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SEMI-CONVERGENCE OF AN ALTERNATING-DIRECTION ITERATIVE METHOD FOR SINGULAR SADDLE POINT PROBLEMS

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Abstract. For large-scale sparse saddle point problems, Peng and Li [12] have recently proposed a new alternating-direction iterative method for solving nonsingular saddle point problems, which is more competitive (in terms of iteration steps and CPU time) than some classical iterative methods such as Uzawa-type and HSS (Hermitian skew splitting) methods. In this paper, we further study this method when it is applied to the solution of singular saddle point problems and prove that it is semi-convergent under suitable conditions.

Key words. saddle point problem, semi-convergence, singular, alternating-direction iterative, iterative method

1. Introduction

Consider the saddle point problem

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \text{ or } \tilde{A}\tilde{x} = b, \tag{1.1}$$

where the matrices $A \in R^{m\times m}$ and $B \in R^{m\times n}$ and the vectors $f \in R^m$ and $g \in R^n$ are given, with $n \leq m$, and B^T is the transpose of the matrix B. We shall assume that A is positive real (i.e., $A^T + A$ is positive definite) and B is a rectangular matrix with rank$(B) = r$. Saddle point problems arise in many scientific computing and engineering applications such as mixed finite element methods for solving elliptic partial differential equations, and Stokes problems, computational fluid dynamics, and constrained least-squares problems; see [7, 8, 9, 11], for example. Benzi et al. [5] gave a comprehensive survey for recent work on the saddle point problems.

When the matrix B is of full column rank, i.e., $r = n$, we know that the coefficient matrix of system (1.1) is nonsingular and this sytem has a unique solution. Because the matrices A and B are usually large and sparse, iterative methods are always considered to be the most suitable candidates for solving system (1.1). So far, a large variety of iterative methods based on the matrix splitting of the coefficient matrix of (1.1) have been studied in the literature, for example, Uzawa-type methods [4, 6], GSSOR (generalized symmetric successive over-relaxation) iterative methods [1, 10], and HSS iterative methods [3]. Recently, Peng and Li [12] has proposed a new alternating-direction iterative method for solving system (1.1). Theoretical analysis and numerical experiments have shown that this new iterative method is more competitive than some classical iterative methods in terms of iteration steps and CPU time, such as the Uzawa-type and HSS iterative methods with optimal parameters.

When $r < n$, the coefficient matrix of system (1.1) is singular and this system has an infinite number of solutions; see [5]. Recently, Zheng et al. [14] has showed that

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the GSOR (generalized successive over-relaxation) iterative method proposed in [4] can be used to solve a singular saddle point problem of type (1.1) and proved that this method is semi-convergent. In this paper, we will study the new alternating-direction iterative (ADI) method presented in [12] when it is applied to the solution of the singular system (1.1), and prove that this new method is semi-convergent under suitable conditions.

2. An Alternating-direction Iterative Method

In this section, we review the new alternating-direction iterative method proposed in [12] for solving the nonsingular saddle point problem (1.1).

Define

$$\tilde{A} = \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} 0 & 0 \\ -B^T & 0 \end{pmatrix}.$$

We then consider the following splittings of $\tilde{A}$:

$$\tilde{A} = (\alpha_1 I + \tilde{H}) - (\alpha_1 I - \tilde{S}) = (\alpha_2 I + \tilde{S}) - (\alpha_2 I - \tilde{H}),$$

where I denotes the identity matrix with the corresponding dimension and α_1 and α_2 are positive parameters.

Now, given the initial guess $\tilde{x}^{(0)}$, a new-ADI method can be described as follows:

$$\begin{cases} (\alpha_1 I + \tilde{H})\tilde{x}^{k+\frac{1}{2}} = (\alpha_1 I - \tilde{S})\tilde{x}^{k} + b, \\ (\alpha_2 I + \tilde{S})\tilde{x}^{k+1} = (\alpha_2 I - \tilde{H})\tilde{x}^{k+\frac{1}{2}} + b. \end{cases}$$

By eliminating the intermediate vector $\tilde{x}^{(k+\frac{1}{2})}$, this method can be equivalently rewritten as

$$\tilde{x}^{(k+1)} = T_{\alpha_1,\alpha_2}\tilde{x}^{(k)} + c,$$

where

$$\begin{aligned} T_{\alpha_1,\alpha_2} &= (\alpha_2 I + \tilde{S})^{-1}(\alpha_2 I - \tilde{H})(\alpha_1 I + \tilde{H})^{-1}(\alpha_1 I - \tilde{S}), \\ c &= (\alpha_2 I + \tilde{S})^{-1}[I + (\alpha_2 I - \tilde{H})(\alpha_1 I + \tilde{H})^{-1}]b. \end{aligned} \tag{2.1}$$

Lemma 2.1. *With the above definition, the matrix T_{α_1,α_2} has the form*

$$T_{\alpha_1,\alpha_2} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned} T_{11} &= \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}(\alpha_1 I + A)^{-1}(\alpha_1^2 I - BB^T) - \frac{\alpha_1}{\alpha_2} I, \\ T_{12} &= -\frac{\alpha_1+\alpha_2}{\alpha_2}(\alpha_1 I + A)^{-1}B, \\ T_{21} &= \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2^2}B^T(\alpha_1 I + A)^{-1}(\alpha_1^2 I - BB^T) + \frac{\alpha_2^2-\alpha_1^1}{\alpha_1\alpha_2^2}B^T, \\ T_{22} &= I - \frac{\alpha_1+\alpha_2}{\alpha_2^2}B^T(\alpha_1 I + A)^{-1}B. \end{aligned}$$

Proof. Note that

$$(\alpha_1 I + \tilde{H})^{-1}(\alpha_2 I - \tilde{H}) = (\alpha_2 I - \tilde{H})(\alpha_1 I + \tilde{H})^{-1},$$

which is true, since each term in this equation is a polynomial of $\tilde{H}$. Then, with the splitting $\tilde{A} = M - N$, where M is a nonsingular matrix, T_{α_1,α_2} is the corresponding iteration matrix

$$T_{\alpha_1,\alpha_2} = M^{-1}N,$$

where

$$M = \frac{1}{\alpha_1+\alpha_2}(\alpha_1 I + \tilde{H})(\alpha_2 I + \tilde{S}),$$
$$N = \frac{1}{\alpha_1+\alpha_2}(\alpha_2 I - \tilde{H})(\alpha_1 I - \tilde{S}),$$

which gives the desired result (2.2). □

When the matrix B is of the full column rank (i.e., $r = n$), the convergence of the above ADI method was carefully analyzed and the choice of the parameters was also discussed in paper [12].

3. Semi-convergence of the ADI Method

In this section, we shall show our main results for the above ADI method when the coefficient matrix of system (1.1) is singular. Before its semi-convergence is discussed, we shall state some basic concepts and lemmas.

Denote $\lambda(A)$ and $\rho(A)$ the spectral set and the spectral radius of a square matrix A, respectively. Also, let $\sigma(A)$ represent the set of singular values of A. Furthermore, define $\vartheta(A) = \max\{|\lambda| : \lambda \in \lambda(A), \lambda \neq 1\}$ and $\mathrm{Index}(A) = \min\{k : \mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1}),$ where k is a nonnegative integer$\}$.

Lemma 3.1 ([2,6]). *Suppose that $A \in R^{n\times n}$ is a square matrix and I is the identity matrix with the same dimension. Then the matrix A is semi-convergent if and only if the following two conditions are fulfilled:*

(1) $\vartheta(A) < 1$,

(2) $\mathrm{Index}(I - A) \leq 1$.

For a matrix $A \in R^{n\times n}$, $A = M - N$ is a splitting if M is nonsingular and the corresponding iteration scheme can be described as follows:

$$x_{k+1} = Tx_k + c, \quad k = 0, 1, 2, \ldots, \tag{3.1}$$

where $T = M^{-1}N$ is the iteration matrix, $c = M^{-1}b$, and x_0 is an initial guess vector.

We recall that a square matrix A is semi-convergent if $\lim\limits_{k\to\infty} A^k$ exists ($k = 0, 1, ...,$), and iteration (3.1) is semi-convergent if the corresponding iteration matrix T is semi-convergent [6,13].

When A is nonsingular, the iterative scheme (3.1) is convergent if and only if $\rho(T) < 1$. When A is singular, the semi-convergence of the iteration matrix T guarantees the semi-convergence of the iterative scheme (3.1).

Lemma 3.2 ([14]). *Let matrix $H \in R^{\ell\times\ell}$, with ℓ a positive integer, and I be the corresponding matrix. Then the partitioned matrix*

$$T = \begin{pmatrix} H & 0 \\ L & I \end{pmatrix}$$

is semi-convergent if and only if either of the following conditions is satisfied:

(1) $L = 0$ and H is semi-convergent,

(2) $\rho(H) < 1$.

With Lemmas 3.1 and 3.2, we shall now extend the results for nonsingular saddle point problems proposed in [12] to those for singular systems.

Theorem 3.3. *Assume that $A \in R^{m\times m}$ is positive real and $B \in R^{m\times n}$ is rank-deficient (i.e., $r < n$). Denote by $[u^*, v^*]^*$ the eigenvector of the matrix T_{α_1,α_2}, where u^* and v^* are the conjugate transposes of u and v, respectively, T_{α_1,α_2} is* *defined in* (2.1), *and* α_1 *and* α_2 *are positive parameters. Then the alternating-direction iterative method is semi-convergent to a solution of the singular saddle point problem* (1.1) *if*

$$\gamma > \frac{u^* BB^T u}{2\alpha_1\alpha_2 Re(u^*Au)}\{(\alpha_1+\alpha_2)^2[Re(u^*Au)]^2 + (\alpha_1-\alpha_2)^2[Im(u^*Au)]^2\}, \quad (3.2)$$

where $\gamma = (\alpha_2 - \alpha_1)|u^*Au|^2 + 2\alpha_1\alpha_2 Re(u^*Au)$. *If* A *is symmetric and positive definite, then the method is semi-convergent if*

$$2\alpha_1\alpha_2[(\alpha_2-\alpha_1)(u^*Au)^2 + 2\alpha_1\alpha_2] > (\alpha_1+\alpha_2)^2 u^* BB^T u$$

Proof. From the definition of semi-convergence of an iteration scheme, it suffices to show that the iteration matrix T_{α_1,α_2} is semi-convergent. Let $B = U(B_1, 0)V^*$ be the singular value decomposition of B, where U and V are unitary matrices, $B_1 = (\Sigma_r, 0)^T \in R^{m\times r}$, with

$$\Sigma_r = \mathrm{diag}(\sigma_1, \sigma_2, ..., \sigma_r),$$

and σ_i is a singular value of B, $i = 1, ..., r$. Then the matrix

$$P = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

is an $(m+n)\times(m+n)$ unitary matrix. Defining $\hat{T} = P^* T_{\alpha_1,\alpha_2} P$, it follows that the matrix $\hat{T}$ is similar to T_{α_1,α_2}. Thus these two matrices have the same eigenvalues. Hence we only need to demonstrate that $\hat{T}$ is semi-convergent.

Define the matrices

$$\hat{A} = U^*AU, \quad \hat{B} = U^*BV.$$

It holds that $\hat{B} = (B_1, 0)$. Then we see that

$$\hat{T} = P^* T_{\alpha_1,\alpha_2} P = \begin{pmatrix} U^*T_{11}U & U^*T_{12}V \\ V^*T_{21}U & V^*T_{22}V \end{pmatrix}.$$

By algebraic computations, we have

$$\begin{aligned} U^*T_{11}U &= U^*[\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}(\alpha_1 I + A)^{-1}(\alpha_1^2 I - BB^T) - \frac{\alpha_1}{\alpha_2}I]U \\ &= \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}U^*(\alpha_1 I + A)^{-1}UU^*(\alpha_1^2 I - BB^T)U - \frac{\alpha_1}{\alpha_2}I \\ &= \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}(\alpha_1 I + \hat{A})^{-1}(\alpha_1^2 I - B_1B_1^T) - \frac{\alpha_1}{\alpha_2}I, \end{aligned}$$

$$\begin{aligned} U^*T_{12}V &= U^*[-\frac{\alpha_1+\alpha_2}{\alpha_2}(\alpha_1 I + A)^{-1}B]V \\ &= -\frac{\alpha_1+\alpha_2}{\alpha_2}(\alpha_1 I + \hat{A})^{-1}U^*BV \\ &= (-\frac{\alpha_1+\alpha_2}{\alpha_2}(\alpha_1 I + \hat{A})^{-1}B_1, 0), \end{aligned}$$

$$\begin{aligned} V^*T_{21}U &= V^*[\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2^2}B^T(\alpha_1 I + A)^{-1}(\alpha_1^2 I - BB^T) + \frac{\alpha_2^2-\alpha_1^2}{\alpha_1\alpha_2^2}B^T]U \\ &= \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2^2}\begin{pmatrix} B_1^T \\ 0 \end{pmatrix}(\alpha_1 I + \hat{A})^{-1}(\alpha_1^2 I - B_1B_1^T) + \frac{\alpha_2^2-\alpha_1^2}{\alpha_1\alpha_2^2}\begin{pmatrix} B_1^T \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2^2}B_1^T(\alpha_1 I + \hat{A})^{-1}(\alpha_1^2 I - B_1B_1^T) + \frac{\alpha_2^2-\alpha_1^2}{\alpha_1\alpha_2^2}B_1^T \\ 0 \end{pmatrix}, \end{aligned}$$

and

$$V^*T_{22}V = V^*[I - \frac{\alpha_1+\alpha_2}{\alpha_2^2}B^T(\alpha_1 I + A)^{-1}B]V$$
$$= \begin{pmatrix} I - \frac{\alpha_1+\alpha_2}{\alpha_2^2}B_1^T(\alpha_1 I + \hat{A})^{-1}B_1 & 0 \\ 0 & I_{n-r} \end{pmatrix}.$$

Therefore, we obtain

$$\hat{T} = \begin{pmatrix} \breve{T} & 0 \\ 0 & I_{n-r} \end{pmatrix},$$

where

$$\breve{T} = \begin{pmatrix} \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}(\alpha_1 I + \hat{A})^{-1}(\alpha_1^2 I - B_1B_1^T) - \frac{\alpha_1}{\alpha_2}I & -\frac{\alpha_1+\alpha_2}{\alpha_2}(\alpha_1 I + \hat{A})^{-1}B_1 \\ \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2^2}B_1^T(\alpha_1 I + \hat{A})^{-1}(\alpha_1^2 I - B_1B_1^T) + \frac{\alpha_2^2-\alpha_1^2}{\alpha_1\alpha_2^2}B_1^T & I - \frac{\alpha_1+\alpha_2}{\alpha_2^2}B_1^T(\alpha_1 I + \hat{A})^{-1}B_1 \end{pmatrix}.$$

From Lemma 3.2, we know that matrix $\hat{T}$ is semi-convergent if and only if $\breve{T}$ is semi-convergent. Now, if $\rho(\breve{T}) < 1$, it holds that $\lambda(I - \breve{T}) \neq 0$; i.e., the matrix $(I - \breve{T})$ has a full rank. Thus Index$(I - \breve{T}) = 0$. It follows from Lemma 3.1 that the matrix $\breve{T}$ is semi-convergent. Furthermore, note that matrix $\breve{T}$ is the corresponding iteration matrix when the alternating-direction method is applied to the nonsingular saddle point problem:

$$\begin{pmatrix} \hat{A} & B_1 \\ -B_1^T & 0 \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \hat{f} \\ -\hat{g} \end{pmatrix},$$

where $\hat{y}, \hat{g} \in R^r$. Hence, by using Theorem 1 in [5], we see that $\rho(\breve{T}) < 1$ if and only if

$$\gamma > \frac{\hat{u}^*B_1B_1^T\hat{u}}{2\alpha_1\alpha_2 Re(\hat{u}^*\hat{A}\hat{u})}\{(\alpha_1+\alpha_2)^2[Re(\hat{u}^*\hat{A}\hat{u})]^2 + (\alpha_1-\alpha_2)^2[Im(\hat{u}^*\hat{A}\hat{u})]^2\}, \quad (3.3)$$

where $\gamma = (\alpha_2 - \alpha_1)|\hat{u}^*\hat{A}\hat{u}|^2 + 2\alpha_1\alpha_2 Re(\hat{u}^*\hat{A}\hat{u})$ and $[\hat{u}^*, \hat{v}^*]^*$ is the eigenvector of $\hat{T}$. Since $\hat{T} = P^*T_{\alpha_1,\alpha_2}P$, $[\hat{u}^*, \hat{v}^*]^* = P^*[u^*, v^*]^*$. Therefore, we have

$$\hat{u}^*\hat{A}\hat{u} = \hat{u}^*U^*AU\hat{u} = (U\hat{u})^*A(U\hat{u}) = u^*Au,$$
$$\hat{u}^*B_1B_1^T\hat{u} = \hat{u}^*U^*BB^TU\hat{u} = u^*BB^Tu,$$

which indicates that (3.3) is equivalent to (3.2) and thus completes the proof. □

Denote by $\sigma_{min}(B^T)$ and $\sigma_{max}(B^T)$ the smallest and largest singular values of B^T, respectively. Similarly, let $\lambda_{min}(A)$ and $\lambda_{max}(A)$ be the smallest and largest eigenvalues of A. Suppose that the following assumptions are established:

(A1) A is symmetric positive definite and ill-conditioned;

(A2) $\alpha_1 = \alpha_2 = \alpha$.

Based on assumption (A1), without loss of generality, we further assume that

(A3) $\lambda_{min}(A) < \sigma_{max}(B^T) < \frac{1}{2}\lambda_{max}(A)$.

Since $\lambda(A) = \lambda(\hat{A})$ and $\sigma(B^T) = \sigma(B_1^T)$, the following theorem immediately follows from Theorem 4.1 in [12]:

Theorem 3.4. *Under the assumptions* (A1)–(A3), *if* $a_1 \geq 2\sigma_{max}(B^T)$ *and* $a_2 > 0$, *then*

$$\rho(\breve{T}) = \begin{cases} \dfrac{2\sigma_{max}^2(B^T) - \alpha^2 + \sqrt{[\lambda_{max}^2(A) - 4\sigma_{max}^2(B^T)]\alpha^2 + 4\sigma_{max}^4(B^T)}}{\alpha[\alpha + \lambda_{max}(A)]}, & \\ \hfill \sigma_{max}(B^T) < \alpha \leq \alpha_0, & \\ \dfrac{\alpha^2 - 2\sigma_{min}^2(B^T) + \sqrt{[\lambda_{min}^2(A) - 4\sigma_{min}^2(B^T)]\alpha^2 + 4\sigma_{min}^4(B^T)}}{\alpha[\alpha + \lambda_{min}(A)]}, & \alpha > \alpha_0, \end{cases}$$

where $\alpha_0 = \sqrt{\sigma_{max}^2(B^T) + \sigma_{min}^2(B^T)}$, $a_1 = u^*Au$, *and* $a_2 = u^*BB^Tu$.

4. Conclusion

In this paper, an alternating-direction iterative method has been studied when it is applied to the solution of a singular saddle point problem. The semi-convergence of this method has been proved with reasonable assumptions, and the spectral radius of the iteration matrix has been obtained for special cases.

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