INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 1, Number 1, Pages 70–92

ASYMPTOTIC BEHAVIOR OF SOLUTION TO NONLINEAR DAMPED p-SYSTEM WITH BOUNDARY EFFECT

CHI-KUN LIN, CHI-TIEN LIN, AND MING MEI

Abstract. For the initial-boundary value problem to the 2×2 damped p-system with nonlinear source,

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u - \beta |u|^{q-1} u, \quad q \geq 2, \\ (v,u)|_{t=0} = (v_0, u_0)(x) \to (v_+, u_+) \quad \text{as } x \to +\infty, \\ u|_{x=0} = 0, \quad u_+ \neq 0, \end{cases} \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

when $\beta > 0$, or $\beta < 0$ but $|\beta| < \frac{\alpha}{|u_+|^{q-1}}$, the solution $(v,u)(x,t)$ is proved to globally exist and converge to the solution of the corresponding porous media equations

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha \bar{u}, \\ \bar{v}|_{t=0} = \bar{v}_0(x) \to v_+ \quad \text{as } x \to +\infty, \\ \bar{u}|_{x=0} = 0, \end{cases} \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

with a specially selected initial data $\bar{v}_0(x)$. The optimal convergence rates $\|\partial_x^k (v - \bar{v}, u - \bar{u})(t)\|_{L^2} = O(1)(t^{-\frac{2k+3}{4}}, t^{-\frac{2k+5}{4}})$, $k = 0, 1$, are also obtained, as the initial perturbation is in $L^1(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$. If the initial perturbation is in the weighted space $L^{1,\gamma}(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$ with the best choice of $\gamma = \frac{1}{4}$, some new and much better decay rates are further obtained: $\|\partial_x^k (v - \bar{v})(t)\|_{L^2} = O(1)(1+t)^{-\frac{2k+3}{4} - \frac{\gamma}{2}}$, $k = 0, 1$. The proof is based on the technical weighted energy method combining with the Green function method. However, when $\beta < 0$ and $|\beta| > \frac{\alpha}{|u_+|^{q-1}}$, then the solution will blow up at a finite time. Finally, numerical simulations are carried out to confirm the theoretical results by using the central-upwind scheme. In particular, the interest phenomenon of coexistence of the global solution $v(x,t)$ and the blow-up solution $u(x,t)$ is observed and numerically demonstrated.

Key words. p-system of hyperbolic conservation laws, nonlinear damping, IBVP, porous equations, diffusion waves, asymptotic behavior, convergence rates, blow-up.

1. Introduction and Main Results

This is a series of study on the hyperbolic p-system with nonlinear source. In the first part [22], we investigated the asymptotic behavior of the solution for the Cauchy problem. Here, as the second part, we are going to treat the initial-boundary value problem. Namely, we study the 2×2 nonlinear damped p-system on the quadrant

$$(1) \qquad \begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u - \beta |u|^{q-1} u, \end{cases} \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

with the initial-boundary conditions

$$(2) \qquad \begin{cases} (v,u)|_{t=0} = (v_0, u_0)(x) \to (v_+, u_+) \quad \text{as } x \to +\infty, \ x \in \mathbb{R}_+, \\ u|_{x=0} = 0. \end{cases}$$

Received by the editors June 2, 2010 and, in revised form, July 20, 2010.
2000 *Mathematics Subject Classification.* 35L50, 35L60, 35L65, 76R50.

This model represents the compressible flow through porous media with nonlinear dissipative external force field in the Lagrangian coordinates. Here, $v = v(x,t) > 0$ is the specific volume, $u = u(x,t)$ is the velocity, the pressure $p(v)$ is a smooth function of v such that $p(v) > 0$, $p'(v) < 0$. As well-known in a hyperbolic system, the typical example in the case of a polytropic gas is $p(v) = v^{-\nu}$ with $\nu \geq 1$. The external term $-\alpha u - \beta|u|^{q-1}u$ appears in the momentum equation, where $\alpha > 0$ is a constant, $\beta \neq 0$ is another constant but can be either negative or positive. The term $-\alpha u$ is called the linear damping, and $-\beta|u|^{q-1}u$ with $q \geq 2$ is regarded as a nonlinear source to the linear damping $-\alpha u$. When $\beta > 0$, the term $-\beta|u|^{q-1}u$ is nonlinear damping, while, when $\beta < 0$, the term $-\beta|u|^{q-1}u$ is regarded as nonlinear accumulating. $v_+ > 0$ and u_+ are the state constants. For compatibility, we need $u_0(0) = 0$.

When $\beta = 0$, the system (1) is linear damping. The asymptotic behavior of the solution for the Cauchy problem or the IVBP for the linear damped 2×2 p-system has been extensively studied. In 1992, Hsiao and Liu [3, 4] first studied the Cauchy problem for the linearly damped p-system, and showed that the solution $(v,u)(x,t)$ converges to its diffusion wave $(\bar{v},\bar{u})(x/\sqrt{1+t})$, a self-similar solution to the following porous media equations

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha\bar{u}, \end{cases} \quad \text{or} \quad \begin{cases} \bar{v}_t = -\frac{1}{\alpha}p(\bar{v})_{xx}, \\ p(\bar{v})_x = -\alpha\bar{u}, \end{cases} \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

in the form of $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1/2}, t^{-1/2})$. Since then, the convergence have been improved by Nishihara [24, 25] as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4}, t^{-5/4})$ for the initial perturbation in $H^3(\mathbb{R})$, and then by Nishihara, Wang and Yang [28, 34] as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-3/2})$ for the initial perturbation in $L^1(\mathbb{R}) \cap H^3(\mathbb{R})$. These convergence results need the initial perturbation around the specified diffusion wave and the wave strength both to be sufficiently small. Such restrictions were then partially released by Zhao [35], where the initial perturbation in L^∞-sense can be arbitrarily large but its first derivative must be sufficiently small, which implies that the wave must also be weak. Furthermore, when $v_+ = v_-$, Nishihara [26] improved the rates as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/2}\log t,\ t^{-2}\log t)$. Very recently, when $v_+ \neq v_-$, by a heuristic analysis, Mei [23] pointed out that the best asymptotic profile to the linearly damped p-system is the particular parabolic solution to the corresponding porous media equation with a specific initial data, rather than the self-similar solutions (the so-called nonlinear diffusion waves), and further proved the convergence as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/2}\log t,\ t^{-2}\log t)$.

For the initial boundary problem on the quadrant in the case of linear damping (i.e., $\beta = 0$), the convergence to the diffusion waves with different boundary conditions has been studied respectively by Marcati and Mei [19] and by Nishihara and Yang [27] with the rate $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4}, t^{-5/4})$ for the initial perturbation in $H^3(\mathbb{R}_+)$, respectively, and then, further improved to $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-3/2})$ by Marcati, Mei and Rubino [20] for the initial perturbation in $L^1(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$. Motivated by [35], the convergence has been improved by Jiang and Zhu [14] for the strong diffusion wave. Recently, Saind-Houari [31] claimed that the decay rate could be improved to $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(t^{-1-\frac{\gamma}{2}}, t^{-\frac{3}{2}-\frac{\gamma}{2}})$, if the initial perturbation is in $L^{1,r}(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$, where $L^{1,r}(\mathbb{R}_+)$ is a weighted L^1-space with the weight $(1+x)^\gamma$ and $0 \leq \gamma \leq 1$. However, this result is not correct in all cases, and the proof is also with some problems. In fact, the author just applied the well-known results from

[20, 21, 14, 12] to try to prove the improved rates for the linear part only, but did not check the nonlinear part. It is important to note that including the nonlinear part is the crucial step in the proof. For details, we refer to Remark 1.3 below.

For other interesting studies in convergence to diffusion waves in many different cases with linear damping, we refer to [5, 6, 7, 8, 10, 11, 14, 16, 25, 26, 29, 33, 35, 36] and the references therein.

When $\beta \neq 0$, the system (1) becomes either nonlinear damping for $\beta > 0$ or nonlinear accumulating for $\beta < 0$. The research related to this topic, so far, is very limited. For the Cauchy problem case, under the stiff condition $u_+ = u_- = 0$, Jiang and Zhu [37, 38] proved the solution to converge the diffusion wave in the form of $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4}, t^{-5/4})$ with the initial perturbation around the diffusion wave in $H^3(\mathbb{R})$. Recently, by technically constructing a pair of correction functions, Mei [22] released the condition $u_+ = u_- = 0$ to the general case $u_+ \neq u_-$, and further proved the convergence to the diffusion wave with the optimal rates $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-3/2})$ when the initial perturbation is in $L^1(\mathbb{R}) \cap H^3(\mathbb{R})$. For the IBVP case, the convergence of the solution to the diffusion wave with the rate $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4}, t^{-5/4})$ has been investigated by Jiang and Zhu in [13] under the condition $u_+ = 0$. However, these results are not satisfied, because the condition $u_+ = 0$ is too special, and the convergence rate is not sufficient. The main purpose in the present paper is to seek the best asymptotic profile for the original solution, and to show a much better convergence in the case $u_+ \neq 0$.

In what follows, we organize our paper as four parts. In the rest of the current section, we first make a heuristic analysis to see what will be the best asymptotic profile for the original solution to the system (1) and (2), and then state our main convergence results. Namely, when $\beta > 0$ (nonlinear damping case) or $\beta < 0$ (nonlinear accumulating case) but $|\beta|$ is small, the solution of (1) and (2) converges to its corresponding diffusion wave with optimal rates. As observed, we further remark that, when $\beta < 0$ (nonlinear accumulating case) but $|\beta|$ is large, the solution $u(x,t)$ will blow up in finite time, but the solution $v(x,t)$ still globally exists. This is the coexistence of global solution and non-global solution for the system. In section 2, we will give some well-known results which are the preparation for the proof of main theorems. Section 3 is devoted to the proof of main theorem. We show the convergence of the original solution to its diffusion wave with optimal decay rates in the case either $\beta > 0$ or $\beta < 0$ with samll $|\beta|$. The adopted approach is the weighted energy method together with Fourier transform and Green function method. In the last section, we present some numerical simulations to confirm our theoretical results. In order to avoid the non-necessary oscillations for the numerical solutions to the hyperbolic p-system, we adopt the central-upwind scheme. In particular, the interest phenomenon of coexistence of the global solution $v(x,t)$ and the blow-up solution $u(x,t)$ is numerically demonstrated too.

Before precisely stating the related results and what the difficulty that we have to face, let us first derive the asymptotic state equations for the IBVP (1). By setting the following scalings to the variables

$$t = \bar{t}/\varepsilon^2, \quad x = \bar{x}/\varepsilon, \quad v = \bar{v}, \quad u = \varepsilon\bar{u}$$

for $0 < \varepsilon \ll 1$, we then scale the damped p-system (1) to the new system (still denote $\bar{t}$ and $\bar{x}$ as t and x, respectively)

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ \varepsilon^2 \bar{u}_t + p(\bar{v})_x = -\alpha\bar{u} - \beta\varepsilon^{q-1}|\bar{u}|^{q-1}\bar{u}. \end{cases}$$

Neglecting the small terms $\varepsilon^2\bar{u}_t$ and $-\beta\varepsilon^{q-1}|\bar{u}|^{q-1}\bar{u}$, we derive the asymptotic state equations for (1) and (2) as follows:

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha\bar{u}, \\ \bar{v}|_{t=0} = \bar{v}_0(x), \\ \bar{u}|_{x=0} = 0, \end{cases} \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{3}$$

where the initial data $\bar{v}_0(x)$ will be carefully selected later. This can be also expected by the Darcy's law. From the Dirichlet boundary condition $\bar{u}|_{x=0} = 0$ and the second equation of (3), we immediately have the Neumann boundary condition $\bar{v}_x|_{x=0} = 0$. It can be verified from (3) that

$$(\bar{v}, \bar{u})(x,t) \to (v_+, 0) \quad \text{as} \quad x \to +\infty. \tag{4}$$

Thus, (3) is equivalent to the (parabolic) porous media equation for $\bar{v}$

$$\begin{cases} \bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx}, \\ \bar{u} = -\frac{1}{\alpha} p(\bar{v})_x, \\ \bar{v}|_{t=0} = \bar{v}_0(x) \to v_+ \quad \text{as } x \to +\infty, \\ \bar{v}_x|_{x=0} = 0, \end{cases} \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{5}$$

In order to release $u_+ = 0$ to the general case $u_+ \neq 0$, as in [22] we will ingeniously construct a pair of the correction functions $(\hat{v}, \hat{u})(x,t)$ to eliminate the gap between $(v,u)(\infty,t)$ and $(\bar{v},\bar{u})(\infty,t)$. On the other hand, to find the best asymptotic profile, the selection of the initial data $\bar{v}_0(x)$ will also play a crucial role.

We are going to answer aforementioned questions. Firstly, we show how to set up the proper equations for the general case $u_+ \neq 0$. From the first equations of (1) and (3), we have

$$(v - \bar{v})_t - (u - \bar{u})_x = 0.$$

Integrating over $\mathbb{R}_+$ with respect to x and noting $u(0,t) = \bar{u}(0,t) = 0$ and $\bar{u}(\infty,t) = 0$, it yields

$$\frac{d}{dt}\int_0^\infty (v - \bar{v})(x,t)dx = (u - \bar{u})\Big|_{x=0}^{\infty} = u(\infty,t).$$

When $u_+ = 0$, obviously, we can expect $u(+\infty,t) = 0$, and then expect $u - \bar{u} \in L^2(\mathbb{R}_+)$. This is the reason for Jiang and Zhu [13] to assume $u_+ = 0$. However, when $u_+ \neq 0$, we no long have $u - \bar{u} \in L^2(\mathbb{R}_+)$ because of $u(+\infty,t) \neq 0$. In order to delete such a gap, we need to apply the procedure initially proposed by Hsiao and Liu [3] then later developed by Mei [22] to construct a pair of the correction functions $(\hat{v}, \hat{u})(x,t)$. To do this, inspired by Mei [22], we need to investigate $u(+\infty,t)$ first.

From the second equation of (1), the solution $u(+\infty,t)$ (denoted by $u^+(t)$) satisfies the following Bernoulli's equation

$$\begin{cases} \frac{d}{dt}u^+(t) = -\alpha u^+(t) - \beta|u^+(t)|^{q-1}u^+(t), \\ u^+(0) = u(+\infty,0) = u_0(+\infty) = u_+, \end{cases}$$

which can be solved explicitly as (see the Appendix in [22] for detail)

$$u(+\infty,t) = u^+(t) = \frac{u_+ e^{-\alpha t}}{\left(1 + \frac{\beta}{\alpha}|u_+|^{q-1}[1 - e^{-\alpha(q-1)t}]\right)^{1/(q-1)}}. \tag{6}$$

Notice that, when

$$\beta < 0 \quad \text{and} \quad |\beta| > \frac{\alpha}{|u_+|^{q-1}}, \tag{7}$$

the solution $u^+(t)$ will blow up at $t_* = \frac{1}{\alpha(q-1)} \ln \frac{|\beta||u_+|^{q-1}}{|\beta||u_+|^{q-1}-\alpha}$. So, in order to guarantee the global existence of $u^+(t)$, we need

$$\text{either } \beta > 0, \quad \text{or } \beta < 0 \text{ but } |\beta| < \frac{\alpha}{|u_+|^{q-1}}. \tag{8}$$

Hence, the gap between $u(\infty, t)$ and $\bar{u}(\infty, t) = 0$ is

$$u(\infty, t) - \bar{u}(\infty, t) = u^+(t) - 0 = O(1)|u_+|e^{-\alpha t},$$

which causes that $u - \bar{u}$ is not in $L^2(\mathbb{R}_+)$.To remove the gap, we construct the correction function $\hat{u}(x, t)$ as follows.

Let $\hat{u}(x, t)$ be such that

$$\begin{cases} \frac{d}{dt}\hat{u} = -\alpha\hat{u} - \beta|\hat{u}|^{q-1}\hat{u}, \ (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ \hat{u}|_{x=\infty} = u^+(t), \\ \hat{u}|_{x=0} = 0. \end{cases} \tag{9}$$

In the similar way of [22], $\hat{u}(x, t)$ can be constructed as

$$\hat{u}(x,t) = \frac{m(x)e^{-\alpha t}}{\Big(1 - \frac{\beta}{\alpha}[|m(x)|e^{-\alpha t}]^{q-1}\Big)^{1/(q-1)}}, \tag{10}$$

where $m(x)$ is an integration constant (with respect to t) given by

$$m(x) = C_+ \int_0^x m_0(y)dy, \qquad m(0) = 0, \qquad m(+\infty) = C_+. \tag{11}$$

Here,

$$C_+ = \frac{u_+}{\Big(1 + \frac{\beta}{\alpha}|u_+|^{q-1}\Big)^{1/(q-1)}}, \tag{12}$$

and $m_0(x)$ satisfies

$$m_0(x) \geq 0, \ m_0(0) = m_0(+\infty) = 0, \ m_0(x) \in C_0^\infty(\mathbb{R}_+), \ \text{ and } \int_{\mathbb{R}_+} m_0(x)dx = 1. \tag{13}$$

Furthermore, let $\hat{v}(x, t)$ be

$$\hat{v}(x,t) = \frac{-m'(x)e^{-\alpha t}}{\alpha\Big(1 - \frac{\beta}{\alpha}[|m(x)|e^{-\alpha t}]^{q-1}\Big)^{1/(q-1)}}. \tag{14}$$

Thus, the correction functions $(\hat{v}, \hat{u})(x, t)$ satisfy

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, \\ \hat{u}_t = -\alpha\hat{u} - \beta|\hat{u}|^{q-1}\hat{u}, \\ (\hat{v}, \hat{u})|_{x=+\infty} = (0, u^+(t)), \\ \hat{u}|_{x=0} = 0. \end{cases} \tag{15}$$

Now we are going to determine the best asymptotic profile $(\bar{v}, \bar{u})(x, t)$ by selecting a suitable initial data $\bar{v}_0(x)$. From $(1)_1$, $(3)_1$ and $(15)_1$, we have

$$(v - \bar{v} - \hat{v})_t - (u - \bar{u} - \hat{u})_x = 0.$$

Here, $(1)_1$ denotes the first equation in (1), and similar notation is used in this paper. Integrating this equation over $\mathbb{R}_+$ with respect to x, and noting that

$u(+\infty,t) = u^+(t)$, $\bar{u}(+\infty,t) = 0$, $\hat{u}(+\infty,t) = u^+(t)$ and $u(0,t) = \bar{u}(0,t) = \hat{u}(0,t) = 0$, we obtain

$$\frac{d}{dt}\int_0^\infty [v(x,t) - \bar{v}(x,t) - \hat{v}(x,t)]dx = 0.$$

Furthermore, integrating the above equation with respect to t yields

$$\int_0^\infty [v(x,t) - \bar{v}(x,t) - \hat{v}(x,t)]dx = \int_0^\infty [v_0(x) - \bar{v}_0(x) - \hat{v}(x,0)]dx. \tag{16}$$

Let us select the initial data $\bar{v}_0(x)$ such that

$$\int_0^\infty [v_0(x) - \bar{v}_0(x) - \hat{v}(x,0)]dx = 0, \tag{17}$$

for example, taking $\bar{v}_0(x) = v_0(x) - \hat{v}(x,0)$, then we have formally

$$\int_0^\infty [v(x,t) - \bar{v}(x,t) - \hat{v}(x,t)]dx = 0 \quad \text{for} \quad t \geq 0. \tag{18}$$

Thus, we can define some possible L^2-functions as

$$V(x,t) := -\int_x^\infty [v(y,t) - \bar{v}(y,t) - \hat{v}(y,t)]dy, \tag{19}$$

$$U(x,t) := u(x,t) - \bar{u}(x,t) - \hat{u}(x,t), \tag{20}$$

then, from (1)-(3) and (15), we reformulate the system as

$$\begin{cases} V_t - U = 0, \\ U_t + (p'(\bar{v})V_x)_x + \alpha U = -F_1 - F_2, \\ (V,U)|_{t=0} = (V_0,U_0)(x), \\ V|_{x=0} = 0, \end{cases} \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{21}$$

where

$$F_1 := \frac{1}{\alpha}p(\bar{v})_{xt} + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})V_x)_x, \tag{22}$$

$$\begin{aligned} F_2 &:= \beta|U + \bar{u} + \hat{u}|^{q-1}(U + \bar{u} + \hat{u}) - \beta|\hat{u}|^{q-1}\hat{u} \\ &= \beta|V_t + \bar{u} + \hat{u}|^{q-1}(V_t + \bar{u} + \hat{u}) - \beta|\hat{u}|^{q-1}\hat{u}, \end{aligned} \tag{23}$$

$$V_0(x) := -\int_x^\infty [v_0(y) - \bar{v}_0(y) - \hat{v}(y,0)]dy, \tag{24}$$

$$U_0(x) := u_0(x) - \bar{u}(x,0) - \hat{u}(x,0). \tag{25}$$

Before stating our main results, we introduce the following notations.

Notations. Throughout the paper, $C > 0$ denotes a generic constant which may change its value from line to line or even in the same line, while $C_i > 0$ ($i = 0,1,2,\cdots$) represents a specific constant. The partial derivatives of f are denoted by f_x, f_{xx}, and so on, or sometimes by $\partial_x^k f$, $k = 0,1,2,\cdots$. $L^p(\mathbb{R}_+)$ $(1 \leq p \leq \infty)$ is the usual Lebesgue space with the norm

$$\|f\|_{L^p} = \Big(\int_{\mathbb{R}_+} |f(x)|^p dx\Big)^{1/p} \quad \text{for} \quad 1 \leq p < \infty, \quad \text{and} \quad \|f\|_{L^\infty} = \sup_{x\in\mathbb{R}_+} |f(x)|,$$

where the integral region $\mathbb{R}_+$ will be omitted without any confusion. $L^{p,\gamma}(\mathbb{R}_+)$ with $\gamma > 0$ and $1 \leq p \leq \infty$ is the weighted $L^p(\mathbb{R}_+)$ space with a weight $(1+x)^\gamma$. Its norm is denoted as

$$\|f\|_{L^{p,\gamma}(\mathbb{R}_+)} = \Big(\int_{\mathbb{R}_+} (1+x)^\gamma |f(x)|^p dx\Big)^{1/p}, \qquad 1 \leq p \leq \infty.$$

$H^k(\mathbb{R}_+)$ $(k \geq 0)$ is the usual Sobolev space with the norm

$$\|f\|_{H^k} = \Big(\sum_{i=0}^{k} \int_{\mathbb{R}_+} |\partial_x^i f|^2 dx \Big)^{1/2}.$$

For the sake of simplicity, we also denote $\|(f,g,h)\|_{L^2}^2 = \|f\|_{L^2}^2 + \|g\|_{L^2}^2 + \|h\|_{L^2}^2$ and $\|(f,g,h)\|_{H^k}^2 = \|f\|_{H^k}^2 + \|g\|_{H^k}^2 + \|h\|_{H^k}^2$. Let $T > 0$ and let $\mathbf{B}$ be a Banach space. We denote by $C^0([0,T];\mathbf{B})$ the space of $\mathbf{B}$-valued continuous functions on $[0,T]$, and $L^2([0,T];\mathbf{B})$ as the space of $\mathbf{B}$-valued L^2-functions on $[0,T]$. The corresponding spaces of $\mathbf{B}$-valued functions on $[0,\infty)$ are defined similarly.

Now we state the convergence results.

Theorem 1.1. *Let β and u_+ satisfy (8), $q \geq 2$, and $\bar{v}_0(x)$ be chosen such that (17) holds, and $\bar{v}_0(x) - v_+ \in L^1(\mathbb{R}_+) \cap H^m(\mathbb{R}_+)$ with $m \geq 3$.*

(1) *If $(V_0, U_0) \in H^3(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$, when*

$$\max_{x\in\mathbb{R}_+} |\bar{v}_0 - v_+| + \|V_0\|_{H^3} + \|U_0\|_{H^2} + |u_+| \ll 1,$$

then the global solution $(V,U)(x,t)$ of (21) uniquely exists and satisfies

$$V(x,t) \in \bigcap_{k=0}^{2} C^k(0,\infty; H^{3-k}(\mathbb{R})), \quad U(x,t) \in \bigcap_{k=0}^{1} C^k(0,\infty; H^{2-k}(\mathbb{R})),$$

and

$$\|\partial_x^k V(t)\|_{L^2} = O(1)(1+t)^{-k/2}, \quad k = 0,1,2,3, \tag{26}$$

$$\|\partial_x^k U(t)\|_{L^2} = O(1)(1+t)^{-(k+2)/2}, \quad k = 0,1, \tag{27}$$

$$\|\partial_x^k V(t)\|_{L^\infty} = O(1)(1+t)^{-(2k+1)/4}, \quad k = 0,1,2, \tag{28}$$

$$\|U(t)\|_{L^\infty} = O(1)(1+t)^{-5/4}. \tag{29}$$

(2) *If $(V_0, U_0) \in (L^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+)) \times (L^1(\mathbb{R}_+) \cap H^1(\mathbb{R}_+))$, then*

$$\|\partial_x^k V(t)\|_{L^2} = O(1)(1+t)^{-(2k+1)/4}, \quad k = 0,1,2, \tag{30}$$

$$\|U(t)\|_{L^2} = O(1)(1+t)^{-5/4}, \tag{31}$$

$$\|\partial_x^k V(t)\|_{L^\infty} = O(1)(1+t)^{-(k+1)/2}, \quad k = 0,1, \tag{32}$$

$$\|U(t)\|_{L^\infty} = O(1)(1+t)^{-3/2}. \tag{33}$$

(3) *If $(V_0, U_0) \in (L^{1,\gamma}(\mathbb{R}_+) \cap H^2(\mathbb{R}_+)) \times (L^{1,\gamma}(\mathbb{R}_+) \cap H^1(\mathbb{R}_+))$ with $0 \leq \gamma \leq \frac{1}{4}$ (the best choice for γ is $\gamma = \frac{1}{4}$), then*

$$\|\partial_x^k V(t)\|_{L^2} = O(1)(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}, \quad k = 0,1,2, \tag{34}$$

$$\|U(t)\|_{L^2} = O(1)(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}, \tag{35}$$

$$\|\partial_x^k V(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{k+1}{2}-\frac{\gamma}{2}}, \quad k = 0,1, \tag{36}$$

$$\|U(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{3}{2}}. \tag{37}$$

Furthermore, notice that $V_x = v - \bar{v} - \hat{v}$, $U = u - \bar{u} - \hat{u}$, and use (10) and (14), i.e., $\|\hat{v}(t)\|_{L^\infty} = O(1)e^{-\alpha t}$ and $\|\hat{u}(t)\|_{L^\infty} = O(1)e^{-\alpha t}$, we immediately obtain the following optimal convergence to the diffusion wave in L^∞-space.

Corollary 1.2 (Convergence to diffusion wave)**.** *Under the conditions in Theorem 1.1, and $(V_0, U_0)(x) \in L^1(\mathbb{R}_+)$, it holds*

$$\|(v - \bar{v})(t)\|_{L^\infty} = O(1)(1+t)^{-1}, \tag{38}$$

$$\|(u - \bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{-3/2}. \tag{39}$$

Furthermore, $(V_0, U_0)(x) \in L^{1,\gamma}(\mathbb{R}_+)$ *with* $0 \le \gamma \le \frac{1}{4}$, *it holds*

$$\|(v - \bar{v})(t)\|_{L^\infty} = O(1)(1+t)^{-1-\frac{\gamma}{2}} = O(1)(1+t)^{-\frac{9}{8}}, \tag{40}$$

$$\|(u - \bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{3}{2}}. \tag{41}$$

Remark 1.3.

(i) *For the Cauchy problem case studied in* [22], *the condition* $(V_0, U_0) \in L^1$ *is not explicitly stated by Mei* [22]. *However, in order to obtain the better decay rates* (30)-(33) *in Theorem 1.1 (see (1.36) and (1.37) on pp. 1282,* [22]*), the condition must be imposed.*

(ii) *In* [22], *without any difficulty, the condition for* $\beta < 0$ *with* $|\beta| < \frac{\alpha}{2|u_\pm|^{q-1}}$ *can be released to* $|\beta| < \frac{\alpha}{|u_\pm|^{q-1}}$.

(iii) *From Theorem 1.1 and Corollary 1.2, we achieve the convergence rates as*

$$\|\partial_x^k V(t)\|_{L^2} = O(1)(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}, \quad k = 0, 1, 2,$$

with the best choice of $\gamma = \frac{1}{4}$ *(see* (87) *below) for* $q \ge 2$, *which are much better than the existing rates. But, unfortunately we cannot improve* $\|U(t)\|_{L^\infty} = \|V_t(t)\|_{L^\infty} = O(1)t^{-\frac{3}{2}}$ *to* $O(1)t^{-\frac{3}{2}-\frac{\gamma}{2}}$ *due to the slow decay of* $\bar{v}_{xt}$ *in the nonlinear term. These results are also true for the case* $\beta = 0$, *namely, the system* (1) *becomes the linear damping. We also note that, when* $\beta = 0$, *Said-Houari* [31] *claimed better decay rates for* $\gamma \in [0, 1]$,

$$\|\partial_x^k V(t)\|_{L^2} = O(1)(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}, \quad k = 0, 1, 2,$$
$$\|\partial_x^k U(t)\|_{L^2} = O(1)(1+t)^{-\frac{2k+3}{4}-\frac{\gamma}{2}}, \quad k = 0, 1,$$

especially, the case of $\frac{1}{4} < \gamma \le 1$. *However, this is not true, and the proof is incorrect. The author never checked how the nonlinear term decays, in particular, the term involving* $\bar{v}_{xt}$ *in the nonlinear term does not lead to any improved rates in* $L^{1,\gamma}(\mathbb{R}_+)$, *because* $\bar{v}(x,t)$ *is the solution to the corresponding porous media equation with the Neumann boundary condition, and the improved rate in the weighted* $L^{1,\gamma}(\mathbb{R}_+)$ *obtained by Ikehata* [12] *for the Dirichlet boundary case is failed for the Neumann boundary case. In another word, the decay rates of the nonlinear term does not decay as fast as we always expect. In fact, we have only* $\|\bar{v}_{xt}(t)\|_{L^1} = O(1)t^{-\frac{3}{2}}$ *and* $\|\bar{v}_{xt}(t)\|_{L^{1,\gamma}} = O(1)t^{-\frac{3}{2}+\frac{\gamma}{2}}$, *which are impossible to ensure the perturbed solution to decay faster as* $\|(u - \bar{u})(t)\|_{L^\infty} = O(1)t^{-\frac{3}{2}-\frac{\gamma}{2}}$. *For detail, we refer to Subsection 3.2 below.*

Remark 1.4. *When the parameters* β *and* u_+ *satisfy* (7), *namely,* $\beta < 0$ *and* $|\beta| > \frac{\alpha}{|u_+|^{q-1}}$, *from* (6), $u(+\infty, t)$ *will blow up at the finite time* t_*. *Thus, the solution* $u(x,t)$ *of* (1) *and* (2) *does not globally exist, and*

$$\lim_{t \to T^{*-}} \|u(t)\|_{L^\infty} = +\infty, \quad \textit{for } 0 < T^* \le t_*. \tag{42}$$

As showed in Section 4, the interesting numerical results presented in Figure 5 indicate that $u(x,t)$ *blows up at the finite time* T^*, *but* $v(x,t)$ *is bounded and never blows up.*

For the Cauchy problem case, we have the following example with a specific initial datum (which is suggested by Huang [9]*) to show that* $u(x,t)$ *will blow up, but not*

$v(x,t)$:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u - \beta |u|^{q-1} u, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \\ (v,u)|_{t=0} = (v_+, u_+), \end{cases}$$

Obviously, it possesses the unique solution

$$\begin{cases} v(x,t) = v_+, \\ u(x,t) = u_+ e^{-\alpha t} \Big(1 + \frac{\beta}{\alpha} |u_+|^{q-1} [1 - e^{-\alpha(q-1)t}] \Big)^{-1/(q-1)}. \end{cases}$$

It is clear that $v(x,t) = v_+$ is never blowing-up, but $u(x,t)$ will blow up at $t_* = \frac{1}{\alpha(q-1)} \ln \frac{|\beta||u_+|^{q-1}}{|\beta||u_+|^{q-1} - \alpha}$ for $\beta < 0$ and $|\beta| > \frac{\alpha}{|u_+|^{q-1}}$.

For the IBVP (1) and (2), it seems hard to construct an example to show the blowing-up for $u(x,t)$ and no blowing-up for $v(x,t)$, but from the following non-rigorous analysis, we may understand this phenomenon.

For $x \gg 1$, using the Taylor's expansion at t_*, we have

$$\begin{aligned} u(x,t) \approx u^+(t) &= u_+ e^{-\alpha t} \Big(1 + \frac{\beta}{\alpha} |u_+|^{q-1} [1 - e^{-\alpha(q-1)t}] \Big)^{-\frac{1}{q-1}} \\ &= u_+ e^{-\alpha t} \Big(1 + \frac{\beta}{\alpha} |u_+|^{q-1} [1 - e^{-\alpha(q-1)(t-t_*)} e^{-\alpha(q-1)t_*}] \Big)^{-\frac{1}{q-1}} \\ &\approx u_+ e^{-\alpha t_*} \Big(1 + \frac{\beta}{\alpha} |u_+|^{q-1} \Big[1 - [1 - \alpha(q-1)(t-t_*)] \\ &\qquad \times \frac{\beta |u_+|^{q-1} + \alpha}{\beta |u_+|^{q-1}} \Big] \Big)^{-\frac{1}{q-1}} \\ &= u_+ e^{-\alpha t_*} [(q-1)(t-t_*)(\beta |u_+|^{q-1} + \alpha)]^{-\frac{1}{q-1}}. \end{aligned}$$

So, it holds

$$|u(x,t)| \approx O(1) |t - t_*|^{-\frac{1}{q-1}},$$

which will blow up at t_*. Let us formally expect also

$$|u_x(x,t)| \approx O(1) |t - t_*|^{-\frac{1}{q-1}},$$

then from the first equation $v_t - u_x = 0$ of (1), we have

$$|v(x,t)| = \Big| v_0(x) + \int_0^t u_x(x,s) ds \Big| = O(1) \Big(1 + |t - t_*|^{1 - \frac{1}{q-1}} \Big),$$

which will not blow up for $1 - \frac{1}{q-1} \geq 0$, namely, $q \geq 2$.

However, as mentioned before, there is no rigorous proof for this interest case (the coexistence of the global solution v and the non-global (blow-up) solution u), so this case still remains open.

2. Preliminaries

In this section, we state some well-known results which will be usful for the proof of the convergence in Section 3.

First of all, we give the existence and the decay rates for the solution to the asymptotic profile equations (the IBVP (3)), which have been given in [20], see also [13].

Lemma 2.1 ([20, 13]). *Let $\bar{v}_0 - v_+ \in L^1(\mathbb{R}_+) \cap H^m(\mathbb{R}_+)$ for some positive integer m. Then the solution $(\bar{v}, \bar{u})(x,t)$ of the IBVP (3) globally and uniquely exists, and satisfies*

$$\|\partial_x^k \partial_t^j (\bar{v} - v_+)(t)\|_{L^2} = O(1)\delta_1 (1+t)^{-\frac{4j+2k+1}{4}}, \quad 0 \le 2k + j \le m, \tag{43}$$

$$\|\partial_x^k \partial_t^j (\bar{v} - v_+)(t)\|_{L^\infty} = O(1)\delta_1 (1+t)^{-\frac{2j+k+1}{2}}, \quad 0 \le 2k + j \le m, \tag{44}$$

$$\|(\bar{v} - v_+)_{xt}(t)\|_{L^1} = O(1)\delta_1 (1+t)^{-\frac{3}{2}}, \tag{45}$$

where $\delta_1 := \max_{x \in \mathbb{R}_+} |\bar{v}_0(x) - v_+|$.

Notice that the IBVP (5) is with the Neumann boundary. Different from the Dirichlet boundary case, even if the initial perturbation $\bar{v}_0 - v_+ \in L^{1,\gamma}(\mathbb{R}_+)$ $(0 \le \gamma \le 1)$, the solution $\bar{v}(x,t)$ does not converge to v_+ faster than in the case of $\bar{v}_0 - v_+ \in L^1(\mathbb{R}_+)$. In fact, as showed in [17], we can get a slower decay rate for the solution in the weighted $L^{1,\gamma}(\mathbb{R}_+)$ as follows.

Lemma 2.2 ([17]). *For $\gamma \in [0,1]$, it holds*

$$\|\partial_x^k \partial_t^j (\bar{v} - v_+)(t)\|_{L^{2,\gamma}} = O(1)\delta_1 (1+t)^{-\frac{4j+2k+1}{4}+\frac{\gamma}{4}}, \quad 0 \le 2k + j \le m, \tag{46}$$

$$\|\partial_x^k \partial_t^j (\bar{v} - v_+)(t)\|_{L^{\infty,\gamma}} = O(1)\delta_1 (1+t)^{-\frac{2j+k+1}{2}+\frac{\gamma}{2}}, \quad 0 \le 2k + j \le m, \tag{47}$$

$$\|(\bar{v} - v_+)_{xt}(t)\|_{L^{1,\gamma}} = O(1)\delta_1 (1+t)^{-\frac{3}{2}+\frac{\gamma}{2}}. \tag{48}$$

From (14) and $m'(x) = m_0(x) \in C_0^\infty(\mathbb{R}_+)$, we immediately obtain the following exponential decay for $\hat{v}(x,t)$.

Lemma 2.3. *For $\gamma \in [0,1]$, it holds that*

$$\|\partial_x^k \hat{v}(t)\|_{L^{2,\gamma}} = O(1)|u_+|e^{-\alpha t}, \quad k = 0, 1, 2, \cdots, \tag{49}$$

$$\|\partial_x^k \hat{v}(t)\|_{L^{\infty,\gamma}} = O(1)|u_+|e^{-\alpha t}, \quad k = 0, 1, 2, \cdots, \tag{50}$$

$$\|\partial_x^k \hat{v}(t)\|_{L^{1,\gamma}} = O(1)|u_+|e^{-\alpha t}, \quad k = 0, 1, 2, \cdots. \tag{51}$$

Furthermore, for the linear damped wave equation on the first quadrant with $\mu > 0$

$$\begin{cases} \phi_{tt} + \alpha\phi_t - \mu\phi_{xx} = g(x,t), & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (\phi, \phi_t)|_{t=0} = (\phi_0, \phi_1)(x), & x \in \mathbb{R}_+, \\ \phi|_{x=0} = 0, & t \in \mathbb{R}_+, \end{cases} \tag{52}$$

as showed in [20], its solution can be expressed as

$$\begin{aligned} \phi(x,t) &= \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\phi_0(y)dy \\ &\quad + \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\phi_1(y)dy \\ &\quad + \int_0^t \int_0^\infty [K_1(x-y,t-s) - K_1(x+y,t-s)]g(y,s)dyds, \end{aligned} \tag{53}$$

where $K_i(x,t)$ $(i = 0,1)$ are the fundamental solutions of the homogenous equation

$$\partial_{tt}K_i + \alpha\partial_t K_i - \mu\partial_{xx}K_i = 0, \quad x \in \mathbb{R}, \ t \in \mathbb{R}_+,$$

with

$$\begin{cases} K_0(x,0) = \delta(x) \\ \frac{\partial}{\partial t}K_0(x,0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} K_1(x,0) = 0 \\ \frac{\partial}{\partial t}K_1(x,0) = \delta(x), \end{cases}$$

where $\delta(x)$ is the Dirac-Delta function. The Fourier transforms of $K_i(x,t)$ $(i=0,1)$, denoted by $\hat{K}_i(\xi,t)$ $(i=0,1)$, are given explicitly by

$$\hat{K}_1(\xi,t)=\begin{cases}\frac{2e^{-\alpha t/2}}{\sqrt{\alpha^2-4\mu\xi^2}}\sinh\left(\frac{\sqrt{\alpha^2-4\mu\xi^2}}{2}t\right), & |\xi|<\frac{\alpha}{2\sqrt{\mu}}\\ te^{-\alpha t/2}, & |\xi|=\frac{\alpha}{2\sqrt{\mu}}\\ \frac{2e^{-\alpha t/2}}{\sqrt{4\mu\xi^2-\alpha^2}}\sin\left(\frac{\sqrt{4\mu\xi^2-\alpha^2}}{2}t\right), & |\xi|>\frac{\alpha}{2\sqrt{\mu}},\end{cases}$$

and

$$\hat{K}_0(\xi,t)=\frac{\alpha}{2}\hat{K}_1(\xi,t)+R_2(\xi,t),$$

where

$$R_2(\xi,t)=\begin{cases}e^{-\alpha t/2}\cosh\left(\frac{\sqrt{\alpha^2-4\mu\xi^2}}{2}t\right), & |\xi|<\frac{\alpha}{2\sqrt{\mu}}\\ e^{-\alpha t/2}, & |\xi|=\frac{\alpha}{2\sqrt{\mu}}\\ e^{-\alpha t/2}\cos\left(\frac{\sqrt{4\mu\xi^2-\alpha^2}}{2}t\right), & |\xi|>\frac{\alpha}{2\sqrt{\mu}}.\end{cases}$$

The decay rates are also obtained in [21] for the Cauchy problem, and extended in [20] for the IBVP.

Lemma 2.4 ([21, 20])**.** *If $f\in L^1(\mathbb{R}_+)\cap H^{j+k}(\mathbb{R}_+)$, then*

$$\left\|\partial_t^j\partial_x^k\int_0^\infty[K_1(x-y,t)-K_1(x+y,t)]f(y)dy\right\|_{L^2}\le C(1+t)^{-j-\frac{2k+1}{4}}\left[\|f\|_{L^1}+\|f\|_{H^{j+k-1}}\right],\tag{54}$$

$$\left\|\partial_t^j\partial_x^k\int_0^\infty[K_1(x-y,t)-K_1(x+y,t)]f(y)dy\right\|_{L^\infty}\le C(1+t)^{-j-\frac{k+1}{2}}\left[\|f\|_{L^1}+\|f\|_{H^{j+k}}\right].\tag{55}$$

If $f\in L^1(\mathbb{R}_+)\cap H^{j+k+1}(\mathbb{R}_+)$, then

$$\left\|\partial_t^j\partial_x^k\int_0^\infty[K_0(x-y,t)-K_0(x+y,t)]f(y)dy\right\|_{L^2}\le C(1+t)^{-j-\frac{2k+1}{4}}\left[\|f\|_{L^1}+\|f\|_{H^{j+k}}\right],\tag{56}$$

$$\left\|\partial_t^j\partial_x^k\int_0^\infty[K_0(x-y,t)-K_0(x+y,t)]f(y)dy\right\|_{L^\infty}\le C(1+t)^{-j-\frac{k+1}{2}}\left[\|f\|_{L^1}+\|f\|_{H^{j+k+1}}\right].\tag{57}$$

Furthermore, as shown in [12, 31], we have the following faster decays if the initial data belongs to the weighted space $L^{1,\gamma}(\mathbb{R}_+)$.

Lemma 2.5 ([12, 31])**.** *Let $\gamma\in[0,1]$. If $f\in L^{1,\gamma}(\mathbb{R}_+)\cap H^{j+k}(\mathbb{R}_+)$, then*

$$\left\|\partial_t^j\partial_x^k\int_0^\infty[K_1(x-y,t)-K_1(x+y,t)]f(y)dy\right\|_{L^2}\le C(1+t)^{-j-\frac{2k+1}{4}-\frac{\gamma}{2}}\left[\|f\|_{L^{1,\gamma}}+\|f\|_{H^{j+k-1}}\right],\tag{58}$$

$$\left\|\partial_t^j\partial_x^k\int_0^\infty[K_1(x-y,t)-K_1(x+y,t)]f(y)dy\right\|_{L^\infty}\le C(1+t)^{-j-\frac{k+1}{2}-\frac{\gamma}{2}}\left[\|f\|_{L^{1,\gamma}}+\|f\|_{H^{j+k}}\right].\tag{59}$$

If $f \in L^{1,\gamma}(\mathbb{R}_+) \cap H^{j+k+1}(\mathbb{R}_+)$, *then*

$$\Big\| \partial_t^j \partial_x^k \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)] f(y) dy \Big\|_{L^2} \leq C(1+t)^{-j-\frac{2k+1}{4}-\frac{\gamma}{2}} \Big[\|f\|_{L^{1,\gamma}} + \|f\|_{H^{j+k}} \Big], \tag{60}$$

$$\Big\| \partial_t^j \partial_x^k \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)] f(y) dy \Big\|_{L^\infty} \leq C(1+t)^{-j-\frac{k+1}{2}-\frac{\gamma}{2}} \Big[\|f\|_{L^{1,\gamma}} + \|f\|_{H^{j+k+1}} \Big]. \tag{61}$$

Finally, we give a well-known and useful auxiliary lemma as follows.

Lemma 2.6 ([32]). *Let* $a > 0$, $b > 0$. *Then*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b} ds \leq \begin{cases} C(1+t)^{-\min(a,b)}, & \max(a,b) > 1, \\ C(1+t)^{-\min(a,b)} \ln(2+t), & \max(a,b) = 1, \\ C(1+t)^{1-a-b}, & \max(a,b) < 1. \end{cases} \tag{62}$$

3. Proof of Theorem 1.1

As shown in [22] for the Cauchy problem case, we can similarly prove the convergence rates (26)-(29) by the elementary energy method. Here, we omit the detail for the proof. The proof (30)-(37) will be our main contribution in this section. Using the energy method with Fourier transform together (c.f. [20]), We derive the convergence rates (30)-(33) and (34)-(37) for the initial perturbation in L^1 and $L^{1,\gamma}$, respectively.

3.1. Proof of (30)-(33)**, the second part of Theorem 1.1.** First of all, substituting $U = V_t$ from the first equation of (21) to the second equation of (21), we then reduce (21) to the following damped wave equation in the quadrant:

$$\begin{cases} V_{tt} + \alpha V_t - \mu V_{xx} = G, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (V, V_t)(x,0) = (V_0, U_0)(x), & x \in \mathbb{R}_+, \\ V(0,t) = 0, \end{cases} \tag{63}$$

where $\mu = -p'(v_+) > 0$ and

$$G(x,t) := -F_1 - F_2 - \{[p'(\bar{v}) - p'(v_+)]V_x\}_x. \tag{64}$$

From (52) and (53), the solution of (63) can be expressed in the integral form as follows

$$\begin{aligned} V(x,t) = & \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)] V_0(y) dy \\ & + \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)] U_0(y) dy \\ & + \int_0^t \int_0^\infty [K_1(x-y,t-s) - K_1(x+y,t-s)] G(y,s) dy ds. \end{aligned} \tag{65}$$

Let $T > 0$, we define the solution space for (65) as

$$X(0,T) := \{V(x,t) | V \in C(0,T; H^2(\mathbb{R}_+)), V_t \in C(0,T; L^\infty(\mathbb{R}_+))\} \tag{66}$$

and the measure $N(T)$ by

$$N(T) = \sup_{0\le t\le T}\Big\{\sum_{k=0}^{2}(1+t)^{\frac{2k+1}{4}}\|\partial_x^k V(t)\|_{L^2}+\sum_{k=0}^{1}(1+t)^{\frac{k+1}{2}}\|\partial_x^k V(t)\|_{L^\infty} +(1+t)^{\frac{5}{4}}\|V_t(t)\|_{L^2}+(1+t)^{\frac{3}{2}}\|V_t(t)\|_{L^\infty}\Big\}. \tag{67}$$

The local existence of the solution to the integral equation (65) can be obtained by the standard iteration technique. We omit the details. The crucial part for the proof of the global existence is to establish *a priori* estimate (see Lemma 3.3 below). Based on the local existence and *a priori* estimate, by using the continuous-extension technique, we can prove the global existence of the solution $V(x,t)$ with the sharp decay rates, which implies Theorem 1.1.

Let

$$\eta := \delta_1+|u_+|+\|V_0\|_{L^1}+\|V_0\|_{H^3}+\|U_0\|_{L^1}+\|U_0\|_{H^2}. \tag{68}$$

By applying Lemma 2.4, we immediately have the following estimates.

Lemma 3.1. *Let* $V_0\in L^1(\mathbb{R}_+)\times H^2(\mathbb{R}_+)$ *and* $U_0\in L^1(\mathbb{R}_+)\times H^1(\mathbb{R}_+)$. *Then*

$$\Big\|\partial_t^j\partial_x^k\int_0^\infty\Big[K_0(x-y,t)-K_0(x+y,t)\Big]V_0(y)dy\Big\|_{L^2} \le C(1+t)^{-j-\frac{2k+1}{4}}\Big[\|V_0\|_{L^1}+\|V_0\|_{H^2}\Big],\quad 0\le 2j+k\le 2, \tag{69}$$

$$\Big\|\partial_t^j\partial_x^k\int_0^\infty\Big[K_1(x-y,t)-K_1(x+y,t)\Big]U_0(y)dy\Big\|_{L^2} \le C(1+t)^{-j-\frac{2k+1}{4}}\Big[\|U_0\|_{L^1}+\|U_0\|_{H^1}\Big],\quad 0\le 2j+k\le 2, \tag{70}$$

$$\Big\|\partial_t^j\partial_x^k\int_0^\infty\Big[K_0(x-y,t)-K_0(x+y,t)\Big]V_0(y)dy\Big\|_{L^\infty} \le C(1+t)^{-j-\frac{k+1}{2}}\Big[\|V_0\|_{L^1}+\|V_0\|_{H^2}\Big],\quad 0\le j+k\le 1, \tag{71}$$

$$\Big\|\partial_t^j\partial_x^k\int_0^\infty\Big[K_1(x-y,t)-K_1(x+y,t)\Big]U_0(y)dy\Big\|_{L^\infty} \le C(1+t)^{-j-\frac{k+1}{2}}\Big[\|U_0\|_{L^1}+\|U_0\|_{H^1}\Big],\quad 0\le j+k\le 1. \tag{72}$$

The main goal in this subsection is to establish the following estimates.

Lemma 3.2. *Let* $V(x,t)\in X(0,T)$. *Then*

$$\int_0^t\Big\|\partial_t^j\partial_x^k\int_0^\infty\Big[K_1(x-y,t-s)-K_1(x+y,t-s)\Big]G(y,s)dy\Big\|_{L^2}ds \le C\eta[1+N(T)](1+t)^{-j-\frac{2k+1}{4}},\quad 0\le 2j+k\le 2, \tag{73}$$

$$\int_0^t\Big\|\partial_t^j\partial_x^k\int_0^\infty\Big[K_1(x-y,t-s)-K_1(x+y,t-s)\Big]G(y,s)dy\Big\|_{L^\infty}ds \le C\eta[1+N(T)](1+t)^{-j-\frac{k+1}{2}},\quad 0\le j+k\le 1. \tag{74}$$

Proof. From (22), (23), (64) and by the Taylor's expansion, we obtain

$$|G| \sim O(1)\Big(|\bar v_{xt}|+|\{(V_x+\hat v)^2\}_x|+|\bar v_x\hat v|+|\hat v_x|+|\hat u| +|V_t|^q+|\bar u|^q+|\hat u|^{q-1}|V_t|+|\hat u|^{q-1}|\bar u|+|\{(\bar v-v_+)V_x\}_x|\Big) \tag{75}$$

and

$$(76)\quad |\partial_x^k G| \sim O(1)\Big(|\partial_x^k \bar{v}_{xt}| + |\partial_x^k\{(V_x+\hat{v})^2\}_x| + |\partial_x^k(\bar{v}_x\hat{v})| + |\partial_x^k\hat{v}_x| + |\partial_x^k\hat{u}| + |\partial_x^k(V_t^q)| + |\partial_x^k(\bar{u}^q)| + |\partial_x^k(\hat{u}^{q-1}V_t)| + |\partial_x^k(\hat{u}^{q-1}\bar{u})| + |\partial_x^k\{(\bar{v}-v_+)V_x\}_x|\Big).$$

Noting the decay rates (43)-(45) for $\bar{v}-v_+$, the exponential decay rates (49)-(51) for $\hat{v}(x,t)$, and $V(x,t)\in X(0,T)$, namely,

$$\sum_{2j+k=0}^{2}(1+t)^{j+\frac{2k+1}{4}}\|\partial_t^j\partial_x^k V(t)\|_{L^2} + \sum_{j+k=0}^{1}(1+t)^{j+\frac{k+1}{2}}\|\partial_t^j\partial_x^k V(t)\|_{L^\infty} \le N(T),$$

then by using the Hölder inequality, we have

$$\begin{aligned}
(77)\qquad \|G(t)\|_{L^1} \le\ & C\Big\{\|\bar{v}_{xt}(t)\|_{L^1} \\
& + \Big(\|V_x(t)\|_{L^2} + \|\hat{v}(t)\|_{L^2}\Big)\Big(\|V_{xx}(t)\|_{L^2} + \|\hat{v}_x(t)\|_{L^2}\Big) \\
& + \|\bar{v}_x(t)\|_{L^2}\|\hat{v}(t)\|_{L^2} + \|\hat{v}_x(t)\|_{L^1} \\
& + \|V_t(t)\|_{L^\infty}^{q-2}\|V_t(t)\|_{L^2}^2 + \|\bar{u}(t)\|_{L^\infty}^{q-2}\|\bar{u}(t)\|_{L^2}^2 \\
& + \|\hat{u}(t)\|_{L^\infty}^{q-1}\|V_t(t)\|_{L^1} + \|\hat{u}(t)\|_{L^\infty}^{q-1}\|\bar{u}(t)\|_{L^1} \\
& + \|(\bar{v}-v_+)(t)\|_{L^2}\|V_{xx}(t)\|_{L^2} \\
& + \|(\bar{v}-v_+)_x(t)\|_{L^2}\|V_x(t)\|_{L^2}\Big\} \\
\le\ & C\eta[1+N(T)]\Big\{(1+t)^{-\frac{3}{2}} \\
& + [(1+t)^{-\frac{3}{4}} + e^{-\alpha t}][(1+t)^{-\frac{5}{4}} + e^{-\alpha t}] \\
& + (1+t)^{-\frac{3}{4}}e^{-\alpha t} + e^{-\alpha t} + (1+t)^{-\frac{3(q-2)}{2}}(1+t)^{-\frac{5}{2}} \\
& + (1+t)^{-(q-2)}(1+t)^{-\frac{3}{2}} + e^{-\alpha t(q-1)} + e^{-\alpha t(q-1)} \\
& + (1+t)^{-\frac{1}{4}}(1+t)^{-\frac{5}{4}} + (1+t)^{-\frac{3}{4}}(1+t)^{-\frac{3}{4}}\Big\} \\
\le\ & C\eta[1+N(T)](1+t)^{-\frac{3}{2}}.
\end{aligned}$$

Similarly, by a straightforward but tedious calculation, it gives

$$(78)\qquad \|G(t)\|_{H^1} \le C\eta[1+N(T)](1+t)^{-\frac{3}{2}}.$$

Thus, by using Lemma 2.4 and Lemma 2.6 and noting $3/2 > (2k+1)/4$ for $k = 0,1,2$, we have

$$\begin{aligned}
(79)\qquad & \int_0^t \Big\|\partial_t^j\partial_x^k\int_0^\infty \Big[K_1(x-y,t-s) - K_1(x+y,t-s)\Big]G(y,s)dy\Big\|_{L^2} ds \\
& \le C\int_0^t (1+t-s)^{-j-\frac{2k+1}{4}}\Big[\|G(s)\|_{L^1} + \|G(s)\|_{H^1}\Big]ds \\
& \le C\eta[1+N(T)]\int_0^t (1+t-s)^{-j-\frac{2k+1}{4}}\Big[(1+s)^{-\frac{3}{2}} + (1+s)^{-\frac{3}{2}}\Big]ds \\
& \le C\eta[1+N(T)](1+t)^{-j-\frac{2k+1}{4}}, \quad 0\le 2j+k\le 2.
\end{aligned}$$

This completes the proof of (73).

Similarly, we prove (74) as follows

$$\begin{aligned}
(80)\qquad & \int_0^t \Big\|\partial_t^j\partial_x^k\int_0^\infty \Big[K_1(x-y,t-s)-K_1(x+y,t-s)\Big]G(y,s)dy\Big\|_{L^\infty}ds \\
&\le C\int_0^t (1+t-s)^{-j-\frac{k+1}{2}}\Big[\|G(s)\|_{L^1}+\|G(s)\|_{H^1}\Big]ds \\
&\le C\eta[1+N(T)]\int_0^t (1+t-s)^{-j-\frac{k+1}{2}}\Big[(1+s)^{-\frac{3}{2}}+(1+s)^{-\frac{3}{2}}\Big]ds \\
&\le C\eta[1+N(T)](1+t)^{-j-\frac{k+1}{2}},\quad 0\le j+k\le 1.
\end{aligned}$$

The proof is completed. □

Applying Lemmas 3.1 and 3.2 to Eq.(65), we immediately establish *a priori* estimate for $V(x,t)$.

Lemma 3.3 (*A priori* estimates)**.** *It holds that*

$$\|\partial_t^j\partial_x^k V(t)\|_{L^2}\le C(1+t)^{-j-\frac{2k+1}{4}},\quad 0\le 2j+k\le 2, \tag{81}$$

$$\|\partial_t^j\partial_x^k V(t)\|_{L^\infty}\le C(1+t)^{-j-\frac{k+1}{2}},\quad 0\le j+k\le 1, \tag{82}$$

provided $\eta+N(T)\ll 1$*.*

3.2. Proof of (34)-(37)**, the third part of Theorem 1.1.** Let

$$N_\gamma(T)=\sup_{0\le t\le T}\Big\{\sum_{2j+k=0}^{2}(1+t)^{j+\frac{2k+1}{4}+\frac{\gamma}{2}}\|\partial_t^j\partial_x^k V(t)\|_{L^2}+\sum_{k=0}^{1}(1+t)^{\frac{k+1}{2}+\frac{\gamma}{2}}\|\partial_x^k V(t)\|_{L^\infty}\Big\}.$$

As showed by Ma-Mei in [17] for the linear damping case (see Theorem 2.2, (2.20) in [17]), we can similarly prove that, for $\gamma\in[0,\frac{1}{2}]$,

$$\|\partial_t^j\partial_x^k V(t)\|_{L^{2,\gamma}}\le C(1+t)^{-(j+k)/2},\quad 0\le j+k\le 2. \tag{83}$$

The proof is standard but long, and we omit the detail. Now, we can further prove

$$\|\partial_t^j\partial_x^k V(t)\|_{L^{2,\gamma}}\le C(1+t)^{-j-\frac{2k+1}{4}+\frac{\gamma}{4}},\quad 0\le j+k\le 2. \tag{84}$$

if $\|\partial_t^j\partial_x^k V(t)\|_{L^2}\le C(1+t)^{-j-\frac{2k+1}{4}}$.

Now, let $V(x,t)\in X(0,T)$ with the norm $N_\gamma(t)$. From (75) and using integration by parts and the Hölder inequality, and applying Lemmas 2.2 and 2.3, we have, for

$\gamma \in [0,1)$,

$$
\begin{aligned}
(85)\quad \|G(t)\|_{L^{1,\gamma}} \quad \le\; & C\Big(\|\bar v_{xt}(t)\|_{L^{1,\gamma}} \\
& +(\|V_x(t)\|_{L^{2,\frac{\gamma}{2}}}+\|\hat v(t)\|_{L^{2,\frac{\gamma}{2}}})(\|V_{xx}(t)\|_{L^{2,\frac{\gamma}{2}}}+\|\hat v_x(t)\|_{L^{2,\frac{\gamma}{2}}}) \\
& +\|\bar v_x(t)\|_{L^\infty}\|\hat v(t)\|_{L^{1,\gamma}}+\|\hat v_x(t)\|_{L^{1,\gamma}} \\
& +\|V_t(t)\|_{L^\infty}^{q-2}\|V_t(t)\|_{L^{2,\gamma}}^2+\|\bar u(t)\|_{L^\infty}^{q-2}\|\bar u(t)\|_{L^{2,\gamma}}^2 \\
& +\|\hat u(t)\|_{L^\infty}^{q-1}\|V_t(t)\|_{L^{1,\gamma}}+\|\hat u(t)\|_{L^\infty}^{q-1}\|\bar u(t)\|_{L^{1,\gamma}} \\
& +\|(\bar v-v_+)(t)\|_{L^{2,\frac{\gamma}{2}}}\|V_{xx}(t)\|_{L^{2,\frac{\gamma}{2}}} \\
& +\|(\bar v-v_+)(t)\|_{L^{2,\frac{\gamma}{2}}}\|V_x(t)\|_{L^{2,\frac{\gamma}{2}}}\Big) \\
\le\; & C\eta[1+N_\gamma(t)]\Big[(1+t)^{-\frac{3}{2}+\frac{\gamma}{2}} \\
& +[(1+t)^{-\frac{3}{4}+\frac{\gamma}{8}}+e^{-\alpha t}][(1+t)^{-\frac{5}{4}+\frac{\gamma}{8}}+e^{-\alpha t}] \\
& +(1+t)^{-\frac{3}{4}}e^{-\alpha t}+e^{-\alpha t} \\
& +(1+t)^{-\frac{3}{2}(q-2)}(1+t)^{-\frac{5}{2}+\frac{\gamma}{2}}+(1+t)^{-(q-2)}(1+t)^{-\frac{3}{2}+\frac{\gamma}{2}} \\
& +e^{-\alpha(q-1)t}+e^{-\alpha(q-1)t}(1+t)^{-\frac{1}{2}+\frac{\gamma}{2}} \\
& +(1+t)^{-\frac{1}{4}+\frac{\gamma}{8}}(1+t)^{-\frac{5}{4}+\frac{\gamma}{8}}+(1+t)^{-\frac{3}{4}+\frac{\gamma}{8}}(1+t)^{-\frac{3}{4}+\frac{\gamma}{8}}\Big] \\
\le\; & C\eta[1+N_\gamma(T)](1+t)^{-\frac{3}{2}+\frac{\gamma}{2}}.
\end{aligned}
$$

Thus, from (65) and by Lemma 2.5 and (85), we have

$$
\begin{aligned}
(86)\quad & \|\partial_x^k V(t)\|_{L^2} \\
& \le \Big\|\partial_x^k\int_0^\infty[K_0(x-y,t)-K_0(x+y,t)]V_0(y)dy\Big\|_{L^2} \\
& \quad +\Big\|\partial_x^k\int_0^\infty[K_1(x-y,t)-K_1(x+y,t)]U_0(y)dy\Big\|_{L^2} \\
& \quad +\int_0^t\Big\|\partial_x^k\int_0^\infty[K_1(x-y,t-s)-K_1(x+y,t-s)]G(y,s)dy\Big\|_{L^2}ds \\
& \le C(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}(\|V_0\|_{L^{1,\gamma}}+\|V_0\|_{H^2}) \\
& \quad +C(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}(\|U_0\|_{L^{1,\gamma}}+\|U_0\|_{H^1}) \\
& \quad +C\int_0^t(1+t-s)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}[\|G(s)\|_{L^{1,\gamma}}+\|G(s)\|_{H^1}]ds \\
& \le C(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}(\|V_0\|_{L^{1,\gamma}}+\|V_0\|_{H^2}) \\
& \quad +C(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}(\|U_0\|_{L^{1,\gamma}}+\|U_0\|_{H^1}) \\
& \quad +C\eta[1+N_\gamma(T)]\int_0^t(1+t-s)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}[(1+s)^{-\frac{3}{2}+\frac{\gamma}{2}}+(1+s)^{-\frac{3}{2}}]ds \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k=0,1,2.
\end{aligned}
$$

Let us balance the orders in the last term of (86):

$$-\frac{2k+1}{4}-\frac{\gamma}{2}=-\frac{3}{2}+\frac{\gamma}{2},\qquad \text{for the largest case } k=2,$$

it gives

$$\gamma=\frac{1}{4}. \tag{87}$$

Applying Lemma 2.6 to (86), we obtain

$$\|\partial_x^k V(t)\|_{L^2} \le C\bar{\delta}(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}, \quad k=0,1,2, \quad \gamma=\frac{1}{4}, \tag{88}$$

where

$$\bar{\delta} := \|V_0\|_{L^{1,\gamma}} + \|U_0\|_{L^{1,\gamma}} + \|V_0\|_{H^2} + \|U_0\|_{H^1} + \eta[1+N_\gamma(T)].$$

Similarly, we have

$$\begin{aligned}
&\|V_t(t)\|_{L^2} \\
&\le \left\|\partial_t \int_0^\infty [K_0(x-y,t)-K_0(x+y,t)]V_0(y)dy\right\|_{L^2} \\
&\quad + \left\|\partial_t \int_0^\infty [K_1(x-y,t)-K_1(x+y,t)]U_0(y)dy\right\|_{L^2} \\
&\quad + \int_0^t \left\|\partial_t \int_0^\infty [K_1(x-y,t-s)-K_1(x+y,t-s)]G(y,s)dy\right\|_{L^2} ds \\
&\le C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}(\|V_0\|_{L^{1,\gamma}} + \|V_0\|_{H^2}) \\
&\quad + C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}(\|U_0\|_{L^{1,\gamma}} + \|U_0\|_{H^1}) \\
&\quad + C\int_0^t (1+t-s)^{-\frac{5}{4}-\frac{\gamma}{2}}[\|G(s)\|_{L^{1,\gamma}} + \|G(s)\|_{H^1}]ds \\
&\le C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}(\|V_0\|_{L^{1,\gamma}} + \|V_0\|_{H^2}) \\
&\quad + C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}(\|U_0\|_{L^{1,\gamma}} + \|U_0\|_{H^1}) \\
&\quad + C\eta[1+N_\gamma(T)]\int_0^t (1+t-s)^{-\frac{5}{4}-\frac{\gamma}{2}}[(1+s)^{-\frac{3}{2}+\gamma} + (1+s)^{-\frac{3}{2}}]ds \\
&\le C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}(\|V_0\|_{L^{1,\gamma}} + \|V_0\|_{H^2}) \\
&\quad + C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}(\|U_0\|_{L^{1,\gamma}} + \|U_0\|_{H^1}) \\
&\quad + C\eta[1+N_\gamma(T)](1+t)^{-\frac{5}{4}-\frac{\gamma}{2}} + C[|v_+-v_-| + N_\gamma(T)](1+t)^{-\frac{3}{2}} \\
&\le C\bar{\delta}(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}.
\end{aligned} \tag{89}$$

However, for the decay rate of $\|V_t(t)\|_{L^\infty}$, even if $(V_0,U_0)\in L^{1,\gamma}$, we cannot expect it to be $O(1)(1+t)^{-\frac{3}{2}-\frac{\gamma}{2}}$. In fact, by a similar calculation as before, the decay rate will be $\|V_t(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{3}{2}+\gamma}$, which is even less than the rate $\|V_t(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{3}{2}}$ we obtained in (33). The reason for such a deficiency is from the slower decay term $\|G(t)\|_{L^{1,\gamma}} = O(1)(1+t)^{-\frac{3}{2}+\gamma} = O(1)(1+t)^{-\frac{4}{3}}$. So, in this case, we take

$$\|V_t(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{3}{2}} \tag{90}$$

as obtained in (33).

4. Numerical Computations

In this section, we carry out numerical simulations, which will confirm the theoretical results. First of all, we adopt the semi-discrete central upwind scheme [15] numerically to solve the system (1) and (2) in subsection 4.1, then we demonstrate numerically that the solution of the p-system converges to the corresponding diffusion wave, when the parameter β is positive, or negative and small in subsection 4.2; otherwise, the solution of p-system blows up to infinite in finite time as shown in subsection 4.3.

4.1. Semi-discrete central upwind scheme. The p-system (1) and (2) is a nonlinear hyperbolic system with source term and the porous media equation (3) and (5) is a parabolic equation. The computational results reported in this section are based on the semi-discrete central upwind scheme together with a second order ODE solver [15] for the p-system, and explicit finite difference methods for the porous media equation. The advantage of this numerical scheme is the simplicity and no approximate Riemann solver is needed. Here, we give a brief description on the construction of the central-upwind scheme as follows.

We design the semi-discrete central upwind scheme for the p-system (1) and (2) which is a nonlinear hyperbolic system with source term. The system can be viewed as a system of hyperbolic conservation laws with source term:

$$\mathbf{U}_t = \mathbf{f}(\mathbf{U})_x + \mathbf{R}(\mathbf{U}), \tag{91}$$

where $\mathbf{U} := (v, u)$, $\mathbf{f}(\mathbf{U}) := (u, -p(v))$ and $\mathbf{R}(\mathbf{U}) := (0, -\alpha u - \beta |u|^{q-1} u)$. The eigenvalues of the Jacobian of $\mathbf{f}$ are $\pm\sqrt{-p'(v)}$.

The common method to solve systems of hyperbolic conservation laws with source terms is splitting method. Generally speaking, one needs to pay more attention on boundary treatment for the splitting method. In this paper, we use the semi-discrete central-upwind scheme as first introduced in [15]. This scheme is simple and robust because no approximate Riemann solver is needed. Furthermore, the semi-discrete central-upwind schemes employ less numerical viscosity which makes it possible to solve problems for longer computed time.

We start our construction with the following notation: $x_j = jdx$ and $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, where Δx is the spatial size. The cell average is

$$\bar{\mathbf{U}}_j = \frac{1}{dx} \int_{I_j} \mathbf{U}(x,t)dx. \tag{92}$$

The *semi-discrete central-upwind scheme* can then be obtained in the following flux form:

$$\frac{d}{dt}\bar{\mathbf{U}}_j(t) = -\frac{\mathbf{H}^x_{j+\frac{1}{2}}(t) - \mathbf{H}^x_{j-\frac{1}{2}}(t)}{\Delta x} + \mathbf{R}_j, \tag{93}$$

where the *second-order numerical fluxes* are:

$$\mathbf{H}^x_{j+\frac{1}{2}}(t) := \frac{\mathbf{f}(\mathbf{U}^{\mathrm{E}}_j) + \mathbf{f}(\mathbf{U}^{\mathrm{W}}_{j+1})}{2} + \frac{a_{j+\frac{1}{2}}}{2}\left[\mathbf{U}^{\mathrm{W}}_{j+1} - \mathbf{U}^{\mathrm{E}}_j\right]. \tag{94}$$

Here $\mathbf{U}^{\mathrm{E,W}}_j$ are the point values of the piecewise linear reconstruction

$$\tilde{\mathbf{U}}(x) := \bar{\mathbf{U}}_j + (\mathbf{U}_x)_j(x - x_j), \qquad x \in I_j \tag{95}$$

at points $x_{j+\frac{1}{2}}$ and $x_{j-\frac{1}{2}}$. That is,

$$\mathbf{U}^{\mathrm{E}}_j = \bar{\mathbf{U}}_j + \frac{\Delta x}{2}(\mathbf{U}_x)_j, \tag{96}$$

$$\mathbf{U}^{\mathrm{W}}_j = \bar{\mathbf{U}}_j - \frac{\Delta x}{2}(\mathbf{U}_x)_j, \tag{97}$$

where the numerical slope $\mathbf{U}_x$ can be evaluated by the following minmod reconstruction:

$$(\mathbf{U}_x)_j = \mathrm{minmod}\Big(\frac{\bar{\mathbf{U}}_j - \bar{\mathbf{U}}_{j-1}}{\Delta x}, \frac{\bar{\mathbf{U}}_{j+1} - \bar{\mathbf{U}}_{j-1}}{2\Delta x}, \frac{\bar{\mathbf{U}}_{j+1} - \bar{\mathbf{U}}_j}{\Delta x}\Big). \tag{98}$$

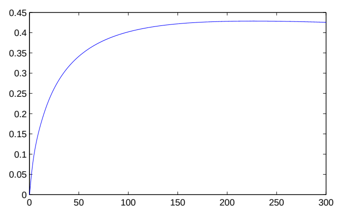
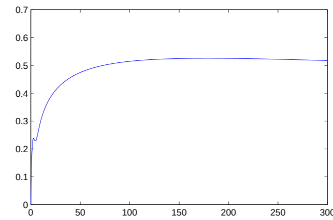
(a) 
(b) 

FIGURE 1. Convergence rates of the p-system solution to the corresponding diffusion wave with $\beta = 4$. (a) is for $F_v(t) := \frac{\|(v-\bar{v})(t)\|_{L^\infty}}{(1+t)^{-9/8}}$ and (b) is for $G_u(t) := \frac{\|(u-\bar{u})(t)\|_{L^\infty}}{(1+t)^{-13/8}}$

The local speed $a_{j+\frac{1}{2}}$ in (94) can be determined by

$$a_{j+\frac{1}{2}} = \max\left(\sqrt{-p'(v_j)}\ ,\sqrt{-p'(v_{j+1})}\right) \tag{99}$$

and the stability criterion is

$$\max(a_{j+\frac{1}{2}})\Delta t \leq \frac{1}{2}\Delta x \tag{100}$$

Although the original model assumes the spatial domain to be the half domain, a finite computational domain $[0, L]$ is imposed. Here, we let $L = 800$, so that the computational domain is sufficiently large and the effect due to the numerical boundary at $x = L$ can be ignored. In each of the following subsections, the initial conditions for p-system are set to $v_0(x) = 2 - \operatorname{sech}(0.1x)$ and $u_0(x) = \tanh(0.1x)$ and the sizes of the space step is chosen as $\Delta x = 0.01$. Note that the right state $u_+ = 1$. Here, the value of q is taken as $q = 2$.

Finally, in our scheme, the ODE system (93) is solved by a second order Runge-Kutta solver. It is also possible to solve the ODEs by other methods. For example, the implicit-explicit Runge-Kutta solver. See [30, 1].

4.2. Positive and small negative β's. In this subsection, we take $\beta = 4$ and $\beta = -0.01$, respectively, which satisfy (8). Hence, the p-system solution is expected to converge to its corresponding diffusion wave. For solving the p-system, the time step is chosen as $\Delta t = 0.001$. When solving the porous media equation, we let $\Delta t = 0.00002$ so that the numerical stability condition is satisfied. The initial condition for the porous media equation is chosen as $\bar{v}(x, 0) = 2 - \operatorname{sech}(0.1x) - \hat{v}(x, 0)$, $\hat{v}(x, 0) = \frac{-2C_+xe^{-x^2}}{\sqrt{1-\beta(C_+(1-e^{-x^2}))^2}}$ and $C_+ = 1/\sqrt{1+\beta}$.

The numerical simulations presented in Figures 1, 2 confirm Corollary 1.3 that the p-system solution converges to the diffusion wave with the optimal rates when measured under the infinite norm. The optimal convergence rate is $(1+t)^{-1-\frac{\gamma}{2}} = (1+t)^{-\frac{9}{8}}$ for $v(x,t) - \bar{v}(x,t)$. Although we show in Theorem 1.1 that $\|(u - \bar{u})(t)\|_{L^\infty} \leq C(1+t)^{-3/2}$ due to the technical reason for the slow decay of $\|\bar{v}_t(t)\|_{L^{1,\gamma}} = O(1)(1+t)^{-\frac{3}{2}+\frac{\gamma}{2}}$, which seems less sufficient. In fact, as we demonstrate numerically, the optimal decay is $\|(u-\bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{3}{2}-\frac{\gamma}{2}} = O(1)(1+t)^{-\frac{3}{2}-\frac{1}{8}} = O(1)(1+t)^{-\frac{13}{8}}$ with the best number $\gamma = \frac{1}{4}$.

4.3. Large negative β. Now, consider $\beta = -4$ which satisfies (7). We expect that the p-system solution $u(x, t)$ will blow up at time $\frac{1}{2}\ln\frac{4}{3} \approx 0.1438410\cdots$.

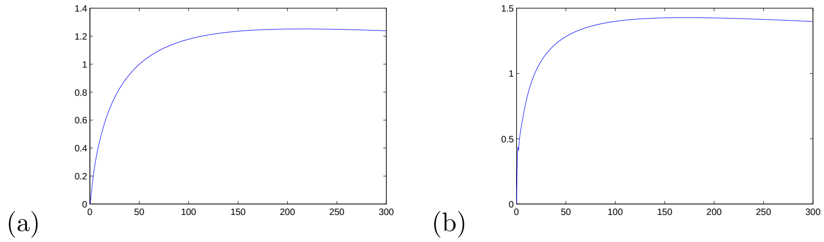


FIGURE 2. Convergence rates of the p-system solution to the corresponding diffusion wave with $\beta = -0.01$. (a) is for $F_v(t) := \frac{\|(v-\bar{v})(t)\|_{L^\infty}}{(1+t)^{-9/8}}$ and (b) is for $G_u(t) := \frac{\|(u-\bar{u})(t)\|_{L^\infty}}{(1+t)^{-13/8}}$

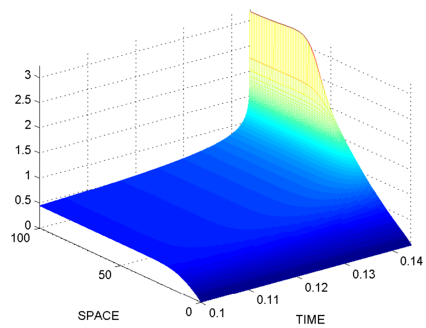


FIGURE 3. Graph of the p-system solution $\log_{10}(1+u)$ at $\beta = -4$.

The numerical simulations presented in Figures 3, 4 and 5 confirm the prediction. However, from Figure 5, we observe that $v(x,t)$ remains bounded and no blow-up occurs. Since $u(x,t)$ blows up at $T_* = \frac{1}{2}\ln\frac{4}{3}$, then from the first equation of (1) $v_t - u_x = 0$, we have formally that

$$\begin{aligned} &|u(x,t)| \sim O(1)|T^* - t|^{-\theta} \to \infty, \\ &v(x,t) \sim O(1)[1 + |T^* - t|^{1-\theta}] \le C \end{aligned}$$

for some constant $0 < \theta < 1$ as $t \to T^{*-}$. However, for this interest phenomenon of coexistence of the global solution $v(x,t)$ and the blow-up solution $u(x,t)$, the rigorous proof is absent, and the question still remains open.

Acknowledgments

The research of CKL was supported in part by National Science Council of Taiwan, R.O.C., under the grant NSC98-2115-M-009-004-MY3, the research of CTL was supported in part by National Science Council of Taiwan, R.O.C., under the grant NSC97-2115-M-126-002, and the research of MM was supported in part by Natural Sciences and Engineering Research Council of Canada under the NSERC grant RGPIN 354724-08.

References

[1] A. Chertock and A. Kurganov, A second-order positivity preserving central-upwind scheme for chemotaxis and haptotaxis models, Num. Math., 111 (2008), 169–205.

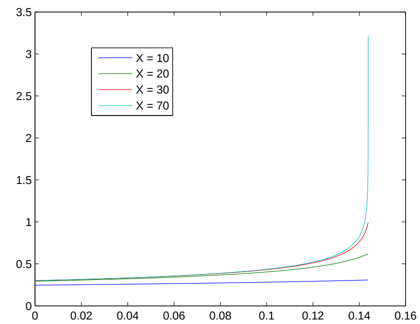


FIGURE 4. Curves of the p-system solution $\log_{10}(1+u)$ with $\beta = -4$ as a function of time at space $x = 10, 20, 30, 70$.

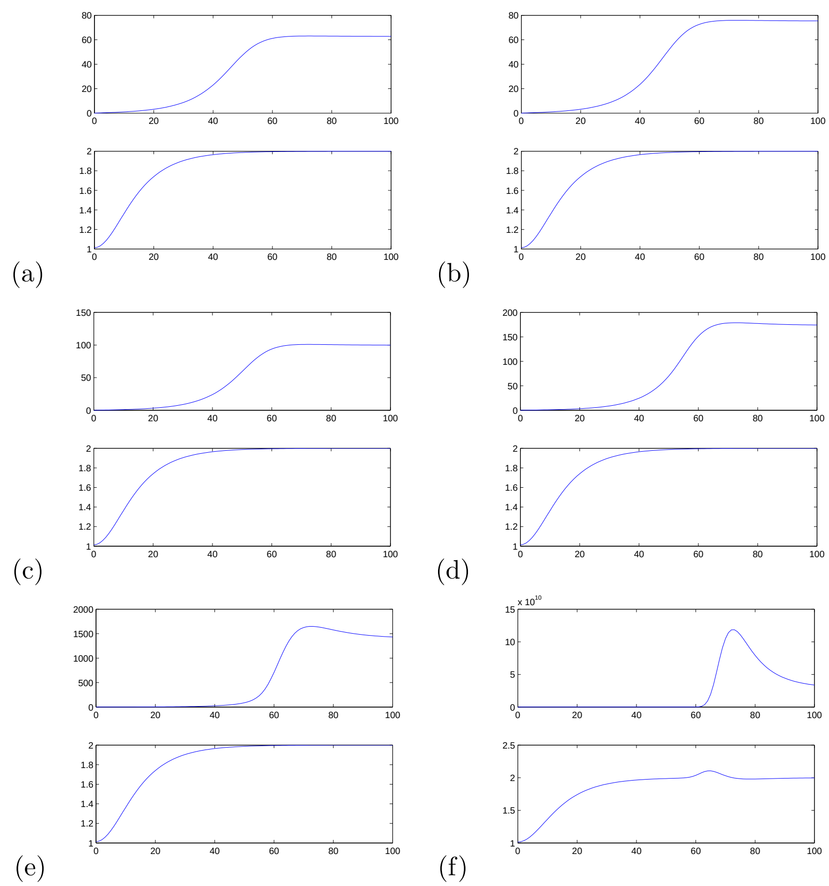


FIGURE 5. From (a) to (f), curves of the p-system solutions u and v with $\beta = -4$ at time from 0.14381 to 0.14386. Here, as showed in (f), $u(x,t)$ will blow up in a finite time, while $v(x,t)$ remains bounded.

[2] C. T. Duyn and L. A. Van Peletier, A class of similarity solutions of the nonlinear diffusion equation, Nonlinear Anal., 1 (1977), 223–233.
[3] L. Hsiao and T.-P. Liu, Convergence to diffusion waves for solutions of a system of hyperbolic conservation laws with damping, Commun. Math. Phys., 143 (1992), 599–605.
[4] L. Hsiao and T.-P. Liu, Nonlinear diffusion phenomena of nonlinear hyperbolic system, Chinese Annals Math. B, 14 (1993), 465–480.
[5] L. Hsiao and T. Luo, Nonlinear diffusion phenomena of solutions for the system of compressible adiabatic flow through porous media, J. Differential Equations, 125 (1996), 329–365.
[6] L. Hsiao and T. Luo, Nonlinear diffusive phenomena of entropy weak solutions for a system of quasilinear hyperbolic conservation laws with damping, Quat. Appl. Math. 56 (1998), 173–189.
[7] L. Hsiao and R. Pan, Initial boundary value problem for the system compressible adiabatic flow through porous media, J. Differential Equations, 159 (1999), 380–305.
[8] L. Hsiao and S. Q. Tang, Construction and qualitative behavior of the solution of the perturbed Riemann problem for the system of one-dimensional isentropic flow with damping, J. Differential Equations, 123 (1995), 480–503.
[9] F. Huang, private communication.
[10] F. Huang, P. Marcati and R. Pan, Convergence to the Barenblatt solution for the compressible Euler equations with damping and vacuum, Arch. Ration. Mech. Anal. 176 (2005) 1–24.
[11] F. Huang and R. Pan, Convergence rate for compressible Euler equations with damping and vacuum, Arch. Ration. Mech. Anal. 166 (2003) 359–376.
[12] R. Ikehata, New decay estimates for linear damped wave equations and its application ro nonlinear problem, Math. Methods Appl. Sci. 27 (2004), 865–889.
[13] M. Jiang and C. Zhu, Convergence rates to nonlinear diffusion waves for p-system with nonlinear damping on quadrant, Discrete and Continuous Dynamical Systems, Series A, 23 (2009) 887–918.
[14] M. Jiang and C. Zhu, Convergence to strong nonlinear diffusion waves for solutions to p-system with damping on quadrant, J. of Differential Equations, 246 (2009), 50–77.
[15] A. Kurganov and E. Tadmor, New high resolution central schemes for nonlinear conservation laws and convection-diffusion equations, J. Comp. Phys., 160 (2000), 241-282.
[16] H.-L. Li and K. Saxton, Asymptotic behavior of solutions to quasilinear hyperbolic equations with nonlinear damping, Quart. Appl. Math., 61 (2003), 295–313.
[17] H. Ma and M. Mei, Best asymptotic profile for linear damped p-system with boundary effect, J. Differential Equations, 249 (2010), 446–484.
[18] P. Marcati and A. Milani, The one-dimensional Darcy's law as the limit of a compressible Euler flow, J. Differential Equations, 84 (1990), 129–147.
[19] P. Marcati and M. Mei, Convergence to nonlinear diffusion waves for solutions of the initial boundary problem to the hyperbolic conservation laws with damping, Quart. Appl. Math. 56 (2000), 763–784.
[20] P. Marcati, M. Mei and B. Rubino, Optimal convergence rates to diffusion waves for solutions of the hyperbolic conservation laws with damping, J. Math. Fluid Mech. 7 (2005), S224-S240.
[21] A. Matsumura, On the asymptotic behavior of solutions of semi-linear wave equation, Publ. RIMS Kyoto Univ. 12 (1976), 169–189.
[22] M. Mei, Nonlinear diffusion waves for hyperbolic p-system with nonlinear damping, J. Differential Equations, 247 (2009), 1275–1269.
[23] M. Mei, Best asymptotic profile for hyperbolic p-system with damping, SIAM J. Math. Anal. 42 (2010), 1-23.
[24] K. Nishihara, Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping, J. Differential Equations, 131 (1996), 171–188.
[25] K. Nishihara, Asymptotic behavior of solutions of quasilinear hyperbolic equations with damping, J. Differential Equations, 137 (1997), 384–396.
[26] K. Nishihara, Asymptotics toward the diffusion wave for a one-dimensional compressible flow through porous media, Proc. Roy. Soc. Edinburgh, Sect. A 133 (2003), 177–196.
[27] K. Nishihara and T. Yang, Boundary effect on asymptotic behavior of solutions to the p-system with linear damping, J. Differential Equations, 156 (1999), 439–458.
[28] K. Nishihara, W. K. Wang and T. Yang, L_p-convergence rates to nonlinear diffusion waves for p-system with damping, J. Differential Equations, 161 (2000), 191–218.
[29] R. Pan, Darcy's law as long-time limit of adiabatic porous media flow, J. Differential Equations, 220 (2006), 121–146.

[30] L. Pareschi and G. Russo, Implicit-explicit Runge-Kutta schemes and applications to hperbolic systems with relaxation, J. Sci. Comp., 25 (2005), 129–155.
[31] B. Said-Houari, Convergence to strong nonlinear diffusion waves for solutions to p-system with damping, J. Differential Equations, 247 (2009), 917–930.
[32] I.E. Segal, Quantization and dispersion for nonlinear relativistic equations, in: Mathematical Theory of Elementary Particles, pp. 79-108, MIT Press, Cambridge, MA, 1966.
[33] D. Serre and L. Xiao (L. Hsiao), Asymptotic behavior of large weak entropy solutions of the damped p-system, J. Partial Differential Equations, 10 (1997), 355–368.
[34] W. Wang and T. Yang, Pointwise estimates and L_p convergence to diffusion waves for p-system with damping, J. Differential Equations, 187 (2003) 310–336.
[35] H. Zhao, Convergence to strong nonlinear diffusion waves for solutions of p-system with damping, J. Differential Equations, 174 (2001), 200–236.
[36] H. Zhao, Asymptotic behavior of solutions of quasilinear hyperbolic equations with damping II, J. Differential Equations, 167 (2000), 467–494.
[37] C. Zhu, Convergence rates to nonlinear diffusion waves for weak entropy solutions to p-system with damping, Sciences in China, Series A, 46 (2003), 562–575.
[38] C. Zhu and M. Jiang, L^p-decay rates to nonlinear diffusion waves for p-system with nonlinear damping, Sciences in China, Series A, 49 (2006), 721–739.

Department of Applied Mathematics, and Center of Mathematical Modeling and Scientific Computing, National Chiao Tung University, Hsinchu, 30010, Taiwan, R.O.C.
E-mail: `cklin@math.nctu.edu.tw`

Department of Applied Mathematics, Providence University, Taichung 43301, Taiwan, R.O.C.
E-mail: `ctlin@pu.edu.tw`

Department of Mathematics, Champlain College Saint-Lambert, Saint-Lambert, Quebec, J4P 3P2, Canada, and Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada
E-mail: `mei@math.mcgill.ca, and mmei@champlaincollege.qc.ca`
URL: `http://www.math.mcgill.ca/~ mei`