_0 \oplus^\perp \mathbb{W}_1 \oplus^\perp \cdots \oplus^\perp \mathbb{W}_n.$$

We assume that $\mathbb{W}_i$ has a basis $\{w_{ij}, j \in \mathbb{Z}_{w(i)}\}$. This means that $\mathbb{X}_n = \operatorname{span}\{w_{ij} : (i,j) \in \mathbb{U}_n\}$, where $\mathbb{U}_n := \{(i,j) : j \in \mathbb{Z}_{w(i)}, i \in \mathbb{Z}_{n+1}\}$.

We now formulate the Galerkin method for solving Eq. (1.3). To this end, for each $n \in N_0$, we let $\mathcal{P}_n$ denote the orthogonal projection from $\mathbb{X}$ onto $\mathbb{X}_n$. The traditional Galerkin method for solving Eq. (1.3) is to find $x_{k,n}^\delta \in \mathbb{X}_n$ such that

$$\begin{cases} x_{0,n}^\delta = \mathcal{P}_n x^*, \\ x_{k+1,n}^\delta = x_{k,n}^\delta + (\alpha_k \mathcal{I} + \mathcal{A}_{k,n}^* \mathcal{A}_{k,n})^{-1}[\mathcal{A}_{k,n}^*(y^\delta - F(x_{k,n}^\delta)) + \alpha_k(\mathcal{P}_n x^* - x_{k,n}^\delta)], \end{cases} \tag{1.6}$$

where $\mathcal{A}_{k,n} := \mathcal{P}_n F'(x_{k,n}^\delta)\mathcal{P}_n$ and $\mathcal{A}_{k,n}^* := \mathcal{P}_n F'(x_{k,n}^\delta)^*\mathcal{P}_n$.

To write (1.6) in its equivalent matrix form, we make use of the multiscale basis functions. We write the solution $x_{k,n}^\delta \in \mathbb{X}_n$ as [14]

$$x_{k,n}^\delta = \sum_{(i,j)\in\mathbb{U}_n} c_{ij}^k w_{ij} \in \mathbb{X}_n.$$

By introducing vectors

$$\mathbf{C}_n^k := [c_{ij}^k : (i,j) \in \mathbb{U}_n]^\top, \quad \mathbf{f}_n^\delta := [(w_{lm}, y^\delta - F(x_{k,n}^\delta)), (k,l) \in \mathbb{U}_n]^\top,$$

and matrices

$$\mathbf{E}_n := [(w_{lm}, w_{ij}) : (l,m), (i,j) \in \mathbb{U}_n],$$

$$\mathbf{A}_\mathbf{n}^\mathbf{k} := [(w_{lm}, F'(x_{k,n}^\delta) w_{ij}) : (i,j), (l,m) \in \mathbb{U}_n].$$

Some calculation shows that (1.6) is equivalent to the following system of equations for determining $\mathbf{C}_n^k$ [14]

$$\begin{cases} (\alpha_k \mathbf{E}_n + (\mathbf{A}_\mathbf{n}^\mathbf{k})^\top \mathbf{E}_n^{-1} \mathbf{A}_\mathbf{n}^\mathbf{k})(\mathbf{C}_n^{k+1} - \mathbf{C}_n^k) = (\mathbf{A}_\mathbf{n}^\mathbf{k})^\top \mathbf{E}_n^{-1} \mathbf{f}_n^\delta + \alpha_k \mathbf{E}_n (\mathbf{C}_n^0 - \mathbf{C}_n^k), \\ \mathbf{C}_n^0 = \mathbf{E}_n^{-1} [(w_{lm}, x^*) : (l,m) \in \mathbb{U}_n]^\top, \quad k = 0, 1, \dots. \end{cases}$$

To analyze the convergence rate of the approximate solution $x_{k,n}^\delta$, we require the following assumptions [9]:

(i) There exists a sequence of positive real numbers $\{\gamma_n : n \in \mathbb{N}_0\}$ satisfying $\lim\limits_{n\to\infty} \gamma_n = 0$ such that

$$\|(\mathcal{I} - \mathcal{P}_n) F'(x)^*\| \le \gamma_n, \quad x \in B_\rho(x^\dagger), \tag{1.7}$$

and

$$\|(\mathcal{I} - \mathcal{P}_n) x^*\| \le \gamma_n. \tag{1.8}$$

(ii) F' satisfies the Lipschitz condition, i.e. there exists a constant L such that

$$\|F'(x) - F'(z)\| \le L \|x - z\| \quad \text{for all} \quad x, z \in B_\rho(x^\dagger). \tag{1.9}$$

(iii) The nonlinear operator F is properly scaled, i.e.

$$\|F'(x)\| \le \alpha_0^{\frac{1}{2}} \quad \text{for all} \quad x \in B_\rho(x^\dagger). \tag{1.10}$$

(iv) There is a $v \in N(F'(x^\dagger)^*)^\perp$ and a positive constant ϱ such that

$$x^* - x^\dagger = F'(x^\dagger)^* v, \quad \|v\| \le \varrho. \tag{1.11}$$

The sequence $x_{k,n} \in \mathbb{X}_n$ appearing in Lemma 2.4 is defined by the discreted iteratively regularized Gauss-Newton method (1.6) corresponding to the noise-free case, that is $x_{k,n} \in \mathbb{X}_n$ is defined successively by

$$\begin{cases} x_{0,n} = \mathcal{P}_n x^*, \\ x_{k+1,n} = x_{k,n} + (\alpha_k \mathcal{I} + \mathcal{T}_{k,n}^* \mathcal{T}_{k,n})^{-1} [\mathcal{T}_{k,n}^* (y - F(x_{k,n})) + \alpha_k (\mathcal{P}_n x^* - x_{k,n})], \end{cases}$$

where $\mathcal{T}_{k,n} := \mathcal{P}_n F'(x_{k,n}) \mathcal{P}_n$ and $\mathcal{T}_{k,n}^* := \mathcal{P}_n F'(x_{k,n})^* \mathcal{P}_n$.

Due to the practical applications, the stopping index of iteration should be designated in an a posteriori manner. To this end, according to the idea used in [8], we propose a modified a posteriori rule in the following form to choose the stopping index k_δ as the first integer satisfing.

Rule 1.1. *Let $\tau > 3$, $d > 0$ and $0 < c_0 \leq \frac{\tau-2}{dr^{\frac{1}{2}}}$ be a large number.*
(i) Choose n to be the first integer such that

$$\gamma_n \leq \min\left\{\sqrt{\frac{\delta}{2(\varrho+1)}},\ \frac{\rho}{10r(\frac{9}{2}\varrho+2)},\ \frac{\sqrt{2c_0\delta}}{2}L, \frac{\sqrt{2c_0\delta}}{4\sqrt{\varrho}},\ \delta\right\}. \tag{1.12}$$

(ii) Choose k_δ to be the first integer such that

$$\alpha_{k_\delta}(\mathcal{P}_n(F(x^\delta_{k_\delta,n}) - y^\delta), (\alpha_{k_\delta}\mathcal{I} + \mathcal{A}_{k_\delta,n}\mathcal{A}^*_{k_\delta,n})^{-1}\mathcal{P}_n(F(x^\delta_{k_\delta,n}) - y^\delta)) \leq \tau^2\delta^2. \tag{1.13}$$

In the following C is always used to denote a generic constant independent of δ, n, we will also use the convention $A \preceq B$ to mean that $A \leq CB$ for some generic constant C.

2. Auxiliary Estimates

Throughout this paper, we will use the notations $\mathcal{A} := F'(x^\dagger)$, $\mathcal{B} := \mathcal{P}_nF'(\mathcal{P}_nx^\dagger)\mathcal{P}_n$, and

$$\tau_k := \|\alpha_k(\alpha_k\mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}(x^* - x^\dagger)\|,$$

$$\vartheta_{k,n} := \|\alpha_k(\alpha_k\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1}\mathcal{P}_n(x^* - x^\dagger)\|.$$

We will also use the notations

$$e_{k,n} := x_{k,n} - \mathcal{P}_nx^\dagger, \quad e^\delta_{k,n} := x^\delta_{k,n} - \mathcal{P}_nx^\dagger.$$

The following elementary result will be used frequently.

Lemma 2.1 ([9]). *Let $\{p_k\}$ be a sequence of positive numbers satisfying $\frac{p_k}{p_{k+1}} \leq p$ with a constant $p > 0$. Suppose that the sequence η_k has the property $\eta_{k+1} \leq p_k + \varepsilon\eta_k$ for all k. If $\varepsilon p < 1$ and $\eta_0 \leq \frac{p}{1-\varepsilon p}p_0$, then $\eta_k \leq \frac{p}{1-\varepsilon p}p_k$ for all k.*

First, we need to give an upper bound on k_δ, to this end, we set $\epsilon := \max\{4\gamma_n^2, \frac{\gamma_n^2}{L^2\varrho^2}, \frac{c_0\delta}{2\varrho}\}$ and define k_* as the integer satisfying

$$\alpha_{k_*} \leq \epsilon < \alpha_k, \quad 0 \leq k < k_*. \tag{2.1}$$

It follows from (1.12) that $\epsilon = \frac{c_0\delta}{2\varrho}$. In order to get the main results of this article, we need a series of auxiliary results. The first one is the following.

Lemma 2.2. *Let conditions* (1.5) *and* (1.7)-(1.11) *hold, let the integers n and k_δ be determined by Rule* 1.1 *with $\tau > 3$, then $k_\delta \leq k_*$, and for all integers $0 \leq k \leq k_*$ there hold $x^\delta_{k,n} \in B_\rho(x^\dagger)$, and*

$$\|e^\delta_{k,n}\| \leq r^{\frac{1}{2}}d_1\varrho\alpha_k^{\frac{1}{2}}, \tag{2.2}$$

where $d_1 = \frac{9}{2} + \frac{2}{c_0} + \frac{7}{4}L(\varrho+1)$.

Proof. We first show that $x^\delta_{k,n} \in B_\rho(x^\dagger)$ for all $0 \leq k \leq k_*$. It is clear from (1.5) and (1.12) that this is true for $k = 0$. Now for any fixed integer $0 < l \leq k_*$, we assume that $x^\delta_{k,n} \in B_\rho(x^\dagger)$ for all $0 \leq k < l$ and we are going to show $x^\delta_{l,n} \in B_\rho(x^\dagger)$. To this end, from the definition of $\{x^\delta_{k,n}\}$ it follows that

$$e^\delta_{k+1,n} = \beta^\delta(k,n) + S^\delta_1(k,n) - S^\delta_2(k,n), \tag{2.3}$$

where

$$\beta^\delta(k,n) := \alpha_k(\alpha_k \mathcal{I} + \mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}\mathcal{P}_n(x^* - x^\dagger),$$
$$S_1^\delta(k,n) := (\alpha_k \mathcal{I} + \mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}\mathcal{A}_{k,n}^*(y^\delta - F(\mathcal{P}_n x^\dagger)),$$
$$S_2^\delta(k,n) := (\alpha_k \mathcal{I} + \mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}\mathcal{A}_{k,n}^* u_{k,n}^\delta$$

with $u_{k,n}^\delta := F(x_{k,n}^\delta) - F(\mathcal{P}_n x^\dagger) - F'(x_{k,n}^\delta)e_{k,n}^\delta$. Using conditions (1.7) and (1.9) it is easy to derive

$$\|S_1^\delta(k,n)\| \le \frac{1}{2\sqrt{\alpha_k}}(\delta + \gamma_n \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|), \tag{2.4}$$

$$\|S_2^\delta(k,n)\| \le \frac{1}{2\sqrt{\alpha_k}} \int_0^1 \|F'(\mathcal{P}_n x^\dagger + t e_{k,n}^\delta) - F'(x_{k,n}^\delta)\|\|e_{k,n}^\delta\| dt \le \frac{L}{4\sqrt{\alpha_k}}\|e_{k,n}^\delta\|^2. \tag{2.5}$$

Define $\beta^\delta(k,n) = t_1 - t_2$, where

$$t_1 = -\alpha_k(\alpha_k \mathcal{I} + \mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}(\mathcal{I} - \mathcal{P}_n)(x^* - x^\dagger),$$

$$t_2 = -\alpha_k(\alpha_k \mathcal{I} + \mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}(x^* - x^\dagger).$$

Let us now estimate t_1 and t_2. It is easy to prove

$$\|t_1\| \le \|(\mathcal{I} - \mathcal{P}_n)(x^* - x^\dagger)\|. \tag{2.6}$$

We can write $t_2 = J_1 + J_2 + J_3$ with

$$J_1 = \alpha_k(\alpha_k \mathcal{I} + \mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}\mathcal{A}_{k,n}^*(\mathcal{A} - \mathcal{A}_{k,n})(\alpha_k \mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}(x^* - x^\dagger),$$

$$J_2 = \alpha_k(\alpha_k \mathcal{I} + \mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}(\mathcal{A}^* - \mathcal{A}_{k,n}^*)\mathcal{A}(\alpha_k \mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}(x^* - x^\dagger),$$

$$J_3 = \alpha_k(\alpha_k \mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}(x^* - x^\dagger).$$

Recall that $x^* - x^\dagger = \mathcal{A}^* v, \|v\| \le \varrho$, and

$$\begin{aligned}\|\mathcal{A} - \mathcal{A}_{k,n}\| &\le \|F'(x^\dagger) - F'(x_{k,n}^\delta)\| + \|F'(x_{k,n}^\delta) - \mathcal{P}_n F'(x_{k,n}^\delta)\mathcal{P}_n\| \\ &\le L(\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \|e_{k,n}^\delta\|) + 2\gamma_n,\end{aligned} \tag{2.7}$$

we have

$$\begin{aligned}\|t_2\| &\le \frac{\alpha_k}{2\sqrt{\alpha_k}}\frac{\varrho}{2\sqrt{\alpha_k}}\|\mathcal{A} - \mathcal{A}_{k,n}\| + \|\mathcal{A} - \mathcal{A}_{k,n}\|\varrho + \tau_k \\ &= \frac{5}{4}\|\mathcal{A} - \mathcal{A}_{k,n}\|\varrho + \tau_k \\ &\le \tau_k + \frac{5}{4}L\varrho\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \frac{5}{2}\varrho\gamma_n + \frac{5}{4}L\varrho\|e_{k,n}^\delta\|.\end{aligned} \tag{2.8}$$

By using (2.6) and (2.8) we have

$$\|\beta^\delta(k,n)\| \le \tau_k + \frac{5}{4}L\varrho\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \|(\mathcal{I} - \mathcal{P}_n)(x^* - x^\dagger)\| + \frac{5}{2}\varrho\gamma_n + \frac{5}{4}L\varrho\|e_{k,n}^\delta\|. \tag{2.9}$$

Combining the estimates (2.4), (2.5) and (2.9) we have

$$\begin{aligned}\|e_{k+1,n}^\delta\| \le \tau_k + \|(\mathcal{I} - \mathcal{P}_n)(x^* - x^\dagger)\| + \left(\frac{5}{4}L\varrho + \frac{\gamma_n}{2\sqrt{\alpha_k}}\right)\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \frac{5}{2}\varrho\gamma_n \\ + \frac{5}{4}L\varrho\|e_{k,n}^\delta\| + \frac{L}{4\sqrt{\alpha_k}}\|e_{k,n}^\delta\|^2 + \frac{\delta}{2\sqrt{\alpha_k}}.\end{aligned} \tag{2.10}$$

Note that $\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| \le \|(\mathcal{I}-\mathcal{P}_n)(x^*-x^\dagger)\| + \|(\mathcal{I}-\mathcal{P}_n)x^*\| \le (\varrho+1)\gamma_n$, it follows (2.1), (1.12) and (2.10) that

$$\frac{\gamma_n}{\sqrt{\alpha_k}} \le \min\Big\{\frac{1}{2}, L\varrho\Big\}, \quad \frac{\delta}{2\sqrt{\alpha_k}} \le \frac{\varrho}{c_0}\sqrt{\alpha_k}, \tag{2.11}$$

and

$$\|e^\delta_{k+1,n}\| \le \left(\frac{1}{2}+\frac{1}{c_0}\right)\varrho\alpha_k^{\frac{1}{2}} + \frac{7}{4}(L\varrho+L+2)\varrho\gamma_n + \frac{5}{4}L\varrho\|e^\delta_{k,n}\| + \frac{L}{4\sqrt{\alpha_k}}\|e^\delta_{k,n}\|^2.$$

This and (1.2) imply

$$\begin{aligned}\left\|\frac{e^\delta_{k+1,n}}{\sqrt{\alpha_{k+1}}}\right\| &\le r^{\frac{1}{2}}\left[\left(\frac{1}{2}+\frac{1}{c_0}\right)\varrho + \frac{7}{4}(L\varrho+L+2)\varrho\frac{\gamma_n}{\sqrt{\alpha_k}} + \frac{5}{4}L\varrho\frac{\|e^\delta_{k,n}\|}{\sqrt{\alpha_k}} + \frac{L}{4}\left(\frac{\|e^\delta_{k,n}\|}{\sqrt{\alpha_k}}\right)^2\right]\\ &\le r^{\frac{1}{2}}\left[\frac{1}{2}d_1\varrho + \frac{5}{4}L\varrho\frac{\|e^\delta_{k,n}\|}{\sqrt{\alpha_k}} + \frac{L}{4}\left(\frac{\|e^\delta_{k,n}\|}{\sqrt{\alpha_k}}\right)^2\right].\end{aligned}$$

Note that (1.10), (1.11) and $e^\delta_{0,n} = \mathcal{P}_n(x^*-x^\dagger)$ imply $\|e^\delta_{0,n}\|/\sqrt{\alpha_0} \le \varrho$. Thus, by induction, we can show that if $L\varrho$ is so small that

$$(5+r^{\frac{1}{2}}d_1)L\varrho < \frac{1}{r}, \tag{2.12}$$

then

$$\left\|\frac{e^\delta_{k,n}}{\sqrt{\alpha_k}}\right\| \le r^{\frac{1}{2}}d_1\varrho, \quad \text{for} \quad 0 \le k \le l. \tag{2.13}$$

Combining this with (2.10) and (2.12), noting that $\frac{\delta}{2\sqrt{\alpha_k}} < \frac{\sqrt{2\varrho}}{2\sqrt{c_0}}\delta^{\frac{1}{2}}$, we have for $0 \le k < l$ that

$$\begin{aligned}\|e^\delta_{k+1,n}\| &\le \tau_k + \|(\mathcal{I}-\mathcal{P}_n)(x^*-x^\dagger)\| + \left(\frac{5}{4}L\varrho+1\right)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{5}{2}\varrho\gamma_n\\ &\quad + \frac{5}{4}L\varrho\|e^\delta_{k,n}\| + \frac{L}{4}r^{\frac{1}{2}}d\varrho\|e^\delta_{k,n}\| + \frac{\sqrt{2\varrho}}{2\sqrt{c_0}}\delta^{\frac{1}{2}}\\ &= w^\delta_{k,n} + \frac{L\varrho}{4}(5+r^{\frac{1}{2}}d)\|e^\delta_{k,n}\|,\end{aligned}$$

where

$$w^\delta_{k,n} := \tau_k + \|(\mathcal{I}-\mathcal{P}_n)(x^*-x^\dagger)\| + \left(\frac{5}{4}L\varrho+1\right)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{5}{2}\varrho\gamma_n + \frac{\sqrt{2\varrho}}{2\sqrt{c_0}}\delta^{\frac{1}{2}}.$$

Using the condition (2.12), we have for $0 \le k \le l$ that

$$\|e^\delta_{k+1,n}\| \le w^\delta_{k,n} + \frac{1}{4r}\|e^\delta_{k,n}\|.$$

It is easy to prove

$$\frac{w^\delta_{k,n}}{w^\delta_{k+1,n}} \le r.$$

We may apply Lemma 2.1 to conclude

$$\|e^\delta_{k,n}\| \le \frac{4}{3}r\left(\tau_k + \|(\mathcal{I}-\mathcal{P}_n)(x^*-x^\dagger)\| + \left(\frac{5}{4}L\varrho+1\right)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{5}{2}\varrho\gamma_n + \frac{\sqrt{2\varrho}}{2\sqrt{c_0}}\delta^{\frac{1}{2}}\right). \tag{2.14}$$

Since

$$\tau_k \leq \|x^\dagger - x^*\| \leq \frac{\rho}{10r},$$

and if we futher assume that

$$\frac{5}{4}L\varrho(\varrho+1) < 1,$$

therefore by using (1.7) and (1.12), it is not difficult to show that for all $0 \leq k \leq l$ there holds

$$\begin{aligned}
\|e_{k,n}^\delta\| &\leq \frac{4}{3}r\Bigg(\tau_k + \left(\frac{5}{4}L\varrho+2\right)\|(\mathcal{I}-\mathcal{P}_n)(x^*-x^\dagger)\| + \left(\frac{5}{4}L\varrho+1\right)\|(\mathcal{I}-\mathcal{P}_n)x^*\| \\
&\qquad + \frac{5}{2}\varrho\gamma_n + \frac{\sqrt{2\varrho}}{2\sqrt{c_0}}\delta^{\frac{1}{2}}\Bigg) \\
&\leq \frac{4}{3}r\left(\frac{\rho}{10r} + \left(\frac{5}{4}L\varrho(\varrho+1) + \frac{9}{2}\varrho + 1\right)\gamma_n + \frac{\sqrt{2\varrho}}{2\sqrt{c_0}}\delta^{\frac{1}{2}}\right) \\
&\leq \frac{4}{3}r\left(\frac{\rho}{10r} + \left(\frac{9}{2}\varrho+2\right)\gamma_n + \frac{\sqrt{2\varrho}}{2\sqrt{c_0}}\delta^{\frac{1}{2}}\right) \\
&\leq \frac{4}{3}r\left(\frac{\rho}{10r} + \frac{2\rho}{10r}\right) = \frac{2}{5}\rho,
\end{aligned} \tag{2.15}$$

provided δ is sufficiently small. Using (1.7) and (2.15), we further conclude $x_{k,n}^\delta \in B_\rho(x^\dagger)$ for all $0 \leq k \leq l$ since

$$\begin{aligned}
\|x_{k,n}^\delta - x^\dagger\| &\leq \|e_{k,n}^\delta\| + \|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| \\
&\leq \frac{2}{5}\rho + \|(\mathcal{I}-\mathcal{P}_n)(x^*-x^\dagger)\| + \|(\mathcal{I}-\mathcal{P}_n)x^*\| \\
&\leq \frac{2}{5}\rho + \gamma_n\varrho + \gamma_n \\
&\leq \frac{2}{5}\rho + \frac{\rho}{10} = \frac{\rho}{2}.
\end{aligned}$$

Using mathematical induction, we thus obtain $x_{k,n}^\delta \in B_\rho(x^\dagger)$ for all $0 \leq k \leq k_*$.

In order to prove $k_\delta \leq k_*$, we use (1.12),(2.1),(2.5) and (2.13) to obtain

$$\begin{aligned}
&\|\alpha_{k_*}^{\frac{1}{2}}(\alpha_{k_*}\mathcal{I} + \mathcal{A}_{k_*,n}\mathcal{A}_{k_*,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k_*,n}^\delta) - y^\delta)\| \\
&\leq \delta + \|F(\mathcal{P}_n x^\dagger) - F(x^\dagger)\| + \|F(x_{k_*,n}^\delta) - F(\mathcal{P}_n x^\dagger) - F'(x_{k_*,n}^\delta)e_{k_*,n}^\delta\| \\
&\quad + \|\alpha_{k_*}^{\frac{1}{2}}(\alpha_{k_*}\mathcal{I} + \mathcal{A}_{k_*,n}\mathcal{A}_{k_*,n}^*)^{-\frac{1}{2}}\mathcal{P}_n F'(x_{k_*,n}^\delta)\mathcal{P}_n e_{k_*,n}^\delta\| \\
&\leq \delta + \gamma_n\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{L}{2}\|e_{k_*,n}^\delta\|^2 + \alpha_{k_*}^{\frac{1}{2}}\|e_{k_*,n}^\delta\| \\
&\leq \delta + \gamma_n + d_1\varrho r^{\frac{1}{2}}\alpha_{k_*} + \frac{L}{2}rd_1^2\varrho^2\alpha_{k_*} \\
&\leq 2\delta + \left(\frac{d_1}{2}r^{\frac{1}{2}} + \frac{d_1^2 r}{4}L\varrho\right)c_0\delta.
\end{aligned}$$

Recall that $\tau > 3$ and $c_0 < \frac{\tau-2}{d_1 r^{\frac{1}{2}}}$. If $L\varrho$ is so small that

$$r^{\frac{1}{2}}d_1 L\varrho < 2,$$

then we have

$$\|\alpha_{k_*}^{\frac{1}{2}}(\alpha_{k_*}\mathcal{I} + \mathcal{A}_{k_*,n}\mathcal{A}_{k_*,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k_*,n}^\delta) - y^\delta)\| \leq \tau\delta.$$

By the difinition of k_δ, we thus conclude that $k_\delta \leq k_*$. $\square$

The second auxiliary result is as follows.

Lemma 2.3. *Let conditions* (1.5) *and* (1.7)-(1.11) *hold and* $x^* - x^\dagger = F'(x^\dagger)^* v$ *for some* $v \in N(F'(x^\dagger)^*)^\perp$. *Let* k_δ *be the integer determinated by Rule* 1.1. *If* $L\varrho \le \varepsilon$, *then*

$$\|e^\delta_{k_\delta,n}\| \le C\delta^{\frac{1}{2}},$$

where ε *and* C *are positive constants depending only on* r *and* τ.

Proof. Recall that in the proof of Lemma 2.2 we have obtained the following two estimates:

$$\|e^\delta_{k,n}\| \le \frac{4}{3} r\tau_k + c_1\delta^{\frac{1}{2}} + c_2\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|, \tag{2.16}$$

$$\|e^\delta_{k+1,n}\| \le \tau_k + \tilde{c}_1\delta^{\frac{1}{2}} + \tilde{c}_2\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{1}{4r}\|e^\delta_{k,n}\| \tag{2.17}$$

for all $0 \le k \le k_*$, where

$$c_1 = \frac{7}{3} r\varrho\sqrt{2c_0} + \frac{2r\sqrt{2\varrho}}{3\sqrt{c_0}}, \quad c_2 = \frac{r}{3}(5L\varrho + 1), \quad \tilde{c}_1 = \frac{\sqrt{2\varrho}}{2\sqrt{c_0}} + \frac{7}{2}\varrho, \quad \tilde{c}_2 = \frac{15}{4}L\varrho + 1.$$

Now we set

$$\tau^\delta_{k,n} = \|\alpha_k(\alpha_k\mathcal{I} + \mathcal{A}^*_{k,n}\mathcal{A}_{k,n})^{-1}\mathcal{P}_n e_0\|.$$

Then it follows from $e_0 := x^* - x^\dagger = F'(x^\dagger)^* v$ and (2.7) that

$$\begin{aligned}
|\tau_k - \tau^\delta_{k,n}| &\le \|\alpha_k[(\alpha_k\mathcal{I} + \mathcal{A}^*_{k,n}\mathcal{A}_{k,n})^{-1} - (\alpha_k\mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}]e_0\| \\
&\quad + \|\alpha_k(\alpha_k\mathcal{I} + \mathcal{A}^*_{k,n}\mathcal{A}_{k,n})^{-1}(\mathcal{P}_n - \mathcal{I})e_0\| \\
&\le \|\alpha_k(\alpha_k\mathcal{I} + \mathcal{A}^*_{k,n}\mathcal{A}_{k,n})^{-1}\mathcal{A}^*_{k,n}(\mathcal{A} - \mathcal{A}_{k,n})(\alpha_k\mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}\mathcal{A}^* v\| \\
&\quad + \|\alpha_k(\alpha_k\mathcal{I} + \mathcal{A}^*_{k,n}\mathcal{A}_{k,n})^{-1}(\mathcal{A}^* - \mathcal{A}^*_{k,n})\mathcal{A}(\alpha_k\mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}\mathcal{A}^* v\| + \gamma_n\|v\| \\
&\le \gamma_n\varrho + \frac{5}{4}L\varrho\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{5}{4}L\varrho\|e^\delta_{k,n}\|.
\end{aligned}$$

By assuming that $10Lr\varrho < 1$, together with (2.16) and (2.17) implies

$$\|e^\delta_{k,n}\| \le \frac{8}{5} r\tau^\delta_{k,n} + C\delta^{\frac{1}{2}} + C\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|, \tag{2.18}$$

$$\|e^\delta_{k+1,n}\| \le \frac{8}{5}\tau^\delta_{k,n} + C\delta^{\frac{1}{2}} + C\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|. \tag{2.19}$$

We need to estimate $\tau^\delta_{k,n}$. We first have

$$\begin{aligned}
(\tau^\delta_{k,n})^2 &= (\alpha_k(\alpha_k\mathcal{I} + \mathcal{A}^*_{k,n}\mathcal{A}_{k,n})^{-1}\mathcal{P}_n e_0, \alpha_k(\alpha_k\mathcal{I} + \mathcal{A}^*_{k,n}\mathcal{A}_{k,n})^{-1}\mathcal{P}_n[\mathcal{A}^*_{k,n} \\
&\qquad + (F'(x^\dagger)^* - \mathcal{A}^*_{k,n})]v) \\
&= (\alpha_k^{\frac{3}{2}}(\alpha_k\mathcal{I} + \mathcal{A}_{k,n}\mathcal{A}^*_{k,n})^{-\frac{3}{2}}\mathcal{A}_{k,n}e_0, \alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I} + \mathcal{A}_{k,n}\mathcal{A}^*_{k,n})^{-\frac{1}{2}}v) \\
&\qquad + (\alpha_k(\alpha_k\mathcal{I} + \mathcal{A}^*_{k,n}\mathcal{A}_{k,n})^{-1}\mathcal{P}_n e_0, \alpha_k(\alpha_k\mathcal{I} + \mathcal{A}^*_{k,n}\mathcal{A}_{k,n})^{-1}\mathcal{P}_n(F'(x^\dagger)^* - \mathcal{A}^*_{k,n})v) \\
&\le \gamma^\delta_{k,n}\|v\| + \tau^\delta_{k,n}\|F'(x^\dagger)^* - \mathcal{A}^*_{k,n}\|\|v\| \\
&\le \gamma^\delta_{k,n}\|v\| + \tau^\delta_{k,n}(2\gamma_n + L\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + L\|e^\delta_{k,n}\|)\|v\|,
\end{aligned}$$

where

$$\gamma^\delta_{k,n} = \|\alpha_k^{\frac{3}{2}}(\alpha_k\mathcal{I} + \mathcal{A}_{k,n}\mathcal{A}^*_{k,n})^{-\frac{3}{2}}\mathcal{A}_{k,n}e_0\|.$$

Therefore,

$$\tau^\delta_{k,n} \le \sqrt{\gamma^\delta_{k,n}}\varrho^{\frac{1}{2}} + (2\gamma_n + L\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + L\|e^\delta_{k,n}\|)\varrho. \tag{2.20}$$

In order to estimates $\gamma_{k,n}^\delta$, we observe that (2.3) implies

$$\begin{aligned}
&\alpha_k^{\frac{3}{2}}(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{3}{2}}\mathcal{A}_{k,n}e_0\\
&=\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}\mathcal{A}_{k,n}e_{k+1,n}^\delta\\
&\quad-\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{3}{2}}\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*(y^\delta-F(\mathcal{P}_nx^\dagger))\\
&\quad+\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{3}{2}}\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*u_{k,n}^\delta\\
&\quad+\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{3}{2}}\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*(F'(x_{k,n}^\delta)-\mathcal{A}_{k,n})e_{k,n}^\delta.
\end{aligned}$$

Thus

$$\begin{aligned}
\gamma_{k,n}^\delta &\le \|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}\mathcal{A}_{k,n}e_{k+1,n}^\delta\|+\|u_{k,n}^\delta\|\\
&\quad+\|y^\delta-F(\mathcal{P}_nx^\dagger)\|+\|\mathcal{P}_nF'(x_{k,n}^\delta)-\mathcal{A}_{k,n}\|\|e_{k,n}^\delta\|\\
&\le \|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k+1,n}^\delta)-y^\delta)\|\\
&\quad+\|\mathcal{P}_n(F(x_{k+1,n}^\delta)-F(\mathcal{P}_nx^\dagger)-F'(x_{k+1,n}^\delta)e_{k+1,n}^\delta)\|\\
&\quad+\|\mathcal{P}_n(y^\delta-F(\mathcal{P}_nx^\dagger))\|+\|(\mathcal{P}_nF'(x_{k+1,n}^\delta)-\mathcal{A}_{k,n})e_{k+1,n}^\delta\|\\
&\quad+(\delta+\gamma_n\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|)+\frac{L}{2}\|e_{k,n}^\delta\|^2+\gamma_n\|e_{k,n}^\delta\|\\
&\le \|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k+1,n}^\delta)-y^\delta))\|\\
&\quad+\frac{L}{2}\|e_{k+1,n}^\delta\|^2+\frac{L}{2}\|e_{k,n}^\delta\|^2+\gamma_n\|e_{k,n}^\delta\|+2(\delta+\gamma_n\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|)+\varrho\gamma_n\\
&\quad+L\|x_{k+1,n}^\delta-x_{k,n}^\delta\|\|e_{k+1,n}^\delta\|\\
&\le \|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k+1,n}^\delta)-y^\delta))\|\\
&\quad+2L(\|e_{k,n}^\delta\|^2+\|e_{k+1,n}^\delta\|^2)+2\delta+\left(\frac{3}{2}\varrho+\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|\right)\gamma_n.
\end{aligned}$$

Now we further assume that

$$L\varrho d_1r^{\frac{1}{2}}<\frac{1}{3}. \tag{2.21}$$

Then by (2.13) we have

$$L\|x_{k+1,n}^\delta-x_{k,n}^\delta\|\le L(\|e_{k,n}^\delta\|+\|e_{k+1,n}^\delta\|)\le 2Lr^{\frac{1}{2}}d_1\varrho\alpha_k^{\frac{1}{2}}\le\frac{2}{3}\alpha_k^{\frac{1}{2}}.$$

Thus we can conclude, by using ([19], proposition 3.4) that there hold

$$\begin{aligned}
&\|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k+1,n}^\delta)-y^\delta))\|\\
&\preceq\|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I}+\mathcal{A}_{k+1,n}\mathcal{A}_{k+1,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k+1,n}^\delta)-y^\delta))\|\\
&\preceq\|\alpha_{k+1}^{\frac{1}{2}}(\alpha_{k+1}\mathcal{I}+\mathcal{A}_{k+1,n}\mathcal{A}_{k+1,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k+1,n}^\delta)-y^\delta))\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma_{k,n}^\delta &\le C\|\alpha_{k+1}^{\frac{1}{2}}(\alpha_{k+1}\mathcal{I}+\mathcal{A}_{k+1,n}\mathcal{A}_{k+1,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k+1,n}^\delta)-y^\delta))\|\\
&\quad+2L(\|e_{k,n}^\delta\|^2+\|e_{k+1,n}^\delta\|^2)+C\delta.
\end{aligned}$$

This together with (2.20) gives

$$\begin{aligned}
\tau_{k,n}^\delta &\le C\|\alpha_{k+1}^{\frac{1}{2}}(\alpha_{k+1}\mathcal{I}+\mathcal{A}_{k+1,n}\mathcal{A}_{k+1,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k+1,n}^\delta)-y^\delta))\|^{\frac{1}{2}}\varrho^{\frac{1}{2}}\\
&\quad+\sqrt{2L\varrho}(\|e_{k,n}^\delta\|+\|e_{k+1,n}^\delta\|)+C\delta^{\frac{1}{2}}\varrho^{\frac{1}{2}}+2\varrho\gamma_n+L\varrho\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|.
\end{aligned}$$

Combining this with (2.18) and (2.19) yields

$$\tau_{k,n}^{\delta} \le C\|\alpha_{k+1}^{\frac{1}{2}}(\alpha_{k+1}\mathcal{I}+\mathcal{A}_{k+1,n}\mathcal{A}_{k+1,n}^{*})^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k+1,n}^{\delta})-y^{\delta}))\|^{\frac{1}{2}}\varrho^{\frac{1}{2}} \\ +C\varrho^{\frac{1}{2}}\delta^{\frac{1}{2}}+CL\varrho\|(\mathcal{I}-\mathcal{P}_n)x^{\dagger}\|+\frac{16}{5}(1+r)\sqrt{L\varrho}\tau_{k,n}^{\delta}.$$

If we assume further that

$$\frac{32}{5}(1+r)\sqrt{L\varrho}\le 1, \tag{2.22}$$

then we obtained

$$\tau_{k,n}^{\delta} \preceq \|\alpha_{k+1}^{\frac{1}{2}}(\alpha_{k+1}\mathcal{I}+\mathcal{A}_{k+1,n}\mathcal{A}_{k+1,n}^{*})^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k+1,n}^{\delta})-y^{\delta}))\|^{\frac{1}{2}}\varrho^{\frac{1}{2}} \\ +\varrho^{\frac{1}{2}}\delta^{\frac{1}{2}}+L\varrho\|(\mathcal{I}-\mathcal{P}_n)x^{\dagger}\|.$$

It follows from (2.19) that for all $0<k\le k_*$ there holds

$$\|e_{k,n}^{\delta}\| \preceq \|\alpha_{k}^{\frac{1}{2}}(\alpha_{k}\mathcal{I}+\mathcal{A}_{k,n}\mathcal{A}_{k,n}^{*})^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k,n}^{\delta})-y^{\delta}))\|^{\frac{1}{2}}\varrho^{\frac{1}{2}} \\ +\varrho^{\frac{1}{2}}\delta^{\frac{1}{2}}+L\varrho\|(\mathcal{I}-\mathcal{P}_n)x^{\dagger}\|.$$

Thus by setting $k=k_\delta$ in the above inequaliity and using the definition of k_δ, and by (1.7), (1.8), (1.11) and (1.12) we can obtain

$$\begin{aligned}\|e_{k_\delta,n}^{\delta}\| &\preceq C_1\varrho^{\frac{1}{2}}\delta^{\frac{1}{2}}+L\varrho\|(\mathcal{I}-\mathcal{P}_n)x^{\dagger}\| \\ &\le C_1\varrho^{\frac{1}{2}}\delta^{\frac{1}{2}}+L\varrho\|(\mathcal{I}-\mathcal{P}_n)(x^*-x^{\dagger})\|+L\varrho\|(\mathcal{I}-\mathcal{P}_n)x^*\| \\ &\le C_1\varrho^{\frac{1}{2}}\delta^{\frac{1}{2}}+L\varrho^2\gamma_n+L\varrho\gamma_n \\ &\le C_1\varrho^{\frac{1}{2}}\delta^{\frac{1}{2}}+L\varrho(\varrho+1)\sqrt{\tfrac{1}{2(\varrho+1)}}\delta^{\frac{1}{2}} \\ &\preceq C\delta^{\frac{1}{2}}.\end{aligned}$$

□

The following three lemmas will be used in the proof of the main results of this paper, We first present a bound on the difference $\|x_{k,n}^{\delta}-x_{k,n}\|$.

Lemma 2.4. *Let conditions* (1.5), (1.7), (1.8) *and* (1.9) *and hold. Then for all integers* $k\le k_*$,

$$\|x_{k,n}^{\delta}-x_{k,n}\|\le\frac{\delta}{\sqrt{\alpha_k}}+\frac{2\gamma_n}{\sqrt{\alpha_k}}\|(\mathcal{I}-\mathcal{P}_n)x^{\dagger}\|.$$

Proof. From the definition of $x_{k,n}$ it follows that

$$e_{k+1,n}=\beta(k,n)+S_1(k,n)-S_2(k,n), \tag{2.23}$$

where

$$\begin{aligned}\beta(k,n) &:= \alpha_k(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^{*}\mathcal{T}_{k,n})^{-1}\mathcal{P}_n(x^*-x^{\dagger}),\\ S_1(k,n) &:= (\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^{*}\mathcal{T}_{k,n})^{-1}\mathcal{T}_{k,n}^{*}(y-F(\mathcal{P}_nx^{\dagger})),\\ S_2(k,n) &:= (\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^{*}\mathcal{T}_{k,n})^{-1}\mathcal{T}_{k,n}^{*}u_{k,n},\end{aligned}$$

and with $u_{k,n}:=F(x_{k,n})-F(\mathcal{P}_nx^{\dagger})-F'(x_{k,n})e_{k,n}$. We subtracts (2.23) from (2.3) to obtain

$$x_{k+1,n}^{\delta}-x_{k,n}=I_0+S_2(k,n)-S_2^{\delta}(k,n)+S_1^{\delta}(k,n)-S_1(k,n),$$

where

$$I_0:=\alpha_k\Big[(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^{*}\mathcal{A}_{k,n})^{-1}-(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^{*}\mathcal{T}_{k,n})^{-1}\Big]\mathcal{P}_n(x^*-x^{\dagger}).$$

From the expressions of $S_2(k,n)$ and $S_2^\delta(k,n)$ we have

$$\begin{aligned}S_2(k,n)-S_2^\delta(k,n) = &\Big[(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1}\mathcal{T}_{k,n}^* - (\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}\mathcal{A}_{k,n}^*\Big]u_{k,n}\\ &+(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}\mathcal{A}_{k,n}^*(u_{k,n}-u_{k,n}^\delta).\end{aligned}$$

Since

$$\begin{aligned}u_{k,n}-u_{k,n}^\delta &= F(x_{k,n})-F(x_{k,n}^\delta)-(F'(x_{k,n})-F'(x_{k,n}^\delta))e_{k,n}-F'(x_{k,n}^\delta)(e_{k,n}-e_{k,n}^\delta)\\ &= \int_0^1\Big[F'(x_{k,n}^\delta+t(x_{k,n}-x_{k,n}^\delta))-F'(x_{k,n}^\delta)\Big](x_{k,n}-x_{k,n}^\delta)dt\\ &\quad -(F'(x_{k,n})-F'(x_{k,n}^\delta))e_{k,n},\end{aligned}$$

clearly, this and (1.9) imply

$$\|u_{k,n}-u_{k,n}^\delta\| \le \frac{L}{2}\|x_{k,n}-x_{k,n}^\delta\|^2 + L\|x_{k,n}-x_{k,n}^\delta\|\|e_{k,n}\|.$$

Since

$$\|(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1}\mathcal{T}_{k,n}^* - (\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}\mathcal{A}_{k,n}^*\| \le J_4+J_5,$$

where

$$\begin{aligned}J_4 &:= \|(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1}(\mathcal{T}_{k,n}^*-\mathcal{A}_{k,n}^*)\|,\\ J_5 &:= \left\|\left[(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1}-(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}\right]\mathcal{A}_{k,n}^*\right\|.\end{aligned}$$

By using condition (1.9) we have

$$\|\mathcal{T}_{k,n}-\mathcal{A}_{k,n}\| = \|\mathcal{P}_n(F'(x_{k,n})-F'(x_{k,n}^\delta))\mathcal{P}_n\| \le L\|x_{k,n}-x_{k,n}^\delta\|. \tag{2.24}$$

From this, we have

$$\begin{aligned}J_4 &\le \frac{1}{\alpha_k}\|\mathcal{T}_{k,n}-\mathcal{A}_{k,n}\| \le \frac{L}{\alpha_k}\|x_{k,n}-x_{k,n}^\delta\|,\\ J_5 &\le \|(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1}\mathcal{T}_{k,n}^*(\mathcal{T}_{k,n}-\mathcal{A}_{k,n})(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}\mathcal{A}_{k,n}^*\|\\ &\quad+\|(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1}(\mathcal{T}_{k,n}^*-\mathcal{A}_{k,n}^*)\mathcal{A}_{k,n}(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}\mathcal{A}_{k,n}^*\|\\ &\le \frac{1}{4\alpha_k}\|\mathcal{T}_{k,n}-\mathcal{A}_{k,n}\| + \frac{1}{\alpha_k}\|\mathcal{T}_{k,n}-\mathcal{A}_{k,n}\|\\ &\le \frac{5L}{4\alpha_k}\|x_{k,n}-x_{k,n}^\delta\|,\end{aligned}$$

and

$$\|u_{k,n}\| \le \frac{L}{2}\|e_{k,n}\|^2.$$

Similar to the proof of (2.13) we have

$$\left\|\frac{e_{k,n}}{\sqrt{\alpha_k}}\right\| \le r^{\frac12}d_2\varrho, \qquad \text{for } \ 0\le k\le l. \tag{2.25}$$

Thus,

$$\begin{aligned}&\|S_2(k,n)-S_2^\delta(k,n)\|\\ &\le (J_4+J_5)\|u_{k,n}\| + \frac{L}{2\sqrt{\alpha_k}}\left(\frac12\|x_{k,n}-x_{k,n}^\delta\|+\|e_{k,n}\|\right)\|x_{k,n}-x_{k,n}^\delta\|\\ &\le \frac{9L}{4\alpha_k}\|x_{k,n}-x_{k,n}^\delta\|\frac{L}{2}\|e_{k,n}\|^2 + \frac{L}{2\sqrt{\alpha_k}}\left(\frac12\|x_{k,n}-x_{k,n}^\delta\|+\|e_{k,n}\|\right)\|x_{k,n}-x_{k,n}^\delta\|\\ &\le \left[\frac{L}{2}\frac{\|e_{k,n}\|}{\sqrt{\alpha_k}} + \frac{9L}{8}\left(\frac{\|e_{k,n}\|}{\sqrt{\alpha_k}}\right)^2 + \frac{L}{4}\left(\frac{\|e_{k,n}\|}{\sqrt{\alpha_k}}+\frac{\|e_{k,n}^\delta\|}{\sqrt{\alpha_k}}\right)\right]\|x_{k,n}-x_{k,n}^\delta\|\\ &\le \frac12\left(\frac12 d_1+\frac32 d_2+\frac94 r^{\frac12}d_2^2L\varrho\right)L\varrho\|x_{k,n}-x_{k,n}^\delta\|,\end{aligned}$$

we can show that if $L\varrho$ is so small that

$$\frac{1}{2}\left(\frac{1}{2}d_1+\frac{3}{2}d_2+\frac{9}{4}r^{\frac{1}{2}}d_2^2L\varrho\right)L\varrho<\frac{1}{4r}, \tag{2.26}$$

then

$$\|S_2(k,n)-S_2^\delta(k,n)\|\le\frac{1}{4}\|x_{k,n}-x_{k,n}^\delta\|.$$

Let us estimate I_0 now. Clearly we can write $I_0=m_1+m_2$, where

$$m_1:=\alpha_k\Big[(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}-(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1}\Big](\mathcal{I}-\mathcal{P}_n)(x^\dagger-x^*),$$

$$m_2:=\alpha_k\Big[(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}-(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1}\Big](x^\dagger-x^*).$$

By using (2.24) we have

$$\begin{aligned}\|m_1\|\le{}&\alpha_k\Big[\|(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}\mathcal{A}_{k,n}^*(\mathcal{T}_{k,n}-\mathcal{A}_{k,n})(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1}\|\\&\quad+\|(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}(\mathcal{T}_{k,n}^*-\mathcal{A}_{k,n}^*)\mathcal{T}_{k,n}(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1}\|\Big]\\&\quad\times\|(\mathcal{I}-\mathcal{P}_n)(x^\dagger-x^*)\|\\\le{}&\frac{L}{2\sqrt{\alpha_k}}(\|\mathcal{T}_{k,n}-\mathcal{A}_{k,n}\|+\|\mathcal{T}_{k,n}^*-\mathcal{A}_{k,n}^*\|)\|(\mathcal{I}-\mathcal{P}_n)(x^\dagger-x^*)\|\\\le{}&\frac{L\varrho}{\sqrt{\alpha_k}}\gamma_n\|x_{k,n}-x_{k,n}^\delta\|.\end{aligned}$$

Since

$$\begin{aligned}m_2={}&\alpha_k\Big[(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}\mathcal{A}_{k,n}^*(\mathcal{T}_{k,n}-\mathcal{A}_{k,n})(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1}\\&+\alpha_k(\alpha_k\mathcal{I}+\mathcal{A}_{k,n}^*\mathcal{A}_{k,n})^{-1}(\mathcal{T}_{k,n}^*-\mathcal{A}_{k,n}^*)\mathcal{T}_{k,n}(\alpha_k\mathcal{I}+\mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1}\Big]\\&\times(\mathcal{T}_{k,n}^*v+(\mathcal{A}^*-\mathcal{T}_{k,n}^*))v,\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{A}-\mathcal{T}_{k,n}\|&=\|F'(x^\dagger)-\mathcal{P}_nF'(x_{k,n})\mathcal{P}_n\|\\&\le\|F'(x^\dagger)(\mathcal{I}-\mathcal{P}_n)\|+\|(F'(x^\dagger)-F'(x_{k,n}))\mathcal{P}_n\|+\|(\mathcal{I}-\mathcal{P}_n)F'(x_{k,n})\mathcal{P}_n\|\\&\le2\gamma_n+L\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|+L\|e_{k,n}\|,\end{aligned}$$

also, this together with (2.11) and (2.24) gives

$$\begin{aligned}\|m_2\|\le{}&\frac{\sqrt{\alpha_k}}{2}\left(\frac{\varrho}{2\sqrt{\alpha_k}}+\frac{1}{\alpha_k}\|\mathcal{A}^*-\mathcal{T}_{k,n}^*\|\varrho\right)\|\mathcal{T}_{k,n}-\mathcal{A}_{k,n}\|\\&+\|\mathcal{T}_{k,n}-\mathcal{A}_{k,n}\|\left(\varrho+\frac{\varrho}{2\sqrt{\alpha_k}}\|\mathcal{A}^*-\mathcal{T}_{k,n}^*\|\right)\\={}&\left(\frac{5}{4}\varrho+\frac{\varrho}{\sqrt{\alpha_k}}\|\mathcal{A}-\mathcal{T}_{k,n}\|\right)\|\mathcal{T}_{k,n}-\mathcal{A}_{k,n}\|\\\le{}&\left[\frac{5}{4}+\frac{1}{\sqrt{\alpha_k}}(2\gamma_n+L\|e_{k,n}\|+L\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|)\right]L\varrho\|x_{k,n}-x_{k,n}^\delta\|\\\le{}&\left(\frac{9}{4}+\frac{1}{2}L(\varrho+1)+r^{\frac{1}{2}}d_2L\varrho\right)L\varrho\|x_{k,n}-x_{k,n}^\delta\|.\end{aligned}$$

and

$$\|I_0\|\le\left(\frac{11}{4}+\frac{1}{2}L(\varrho+1)+r^{\frac{1}{2}}d_2L\varrho\right)L\varrho\|x_{k,n}-x_{k,n}^\delta\|.$$

It is easy to prove

$$\|S_1^\delta(k,n)\| \le \frac{1}{2\sqrt{\alpha_k}}\|y^\delta - F(\mathcal{P}_n x^\dagger)\| \le \frac{1}{2\sqrt{\alpha_k}}(\delta + \gamma_n\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|),$$

$$\|S_1(k,n)\| \le \frac{1}{2\sqrt{\alpha_k}}\|y - F(\mathcal{P}_n x^\dagger)\| \le \frac{1}{2\sqrt{\alpha_k}}\gamma_n\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|. \tag{2.27}$$

Thus we have

$$\|x_{k,n}^\delta - x_{k,n}\| \le \left(\frac{11}{4} + \frac{1}{2}L(\varrho+1) + r^{\frac{1}{2}}d_2 L\varrho\right)L\varrho\|x_{k,n} - x_{k,n}^\delta\| + \frac{1}{4}\|x_{k,n} - x_{k,n}^\delta\| + \frac{\delta}{2\sqrt{\alpha_k}} + \frac{\gamma_n}{\sqrt{\alpha_k}}\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|.$$

Similar to the proof of (2.3) we have

$$\|e_{k,n}\| \le \frac{\rho}{2} \quad \text{for} \quad 0 \le k \le k_*. \tag{2.28}$$

Together with (2.3) and (2.28) we have

$$\|e_{k,n}\| + \|e_{k,n}^\delta\| \le \rho.$$

If $L\varrho$ is so small that

$$\left(\frac{11}{4} + \frac{1}{2}L(\varrho+1) + r^{\frac{1}{2}}d_2 L\varrho\right)L\varrho \le \frac{1}{4},$$

then there holds

$$\begin{aligned}\|x_{k,n}^\delta - x_{k,n}\| &\le \frac{1}{2}\|x_{k,n} - x_{k,n}^\delta\| + \frac{\delta}{2\sqrt{\alpha_k}} + \frac{\gamma_n}{\sqrt{\alpha_k}}\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| \\ &= \eta_k + \frac{1}{2}\|x_{k,n} - x_{k,n}^\delta\|,\end{aligned}$$

where $\eta_k := \frac{\delta}{2\sqrt{\alpha_k}} + \frac{\gamma_n}{\sqrt{\alpha_k}}\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|$. Since

$$\frac{\eta_{k,n}}{\eta_{k+1,n}} \le \sqrt{\frac{\alpha_{k+1}}{\alpha_k}} \le 1 = p, \quad \varepsilon = \frac{1}{2}.$$

We may apply Lemma 2.1 to conclude

$$\|x_{k,n}^\delta - x_{k,n}\| \le \frac{\delta}{\sqrt{\alpha_k}} + \frac{2\gamma_n}{\sqrt{\alpha_k}}\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|. \qquad \square$$

Lemma 2.5. *Let conditions* (1.5), (1.7), (1.8) *and* (1.9) *hold. Then for all integers* $0 \le k \le k_*$ *there hold*

$$\|e_{k+1,n} - \alpha_k(\alpha_k\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1}\mathcal{P}_n(x^* - x^\dagger)\| \le \frac{1}{4}\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{1}{4r}\|e_{k,n}\|,$$

and

$$\|x_{k,n} - \mathcal{P}_n x^\dagger\| \le \frac{4r}{3}\tau_k + C\Big\{\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \gamma_n\Big\}.$$

Moreover,

$$\|e_{k,n}\| \le 2r\|e_{k+1,n}\| + 2r\varrho\gamma_n + \tfrac{r}{2}(1+5L\varrho)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|,$$

$$\|e_{k+1,n}\| \le 2\|e_{k,n}\| + 7\varrho\gamma_n + \frac{3}{4}(1+5L\varrho)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|.$$

Proof. From the definition of $S_2(k,n)$ it follows that

$$\|S_2(k,n)\| \leq \frac{1}{2\sqrt{\alpha_k}}\|u_{k,n}\| \leq \frac{L}{4\sqrt{\alpha_k}}\|e_{k,n}\|^2. \tag{2.29}$$

This together with (2.23), (2.27) and (2.29) gives

$$\begin{aligned} &\|e_{k+1,n} - \alpha_k(\alpha_k\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1}\mathcal{P}_n(x^* - x^\dagger)\| \\ &\leq J_0 + \|S_2(k,n)\| + \|S_1(k,n)\| \\ &\leq J_0 + \frac{L}{4\sqrt{\alpha_k}}\|e_{k,n}\|^2 + \frac{\gamma_n}{2\sqrt{\alpha_k}}\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|, \end{aligned} \tag{2.30}$$

where

$$J_0 := \alpha_k\left\|\left[(\alpha_k\mathcal{I} + \mathcal{T}_{k,n}^*\mathcal{T}_{k,n})^{-1} - (\alpha_k\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1}\right]\mathcal{P}_n(x^* - x^\dagger)\right\|.$$

Error estimates J_0 are similar to I_0, and we have

$$\begin{aligned} J_0 \leq & \frac{1}{\sqrt{\alpha_k}}\|(\mathcal{I} - \mathcal{P}_n)(x^\dagger - x^*)\|\|\mathcal{T}_{k,n} - \mathcal{B}\| + \frac{5}{4}\varrho\|\mathcal{T}_{k,n} - \mathcal{B}\| \\ &+ \frac{\varrho}{\sqrt{\alpha_k}}\|\mathcal{T}_{k,n} - \mathcal{B}\|\|\mathcal{A} - \mathcal{B}\|. \end{aligned}$$

Since

$$\|\mathcal{T}_{k,n} - \mathcal{B}\| = \|\mathcal{P}_n(F'(x_{k,n}) - F'(\mathcal{P}_n x^\dagger))\mathcal{P}_n\| \leq L\|e_{k,n}\|, \tag{2.31}$$

and

$$\begin{aligned} \|\mathcal{A} - \mathcal{B}\| &= \|F'(x^\dagger) - \mathcal{P}_n F'(\mathcal{P}_n x^\dagger)\mathcal{P}_n\| \\ &\leq \|F'(x^\dagger)(\mathcal{I} - \mathcal{P}_n)\| + \|(F'(x^\dagger) - F'(\mathcal{P}_n x^\dagger))\mathcal{P}_n\| + \|(\mathcal{I} - \mathcal{P}_n)F'(\mathcal{P}_n x^\dagger)\mathcal{P}_n\| \\ &\leq 2\gamma_n + L\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|, \end{aligned} \tag{2.32}$$

then

$$\begin{aligned} J_0 &\leq L\|e_{k,n}\|\left(\frac{1}{\sqrt{\alpha_k}}\|(\mathcal{I} - \mathcal{P}_n)(x^\dagger - x^*)\| + \frac{5}{4}\varrho + \frac{\varrho}{\sqrt{\alpha_k}}(2\gamma_n + L\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|)\right) \\ &\leq L\|e_{k,n}\|\left(\frac{\varrho\gamma_n}{\sqrt{\alpha_k}} + \frac{5}{4}\varrho + \frac{\varrho\gamma_n}{\sqrt{\alpha_k}}(2 + L(\varrho + 1))\right) \\ &\leq \left(\frac{11}{4} + \frac{L}{2}(\varrho + 1)\right)L\|e_{k,n}\|. \end{aligned} \tag{2.33}$$

Therefore by using (2.30) and (2.33), we obtain

$$\begin{aligned} &\|e_{k+1,n} - \alpha_k(\alpha_k\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1}\mathcal{P}_n(x^* - x^\dagger)\| \\ &\leq \left(\frac{11}{4} + \frac{L}{2}(\varrho + 1)\right)\varrho L\|e_{k,n}\| + \frac{L}{4\sqrt{\alpha_k}}\|e_{k,n}\|^2 + \frac{\gamma_n}{2\sqrt{\alpha_k}}\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| \\ &\leq \left(\frac{11}{4} + \frac{L}{2}(\varrho + 1)\right)\varrho L\|e_{k,n}\| + \frac{1}{4}\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \frac{1}{4}r^{\frac{1}{2}}d_2 L\varrho\|e_{k,n}\| \\ &= \frac{1}{4}\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \left(\frac{11}{4} + \frac{L}{2}(\varrho + 1) + \frac{1}{4}r^{\frac{1}{2}}d_2\right)L\varrho\|e_{k,n}\|. \end{aligned} \tag{2.34}$$

If $L\varrho$ is so small that

$$\left(\frac{11}{4} + \frac{L}{2}(\varrho + 1) + \frac{1}{4}r^{\frac{1}{2}}d_2\right)L\varrho \leq \frac{1}{4r},$$

then we have

$$\begin{aligned}&\|e_{k+1,n} - \alpha_k(\alpha_k\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1}\mathcal{P}_n(x^* - x^\dagger)\|\\ \le\ &\frac{1}{4}\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \frac{1}{4r}\|e_{k,n}\|,\end{aligned}$$

and

$$|\|e_{k+1,n}\| - \vartheta_{k,n}| \le \frac{1}{4}\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \frac{1}{4r}\|e_{k,n}\|. \tag{2.35}$$

We obviously have

$$\vartheta_{k,n} \le \|I_1\| + \|I_2\|,$$

where

$$I_1 := \alpha_k(\alpha_k\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1}(\mathcal{P}_n - \mathcal{I})(x^* - x^\dagger),$$

$$I_2 := \alpha_k(\alpha_k\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1}(x^* - x^\dagger).$$

In order to estimate I_2, we write $I_2 = I_2^1 + I_2^2$, where

$$I_2^1 := \alpha_k[(\alpha_k\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1} - (\alpha_k\mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}](x^* - x^\dagger),$$

$$I_2^2 := \alpha_k(\alpha_k\mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}(x^* - x^\dagger).$$

Therefore by using (2.32) we have

$$\begin{aligned}\|I_2^1\| &= \|\alpha_k(\alpha_k\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1}[\mathcal{B}^*(\mathcal{A} - \mathcal{B}) + (\mathcal{A}^* - \mathcal{B}^*)\mathcal{A}](\alpha_k\mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}](x^* - x^\dagger)\|\\ &\le \alpha_k\|(\alpha_k\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1}\mathcal{B}^*\|\|\mathcal{A} - \mathcal{B}\|\|(\alpha_k\mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}\mathcal{A}^*v\|\\ &\quad +\alpha_k\|(\alpha_k\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1}\|\|\mathcal{A}^* - \mathcal{B}^*\|\|\mathcal{A}(\alpha_k\mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}\mathcal{A}^*v\|\\ &\le \frac{5\varrho}{4}\|\mathcal{A} - \mathcal{B}\| \le \frac{5\varrho}{4}(L\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + 2\gamma_n).\end{aligned}$$

Therefore we have

$$\vartheta_{k,n} \le \frac{7}{2}\varrho\gamma_n + \frac{5}{4}L\varrho\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \tau_k, \tag{2.36}$$

and

$$\begin{aligned}\vartheta_{k,n} &\ge \|I_2\| - \|I_1\| \ge \|I_2^2\| - \|I_2^1\| - \|I_1\|\\ &\ge \tau_k - \frac{7}{2}\varrho\gamma_n - \frac{5}{4}L\varrho\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|.\end{aligned} \tag{2.37}$$

As a direct consequence of (2.35) we have

$$\begin{aligned}\|e_{k+1,n}\| &\le \vartheta_{k,n} + \frac{1}{4}\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \frac{1}{4r}\|e_{k,n}\|\\ &\le \tau_k + \frac{7}{2}\varrho\gamma_n + \left(\frac{1}{4} + \frac{5}{4}L\varrho\right)\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \frac{1}{4r}\|e_{k,n}\|.\end{aligned}$$

In order to use Lemma 2.1, we set

$$p_{k,n} = \tau_k + \frac{7}{2}\varrho\gamma_n + \left(\frac{1}{4} + \frac{5}{4}L\varrho\right)\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|.$$

Similar to the proof of ([8], Lemma 4.1) we have $\tau_k \le r\tau_{k+1}$, therefore $\frac{p_{k,n}}{p_{k+1,n}} \le r$. Thus, by Lemma 2.1, we have

$$\|e_{k,n}\| \le \frac{4}{3}rp_{k,n} \le \frac{4}{3}r\tau_k + \frac{14}{3}r\varrho\gamma_n + \frac{1}{3}r(1 + 5L\varrho)\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|. \tag{2.38}$$

It follows from (2.35) that we have

$$\vartheta_{k,n} \le \|e_{k+1,n}\| + \frac{1}{4}\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{1}{4r}\|e_{k,n}\|.$$

This together with (2.37) gives

$$\tau_k \le \|e_{k+1,n}\| + \frac{7}{2}\varrho\gamma_n + (\frac{1}{4}+\frac{5}{4}L\varrho)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{1}{4r}\|e_{k,n}\|.$$

The combining of this and (2.38) gives

$$\|e_{k,n}\| \le \frac{4}{3}r\|e_{k+1,n}\| + \frac{14}{3}r\varrho\gamma_n + \frac{1}{3}r(1+5L\varrho)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{1}{3}\|e_{k,n}\|,$$

and thus we have

$$\|e_{k,n}\| \le 2r\|e_{k+1,n}\| + 7r\varrho\gamma_n + \frac{r}{2}(1+5L\varrho)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|.$$

It follows from (2.35),(2.37) and (2.38) that we have

$$\begin{aligned}
\|e_{k+1,n}\| &\ge \vartheta_{k,n} - \left(\frac{1}{4}\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{1}{4r}\|e_{k,n}\|\right) \\
&\ge \tau_k - \frac{7}{2}\varrho\gamma_n - \left(\frac{1}{4}+\frac{5}{4}L\varrho\right)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| \\
&\qquad -\frac{1}{4r}\left(\frac{4}{3}r\tau_k + \frac{14}{3}r\varrho\gamma_n + \frac{4}{3}r\left(\frac{1}{4}+\frac{5}{4}L\varrho\right)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|\right) \\
&= \frac{2}{3}\tau_k - \frac{7}{3}\varrho\gamma_n - \frac{1}{3}(1+5L\varrho)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|.
\end{aligned}$$

Note that τ_k is non-increasing, we have

$$\tau_{k+1} \le \tau_k \le \frac{3}{2}\|e_{k+1,n}\| + \frac{7}{2}\varrho\gamma_n + \frac{1}{2}(1+5L\varrho)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|.$$

This together with (2.35) and (2.36) then implies

$$\begin{aligned}
\|e_{k+1,n}\| &\le \vartheta_{k,n} + \frac{1}{4}\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{1}{4r}\|e_{k,n}\| \\
&\le \tau_k + \frac{7}{2}\varrho\gamma_n + \left(\frac{1}{4}+\frac{5}{4}L\varrho\right)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \frac{1}{4r}\|e_{k,n}\| \\
&\le 2\|e_{k,n}\| + 7\varrho\gamma_n + \frac{3}{4}(1+5L\varrho)\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|.
\end{aligned}$$

□

Lemma 2.6. *Let conditions* (1.5), (1.7), (1.8) *and* (1.9) *hold. Then for all integers $k_\delta \le l \le k_*$ there holds*

$$\begin{aligned}
\|x_{k_\delta,n} - \mathcal{P}_n x^\dagger\| \le C\Bigg\{ &\frac{1}{\sqrt{\alpha_l}}\|\alpha_{k_\delta}^{\frac{1}{2}}(\alpha_{k_\delta}\mathcal{I}+\mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k_\delta,n})-y)\| + \frac{\delta}{\sqrt{\alpha_l}} + \gamma_n \\
&+\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| + \|x_{l,n}-\mathcal{P}_n x^\dagger\|\Bigg\}.
\end{aligned}$$

Proof. By setting k in (2.23) to be $k_\delta - 1$ and $l-1$ respectively and then subtracting them it follows that

$$\begin{aligned}
x_{l,n} - x_{k_\delta,n} = &\beta(l-1,n) - S_2(l-1,n) + S_1(l-1,n) \\
&-\Big[\beta(k_\delta-1,n) - S_2(k_\delta-1,n) + S_1(k_\delta-1,n)\Big].
\end{aligned}$$

Using the estimates (2.27) and (2.29) we have

$$\begin{aligned}\|x_{l,n}-x_{k_\delta,n}\| \le \|\beta(l-1,n)-\beta(k_\delta-1,n)\| + \frac{L}{4\sqrt{\alpha_{l-1}}}\|e_{l-1,n}\|^2 \\ + \frac{L}{4\sqrt{\alpha_{k_\delta-1}}}\|e_{k_\delta-1,n}\|^2 + \frac{1}{2}\left(\frac{1}{\sqrt{\alpha_{l-1}}}+\frac{1}{\sqrt{\alpha_{k_\delta-1}}}\right)\gamma_n\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|.\end{aligned} \tag{2.39}$$

Note that

$$\beta(l-1,n)-\beta(k_\delta-1,n) = J_6+J_7+J_8,$$

where

$$\begin{aligned} J_6 &:= \alpha_{l-1}\Big[(\alpha_{l-1}\mathcal{I}+\mathcal{T}^*_{l-1,n}\mathcal{T}_{l-1,n})^{-1}-(\alpha_{l-1}\mathcal{I}+\mathcal{B}^*\mathcal{B})^{-1}\Big]\mathcal{P}_n(x^*-x^\dagger),\\ J_7 &:= -\alpha_{k_\delta-1}\Big[(\alpha_{k_\delta-1}\mathcal{I}+\mathcal{T}^*_{k_\delta-1,n}\mathcal{T}_{k_\delta-1,n})^{-1}-(\alpha_{k_\delta-1}\mathcal{I}+\mathcal{B}^*\mathcal{B})^{-1}\Big]\mathcal{P}_n(x^*-x^\dagger),\\ J_8 &:= \Big[\alpha_{l-1}(\alpha_{l-1}\mathcal{I}+\mathcal{B}^*\mathcal{B})^{-1}-\alpha_{k_\delta-1}(\alpha_{k_\delta-1}\mathcal{I}+\mathcal{B}^*\mathcal{B})^{-1}\Big]\mathcal{P}_n(x^*-x^\dagger).\end{aligned}$$

Similary to estimate I_0, we have

$$\begin{aligned}\|J_6\| &\le \|\mathcal{T}_{l-1,n}-\mathcal{B}\|\left(\frac{1}{\sqrt{\alpha_{l-1}}}\|(\mathcal{I}-\mathcal{P}_n)(x^*-x^\dagger)\|+\frac{5\varrho}{4}+\frac{\varrho}{\sqrt{\alpha_{l-1}}}\|\mathcal{A}-\mathcal{B}\|\right)\\ &\le L\|e_{l-1,n}\|\left(\frac{1}{\sqrt{\alpha_{l-1}}}\gamma_n\varrho+\frac{5\varrho}{4}+\frac{\varrho}{\sqrt{\alpha_{l-1}}}(2\gamma_n+L\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|)\right)\\ &\le \left(\frac{11}{4}+\frac{L}{2}(\varrho+1)\right)L\varrho\|e_{l-1,n}\|,\\ \|J_7\| &\le \left(\frac{11}{4}+\frac{L}{2}(\varrho+1)\right)L\varrho\|e_{k_\delta-1,n}\|.\end{aligned}$$

In order to estimate J_8, we can write $J_8 := \sum_1^4 N_i$ with

$$\begin{aligned} N_1 &= -\left(1-\frac{\alpha_{l-1}}{\alpha_{k_\delta-1}}\right)(\alpha_{l-1}\mathcal{I}+\mathcal{B}^*\mathcal{B})^{-1}\mathcal{B}^*\mathcal{B}\{\alpha_{k_\delta-1}(\alpha_{k_\delta-1}\mathcal{I}+\mathcal{B}^*\mathcal{B})^{-1}\mathcal{P}_n(x^*-x^\dagger)-e_{k_\delta,n}\},\\ N_2 &= -\left(1-\frac{\alpha_{l-1}}{\alpha_{k_\delta-1}}\right)(\alpha_{l-1}\mathcal{I}+\mathcal{B}^*\mathcal{B})^{-1}\mathcal{B}^*\mathcal{P}_n(F'(\mathcal{P}_nx^\dagger)e_{k_\delta,n}-F(x_{k_\delta,n})+F(\mathcal{P}_nx^\dagger)),\\ N_3 &= -\left(1-\frac{\alpha_{l-1}}{\alpha_{k_\delta-1}}\right)(\alpha_{l-1}\mathcal{I}+\mathcal{B}^*\mathcal{B})^{-1}\mathcal{B}^*\mathcal{P}_n(y-F(\mathcal{P}_nx^\dagger)),\\ N_4 &= -\left(1-\frac{\alpha_{l-1}}{\alpha_{k_\delta-1}}\right)(\alpha_{l-1}\mathcal{I}+\mathcal{B}^*\mathcal{B})^{-1}\mathcal{B}^*\mathcal{P}_n(F(x_{k_\delta,n})-y).\end{aligned}$$

Recall that $k_\delta \le k_*$ which implies $\frac{\delta}{\sqrt{\alpha_{k_\delta}}} \le C\delta^{1/2}$. It follows from (2.1), (1.12), Lemma 2.3 and Lemma 2.4 that there is a constant c_1 such that

$$\begin{aligned}\|e_{k_\delta,n}\| &\le \|e^\delta_{k_\delta,n}\|+\|x^\delta_{k_\delta,n}-x_{k_\delta,n}\|\\ &\le C\delta^{\frac12}+C(\delta+2\gamma_n\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|)/\sqrt{\alpha_{k_\delta}}\\ &\le c_1\delta^{\frac12}.\end{aligned}$$

Using (2.1), conditions (1.7) and (1.9) we have

$$\begin{aligned}\|N_2\| &\le \frac{1}{2\sqrt{\alpha_{l-1}}}\left\|\int_0^1[F'(\mathcal{P}_nx^\dagger+te_{k_\delta,n})-F'(\mathcal{P}_nx^\dagger)]e_{k_\delta,n}dt\right\|\\ &\le \frac{L}{4\sqrt{\alpha_{l-1}}}\|e_{k_\delta,n}\|^2 \le \frac{Lc_1^2\delta}{4\sqrt{\alpha_{l-1}}},\\ \|N_3\| &\le \frac{\gamma_n}{2\sqrt{\alpha_{l-1}}}\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\| \le \frac{1}{4}\|(\mathcal{I}-\mathcal{P}_n)x^\dagger\|.\end{aligned}$$

By using the argument in the proof of ([8], Lemma 4.4) we can see that

$$\|N_4\| \leq \frac{r^{\frac{1}{2}}}{\sqrt{\alpha_l}} \|\alpha_{k_\delta}^{\frac{1}{2}}(\alpha_k \mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}} \mathcal{P}_n(F(x_{k_\delta,n}) - y)\|.$$

Using (2.34), the above estimates on N_2, N_3, and N_4 we obtain

$$\begin{aligned}
\|J_8\| &\leq \|\alpha_{k_\delta-1}(\alpha_{k_\delta-1}\mathcal{I} + \mathcal{B}^*\mathcal{B})^{-1}\mathcal{P}_n(x^* - x^\dagger) - e_{k_\delta,n}\| + \frac{Lc_1^2}{4\sqrt{\alpha_{l-1}}}\delta \\
&\quad + \frac{1}{4}\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \frac{r^{\frac{1}{2}}}{\sqrt{\alpha_l}}\|\alpha_{k_\delta}^{\frac{1}{2}}(\alpha_{k_\delta}\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k_\delta,n}) - y)\| \\
&\leq \frac{1}{2}\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \frac{1}{4r}\|e_{k_\delta-1,n}\| + \frac{Lc_1^2}{4\sqrt{\alpha_l}}\delta \\
&\quad + \frac{r^{\frac{1}{2}}}{\sqrt{\alpha_l}}\|\alpha_{k_\delta}^{\frac{1}{2}}(\alpha_{k_\delta}\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k_\delta,n}) - y)\|.
\end{aligned}$$

Combining the above estimates and if $L\varrho$ is so small that

$$\left(\frac{11}{4} + \frac{L}{2}(\varrho+1) + \frac{1}{4}r^{\frac{1}{2}}d_2\right)L\varrho \leq \frac{1}{8r}, \tag{2.40}$$

and using (2.39) and Lemma 2.5 we finally obtain that there is a constant C such that

$$\begin{aligned}
&\|x_{l,n} - x_{k_\delta,n}\| \\
&\leq J_6 + J_7 + J_8 + \frac{L}{4\sqrt{\alpha_{l-1}}}\|e_{l-1,n}\|^2 + \frac{L}{4\sqrt{\alpha_{k_\delta-1}}}\|e_{k_\delta-1,n}\|^2 \\
&\quad + \frac{1}{2}\left(\frac{1}{\sqrt{\alpha_{l-1}}} + \frac{1}{\sqrt{\alpha_{k_\delta-1}}}\right)\gamma_n\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| \\
&\leq \left(\frac{11}{4} + \frac{L}{2}(\varrho+1)\right)L\varrho\|e_{l-1,n}\| + \left(\frac{11}{4} + \frac{L}{2}(\varrho+1)\right)L\varrho\|e_{k_\delta-1,n}\| + \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| \\
&\quad + \frac{1}{4r}\|e_{k_\delta-1,n}\| + \frac{Lc_1^2}{4\sqrt{\alpha_l}}\delta + \frac{r^{\frac{1}{2}}}{\sqrt{\alpha_l}}\|\alpha_{k_\delta}^{\frac{1}{2}}(\alpha_{k_\delta}\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k_\delta,n}) - y)\| \\
&\quad + \frac{1}{4}r^{\frac{1}{2}}d_2L\varrho\|e_{l-1,n}\| + \frac{1}{4}r^{\frac{1}{2}}d_2L\varrho\|e_{k_\delta-1,n}\| \\
&= \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \left(\frac{11}{4} + \frac{L}{2}(\varrho+1) + \frac{1}{4}r^{\frac{1}{2}}d_2\right)L\varrho\|e_{k_\delta-1,n}\|\|\| + \frac{1}{4r}\|e_{k_\delta-1,n}\| \\
&\quad + \left(\frac{11}{4} + \frac{L}{2}(\varrho+1) + \frac{1}{4}r^{\frac{1}{2}}d_2\right)L\varrho\|e_{l-1,n}\| + \frac{Lc_1^2\delta}{4\sqrt{\alpha_l}} \\
&\quad + \frac{r^{\frac{1}{2}}}{\sqrt{\alpha_l}}\|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k,n}) - y)\| \\
&\leq C\left\{\frac{1}{\sqrt{\alpha_l}}\|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k,n}) - y)\| + \frac{\delta}{\sqrt{\alpha_l}} + \gamma_n\right\} \\
&\quad + \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + C\|e_{l-1,n}\| + \frac{3}{8r}\|e_{k_\delta-1,n}\|.
\end{aligned}$$

Using this and Lemma 2.5 we have

$$\begin{aligned}
\|x_{l,n} - x_{k_\delta,n}\| \leq C&\left\{\frac{1}{\sqrt{\alpha_l}}\|\alpha_{k_\delta}^{\frac{1}{2}}(\alpha_{k_\delta}\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k_\delta,n}) - y)\| + \frac{\delta}{\sqrt{\alpha_l}} + \gamma_n\right\} \\
&+ C\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + C\|e_{l,n}\| + \frac{3}{4}\|e_{k_\delta,n}\|,
\end{aligned}$$

which gives the result immediately. $\square$

3. Convergence Rates

In this section we will complete the proof of the convergence rates. In order to apply Lemma 2.6, we need the following estimates.

Lemma 3.1. *Assume that conditions* (1.5), (1.7), (1.9) *and* (1.11) *hold. Let* n, k_δ *be the integer determined by Rule* 1.1 *with* $\tau > 3$, *then we have*

$$\|\alpha_{k_\delta}^{\frac{1}{2}}(\alpha_{k_\delta}\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k_\delta,n}) - y)\| \preceq \delta,$$

and

$$\delta \preceq \|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k,n}) - y)\|,$$

for all $0 \le k < k_\delta$.

Proof. For $0 \le k < k_\delta$ we set

$$a_{k,n} := \|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k,n}) - y)\|^2,$$

$$b_{k,n} := \|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I} + \mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k,n}) - y)\|^2,$$

$$\mathcal{F} := (\alpha_k\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}(\mathcal{A}_{k,n}\mathcal{A}_{k,n}^* - \mathcal{B}\mathcal{B}^*)(\alpha_k\mathcal{I} + \mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}},$$

then we have

$$\begin{aligned}
&|a_{k,n} - b_{k,n}| \\
&= \alpha_k|((\alpha_k\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k,n}) - y), \mathcal{F}(\alpha_k\mathcal{I} + \mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k,n}) - y)| \\
&\le \|\mathcal{F}\|\|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k,n}) - y)\| \times \|\alpha_k^{\frac{1}{2}}(\alpha_k\mathcal{I} + \mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k,n}) - y)\| \\
&\le \frac{1}{2}\|\mathcal{F}\|(a_{k,n} + b_{k,n}).
\end{aligned}$$

Let us estimate $\mathcal{F}$ now. Clearly we can write $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, where

$$\begin{aligned}
\mathcal{F}_1 &:= (\alpha_k\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{B}(\mathcal{A}_{k,n}^* - \mathcal{B}^*)(\alpha_k\mathcal{I} + \mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}, \\
\mathcal{F}_2 &:= (\alpha_k\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}(\mathcal{A}_{k,n} - \mathcal{B})\mathcal{A}_{k,n}^*(\alpha_k\mathcal{I} + \mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}.
\end{aligned}$$

Using the hypothesis (1.9) and (2.31) we have

$$\begin{aligned}
\|\mathcal{F}_1\| &\le \frac{\|\mathcal{A}_{k,n}^* - \mathcal{B}^*\|}{\sqrt{\alpha_k}} \le \frac{L\|e_{k,n}^\delta\|}{\sqrt{\alpha_k}} \le Lr^{\frac{1}{2}}d_1\varrho, \\
\|\mathcal{F}_2\| &\le \frac{\|\mathcal{A}_{k,n} - \mathcal{B}\|}{\sqrt{\alpha_k}} \le \frac{L\|e_{k,n}^\delta\|}{\sqrt{\alpha_k}} \le Lr^{\frac{1}{2}}d_1\varrho,
\end{aligned}$$

It then follows from above and the smallness condition (2.12) that

$$|a_{k,n} - b_{k,n}| \le Lr^{\frac{1}{2}}d_1\varrho(a_{k,n} + b_{k,n}) \le \frac{1}{r}(a_{k,n} + b_{k,n}).$$

This implies $b_{k,n} \preceq a_{k,n} \preceq b_{k,n}$. Thus it is suffices to show that

$$\sqrt{b_{k_\delta,n}} \preceq \delta \quad \text{and} \quad \delta \preceq \sqrt{b_{k,n}} \quad \text{for} \quad 0 \le k < k_\delta.$$

By assuming that $4L\varrho < \frac{c_0(\tau-3)}{2}$, and using (1.12), (2.1) and Lemma 2.4 we have for $0 \le k < k_\delta$,

$$\begin{aligned}
\sqrt{b_{k,n}} &\ge \|\alpha_k^{\frac{1}{2}}(\alpha_k \mathcal{I} + \mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k,n}^\delta) - y^\delta)\| - \delta \\
&\quad -\|F(x_{k,n}) - F(x_{k,n}^\delta) - F'(x_{k,n}^\delta)(x_{k,n} - x_{k,n}^\delta)\| \\
&\quad -\|\alpha_k^{\frac{1}{2}}(\alpha_k \mathcal{I} + \mathcal{A}_{k,n}\mathcal{A}_{k,n}^*)^{-\frac{1}{2}}\mathcal{P}_n F'(x_{k,n}^\delta)(x_{k,n} - x_{k,n}^\delta)\| \\
&\ge (\tau-1)\delta - \alpha_k^{\frac{1}{2}}\|x_{k,n} - x_{k,n}^\delta\| - \frac{L}{2}\|x_{k,n} - x_{k,n}^\delta\|^2 \\
&\ge (\tau-1)\delta - (\delta + 2\gamma_n\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|) - \frac{L}{2\alpha_k}(\delta + 2\gamma_n\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|)^2 \\
&\ge (\tau-1)\delta - [\delta + 2\gamma_n^2(\varrho+1)] - \frac{L}{2\alpha_k} \times [\delta + 2\gamma_n^2(\varrho+1)]^2 \\
&\ge (\tau-3)\delta - \frac{4L\varrho}{c_0}\delta \\
&\ge \frac{(\tau-3)\delta}{2} \succeq \delta.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sqrt{b_{k_\delta,n}} &\le \|\alpha_{k_\delta}^{\frac{1}{2}}(\alpha_{k_\delta} \mathcal{I} + \mathcal{A}_{k_\delta,n}\mathcal{A}_{k_\delta,n}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k_\delta,n}^\delta) - y^\delta)\| + \delta \\
&\quad +\|F(x_{k_\delta,n}) - F(x_{k_\delta,n}^\delta) - F'(x_{k_\delta,n}^\delta)(x_{k_\delta,n} - x_{k_\delta,n}^\delta)\| \\
&\quad +\|\alpha_{k_\delta}^{\frac{1}{2}}(\alpha_{k_\delta} \mathcal{I} + \mathcal{A}_{k_\delta,n}\mathcal{A}_{k_\delta,n}^*)^{-\frac{1}{2}}\mathcal{P}_n F'(x_{k_\delta,n}^\delta)(x_{k_\delta,n} - x_{k_\delta,n}^\delta)\| \\
&\le (\tau+1)\delta + \alpha_{k_\delta}^{\frac{1}{2}}\|x_{k_\delta,n} - x_{k_\delta,n}^\delta\| + \frac{L}{2}\|x_{k_\delta,n} - x_{k_\delta,n}^\delta\|^2 \\
&\le (\tau+1)\delta + (\delta + 2\gamma_n\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|) + \frac{L}{2}\frac{(\delta + 2\gamma_n\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|)^2}{\alpha_{k_\delta}} \\
&\le (\tau+1)\delta + [\delta + 2\gamma_n^2(\varrho+1)] + \frac{L}{2\alpha_k} \times [\delta + 2\gamma_n^2(\varrho+1)]^2 \\
&\le (\tau+1)\delta + 2\delta + \frac{4L\varrho}{c_0}\delta \preceq \delta.
\end{aligned}$$

The proof is complete. □

The following theorem gives the main results of this article.

Theorem 3.1. *Assume that conditions* (1.5), (1.7), (1.8), (1.9) *and* (1.11) *hold and that the noise level δ shall be sufficiently small. Let n, k_δ be the integer determined by Rule* 1.1. *If $L\varrho \le \varepsilon_0$, then*

$$\|x_{k_\delta,n}^\delta - x^\dagger\| \le C \inf\left\{\|e_{k,n}\| + \frac{\delta}{\sqrt{\alpha_k}} + \gamma_n : k = 0, 1, \ldots\right\},$$

where ε_0 and C are some positive constants depending only on r and τ.

In particular, if, in addition, $x^* - x^\dagger = (F'(x^\dagger)^* F'(x^\dagger))^{\frac{\nu}{2}} w$ for some $w \in \mathbb{X}$ and some $1 \le \nu \le 2$, then

$$\|x_{k_\delta,n}^\delta - x^\dagger\| \preceq \delta^{\nu/(1+\nu)}.$$

Proof. Note that for $k \ge k_\delta$, we have from Lemma 2.4, Lemma 2.5, Lemma 2.6 and Lemma 3.1 that

$$\begin{aligned}
\|x^\delta_{k_\delta,n} - x^\dagger\| &\le \|x^\delta_{k_\delta,n} - x_{k_\delta,n}\| + \|x_{k_\delta,n} - \mathcal{P}_n x^\dagger\| + \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| \\
&\le \frac{1}{\sqrt{\alpha_{k_\delta}}}(\delta + 2\gamma_n\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|) + \|e_{k_\delta,n}\| + \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| \\
&\preceq \frac{1}{\sqrt{\alpha_{k_\delta}}}(\delta + 2\gamma_n\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|) + \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| \\
&\qquad + \frac{\|\alpha_{k_\delta}^{\frac{1}{2}}(\alpha_{k_\delta}\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k_\delta,n}) - y))\|}{\sqrt{\alpha_k}} + \frac{\delta}{\sqrt{\alpha_k}} + \gamma_n + \|e_{k,n}\| \\
&\preceq \frac{\delta}{\sqrt{\alpha_k}} + \gamma_n + \|(\mathcal{I} - \mathcal{P}_n)(x^* - x^\dagger)\| + \|(\mathcal{I} - \mathcal{P}_n)x^*\| + \|e_{k,n}\| \\
&\preceq \frac{\delta}{\sqrt{\alpha_k}} + \gamma_n + \|(\mathcal{I} - \mathcal{P}_n)F'(x^\dagger)^*\|^\nu + \|(\mathcal{I} - \mathcal{P}_n)x^*\| + \|e_{k,n}\| \\
&\preceq \frac{\delta}{\sqrt{\alpha_k}} + \gamma_n + \|e_{k,n}\|,
\end{aligned}$$

where for $0 \le k < k_\delta$, we have from Lemmas 2.4, Lemma 2.5 and (2.35) that

$$\begin{aligned}
&\|x^\delta_{k_\delta,n} - x^\dagger\| \\
&\le \|x^\delta_{k_\delta,n} - x_{k_\delta,n}\| + \|x_{k_\delta,n} - \mathcal{P}_n x^\dagger\| + \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| \\
&\le \frac{1}{\sqrt{\alpha_{k_\delta}}}(\delta + 2\gamma_n\|(\mathcal{I} - \mathcal{P}_n)x^\dagger\|) + \|e_{k_\delta,n}\| + \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| \\
&\preceq \|e_{k,n}\| + \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \gamma_n + \frac{1}{\sqrt{\alpha_{k_\delta}}}\|\alpha_{k_\delta-1}^{\frac{1}{2}}(\alpha_{k_\delta-1}\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k_\delta-1,n}) - y))\|
\end{aligned}$$

$$\begin{aligned}
&\preceq \|e_{k,n}\| + \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \gamma_n + \|(\alpha_{k_\delta-1}\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{B}e_{k_\delta-1,n}\| \\
&\qquad + \|(\alpha_{k_\delta-1}\mathcal{I} + \mathcal{B}\mathcal{B}^*)^{-\frac{1}{2}}\mathcal{P}_n(F(x_{k_\delta-1,n}) - F(x^\dagger) - F'(\mathcal{P}_n x^\dagger)e_{k_\delta-1,n})\| \\
&\preceq \|e_{k,n}\| + \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \gamma_n + \|e_{k_\delta-1,n}\| \\
&\qquad + \frac{1}{\sqrt{\alpha_{k_\delta-1}}}\|F(x_{k_\delta-1,n}) - F(\mathcal{P}_n x^\dagger) - F'(\mathcal{P}_n x^\dagger)e_{k_\delta-1,n}\| + \frac{1}{\sqrt{\alpha_{k_\delta-1}}}\|F(\mathcal{P}_n x^\dagger) - F(x^\dagger)\| \\
&\preceq \|e_{k,n}\| + \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \gamma_n + \frac{L}{\sqrt{\alpha_{k_\delta-1}}}\|e_{k_\delta-1,n}\|^2 + \frac{\gamma_n}{\sqrt{\alpha_{k_\delta-1}}}\|(\mathcal{I} - \mathcal{P}_n)x^*\| \\
&\preceq \|e_{k,n}\| + \|(\mathcal{I} - \mathcal{P}_n)x^*\| + \gamma_n + \gamma_n^\nu \\
&\preceq \|e_{k,n}\| + \gamma_n.
\end{aligned}$$

Note that $\tau_k \le \alpha_k^{\nu/2}\|w\|$ under the condition on $x^* - x^\dagger$. We have from Lemma 2.5 that

$$\|e_{k,n}\| \le \alpha_k^{\nu/2}\|w\| + \|(\mathcal{I} - \mathcal{P}_n)x^\dagger\| + \gamma_n \preceq \alpha_k^{\nu/2}\|w\| + \gamma_n.$$

Thus we have

$$\|x^\delta_{k_\delta,n} - x^\dagger\| \preceq \inf\left\{\alpha_k^{\nu/2}\|w\| + \frac{\delta}{\sqrt{\alpha_k}} + \gamma_n : k = 0, 1, \dots\right\}.$$

Now we introduce the integer $\bar{k}_\delta$ such that

$$\alpha_{\bar{k}_\delta} \le \left(\frac{\delta}{\|w\|}\right)^{2/(1+\nu)} < \alpha_k, \quad 0 \le k < \bar{k}_\delta.$$

Then it is readily seen that

$$\begin{aligned}
\|x^\delta_{k_\delta,n} - x^\dagger\| &\preceq \alpha_{\bar{k}_\delta}^{\nu/2}\|w\| + \frac{\delta}{\sqrt{\alpha_{\bar{k}_\delta}}} + \gamma_n \\
&\preceq \delta^{\nu/(1+\nu)} + \gamma_n \preceq \delta^{\nu/(1+\nu)} + \delta \preceq \delta^{\nu/(1+\nu)}.
\end{aligned}$$

The proof is complete. $\square$

4. Numerical Examples

In this section, we present numerical examples to demonstrate the efficiency and accuracy of the proposed method.

Example 4.1. Consider the problem of solving the nonliear integral equation (1.1) with kernel [16]

$$k(s,t,x(t)) = \bar{k}(s,t,x(t)) + \bar{k}(s,-t,x(-t)),$$

where

$$\bar{k}(s,t,x(t)) = \ln \frac{(t-s)^2 + H^2}{(t-s)^2 + (H - x(s))^2}, \quad s,t \in [-1,1]$$

where $H = 2$.

In the numerical tests the gravity strength anomaly $y(t)$ in (1.1) was chosen as the solution of the direct problem for the model function

$$x(t) = 0.5(1-t^2)^2.$$

The integral in (1.1) was calculated by piecewise trapezoidal's formula

$$y(s_i) = \sum_{j=0}^{2^n} w_j k(s_i, t_j, x(t_j)) \triangle_j,$$

where

$$w_0 = w_{2^n} = \frac{1}{2}, \quad w_j = 1, j = 1, \ldots, 2^n - 1, \quad \triangle_j = \frac{2}{2^n}, \quad j = 0, 1, \ldots, 2^n.$$

$$s_i = -1 + ih, \quad i = 0, 1, \ldots, 2^n, \quad t_j = -1 + jh, \quad j = 0, 1, \ldots, 2^n, \ \ h = \frac{2}{2^n}.$$

Example 4.2. Consider the problem of solving the nonliear integral equation (1.1) while F is defined as [6, 20]

$$F(x)(s) = \int_0^s x^3(t)dt.$$

The Fréchet derivative of the operator F is

$$(F'(x)h)(s) = \int_0^s 3x^2(t)h(t)dt.$$

Let $x^\dagger := 1 + \pi/100 \cos(\pi t)$ be a minimum-norm solution of $F(x) = y$. Thus, we have

$$y(s) = \left(1 + \frac{3E^2}{2}\right) s + \frac{3E + E^3}{\pi} \sin(\pi s) + \frac{3E^2}{4\pi} \sin(2\pi s) - \frac{E^3}{3\pi} \sin^3(\pi s)$$

with $E = \pi/10^{-2}$. We choose the initial function $x^*(t) = 0.969$. Therefore, we can obtain $x^* - x^\dagger \in R(F'(x^\dagger)^*)$. We can conclude that the rates of convergence are $O(\sqrt{\delta})$.

We use a perturbed right-hand side $y^\delta = y + \delta \cdot v$, where $v \in \mathbb{X}$ has uniformly distributed random values so that $\|v\| \leq 1$, and where $\delta = 10^{-j}, j = 2, \ldots, 6$.

Let $\mathbb{X}_n$ be the space of piecewise linear polynomials on E with knots at $j/2^n, j = 1, 2, \ldots, 2^n - 1$. As in [4], we decompose $\mathbb{X}_n$ into the form of the orthogonal direct sum of subspaces

$$\mathbb{X}_n = \mathbb{W}_0 \oplus^\perp \mathbb{W}_1 \oplus^\perp \cdots \oplus^\perp \mathbb{W}_n,$$

where $\mathbb{X}_0$ is the linear polynomial space on E, and for $i \in \mathbb{N}, \mathbb{W}_i$ is the orthogonal complement of $\mathbb{X}_{i-1}$ in $\mathbb{X}_i$. The basis for $\mathbb{W}_i, i = 2, 3, \ldots,$ can be constructed recursively once the basis for $\mathbb{W}_1$ is given [12, 13]. We choose a basis for $\mathbb{X}_0$

$$w_{0,0} = 1, \quad w_{0,1} = \sqrt{3}(2t-1), \quad t \in [0,1],$$

and a basis for $\mathbb{W}_1$

$$w_{1,0} = \begin{cases} -6t+1, & t \in \left[0, \frac{1}{2}\right], \\ -6t+5, & t \in \left(\frac{1}{2}, 1\right], \end{cases} \qquad w_{1,1} = \begin{cases} \sqrt{3}(1-4t), & t \in \left[0, \frac{1}{2}\right], \\ \sqrt{3}(4t-3), & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Then the subspaces $\mathbb{W}_i = \operatorname{span}\{w_{i,j} : j = 0, 1, \ldots, 2^i - 1\}$ with the basis recursively generated by

$$w_{i,j}(t) = \begin{cases} \sqrt{2} w_{i-1,j}(2t), & t \in \left[0, \frac{1}{2}\right], \\ 0, & t \in \left(\frac{1}{2}, 1\right], \end{cases} \qquad j = 0, 1, \ldots, 2^{i-1} - 1,$$

and

$$w_{i,2^{i-1}+j}(t) = \begin{cases} 0, & t \in \left[0, \frac{1}{2}\right], \\ \sqrt{2} w_{i-1,j}(2t-1), & t \in \left(\frac{1}{2}, 1\right], \end{cases} \qquad j = 0, 1, \ldots, 2^{i-1} - 1.$$

Tables 4.1 and 4.2 contain the results for the discrete Gauss-Newton iteration (1.6), which is terminated by Rule 1.1 with $\tau = 3.2$. In order to indicate the dependence of the convergence rates on noise level and discretized level, different values of δ are selected. The rates in Tables 4.1 and 4.2 coincide with Theorem 3.1 very well.

Table 4.1: Numerical results for Example 4.1.

n	δ	k_δ	α_{k_δ}	$\lVert x^\delta_{k_\delta,n} - x^\dagger \rVert$
8	1.0e-02	3	1.25e-01	0.171208
8	1.0e-03	10	9.77e-04	0.030846
8	1.0e-04	12	2.44e-04	0.024659
8	1.0e-05	18	3.81e-06	0.007251
8	1.0e-06	20	9.54e-07	0.006447

Table 4.2: Numerical results for Example 4.2.

n	δ	k_δ	α_{k_δ}	$\lVert x^\delta_{k_\delta,n} - x^\dagger \rVert$	$\lVert x^\delta_{k_\delta,n} - x^\dagger \rVert / \sqrt{\delta}$
8	1.00e-02	1	5.00e-01	0.028023	0.280233
8	1.00e-03	6	1.56e-02	0.003834	0.121242
8	1.00e-04	10	9.77e-04	0.000432	0.043245
8	1.00e-05	13	1.22e-04	0.000141	0.044709
8	1.00e-06	16	1.53e-05	0.000059	0.059113

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