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Nearly Comonotone Approximation of Periodic Functions

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Abstract. Suppose that a continuous 2π-periodic function f on the real axis changes its monotonicity at points $y_i : -\pi \le y_{2s} < y_{2s-1} < \cdots < y_1 < \pi$, $s \in \mathbb{N}$. In this paper, for each $n \ge N$, a trigonometric polynomial P_n of order cn is found such that: P_n has the same monotonicity as f, everywhere except, perhaps, the small intervals

$$(y_i - \pi/n, y_i + \pi/n)$$

and

$$\|f - P_n\| \le c(s)\omega_3(f, \pi/n),$$

where N is a constant depending only on $\min\limits_{i=1,\cdots,2s}\{y_i - y_{i+1}\}$, c, $c(s)$ are constants depending only on s, $\omega_3(f,\cdot)$ is the modulus of smoothness of the 3-rd order of the function f, and $\|\cdot\|$ is the max-norm.

Key Words: Periodic functions, comonotone approximation, trigonometric polynomials, Jackson-type estimates.

AMS Subject Classifications: 41A10, 41A17, 41A25, 41A29

1 Introduction and the main theorem

By C we denote the space of continuous 2π-periodic functions $f : \mathbb{R} \to \mathbb{R}$ with the uniform norm

$$\|f\| := \|f\|_{\mathbb{R}} = \max_{x \in \mathbb{R}} |f(x)|,$$

and by $\mathbb{T}_n, n \in \mathbb{N}$, denote the space of trigonometric polynomials

$$P_n(x) = a_0 + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx), \quad a_j \in \mathbb{R}, \quad b_j \in \mathbb{R},$$

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of degree $\leq n$. Recall the classical Jackson-Zygmund-Akhiezer-Stechkin estimate (obtained by Jackson for $k=1$, Zygmund and Akhiezer for $k=2$, and Stechkin for $k\geq 3$, $k\in\mathbb{N}$): *if a function $f\in C$, then for each $n\in\mathbb{N}$ there is a polynomial $P_n\in\mathbb{T}_n$ such that*

$$\|f-P_n\|\leq c(k)\omega_k(f,\pi/n), \tag{1.1}$$

where $c(k)$ is a constant depending only on k, and $\omega_k(f,\cdot)$ is the modulus of continuity of order k of the function f. For details, see, for example, [2].

In 1968 Lorentz and Zeller [7] for $k=1$ obtained a bell-shaped analogue of the inequality (1.1), i.e., when bell-shaped (even and nonincreasing on $[0,\pi]$) 2π-periodic functions are approximated by bell-shaped polynomials.

In papers [9] and [4] a comonotone analogue of the inequality (1.1) was proved for $k=1$ and $k=2$, respectively. Moreover, in [8] arguments from the papers [12,13] of Shvedov and [1] of DeVore, Leviatan and Shevchuk were used to show that for $k>2$ there is no comonotone analogue of the inequality (1.1).

Nevertheless, as we know from the comonotone approximation on a closed interval (by algebraic polynomials, see, for details [5]) *if some relaxation of the condition of comonotonicity for approximating polynomials is allowed, then an extra order of the approximation can be achieved, and no more than one extra order*, see the corresponding counterexample in [6].

So, in this paper in Theorem 1.1 we prove a trigonometric analogue of this algebraic result by Leviatan and Shevchuk [5]. To write it we give necessary notations.

Suppose that on $[-\pi,\pi)$ there are $2s$, $s\in\mathbb{N}$, fixed points y_i:

$$-\pi\leq y_{2s}<y_{2s-1}<\cdots<y_1<\pi,$$

while for other indices $i\in\mathbb{Z}$, the points y_i are defined periodically by the equality

$$y_i=y_{i+2s}+2\pi \quad (\text{i.e., } y_0=y_{2s}+2\pi,\cdots,y_{2s+1}=y_1-2\pi,\cdots).$$

Denote $Y:=\{y_i\}_{i\in\mathbb{Z}}$. By $\Delta^{(1)}(Y)$ we denote the set of all functions $f\in C$ which are nondecreasing on $[y_1,y_0]$, nonincreasing on $[y_2,y_1]$, nondecreasing on $[y_3,y_2]$, and so on. The functions in $\Delta^{(1)}(Y_s)$ are *comonotone* with one another. Note, if a function f is differentiable, then $f\in\Delta^{(1)}(Y)$ if and only if

$$f'(x)\Pi(x)\geq 0, \quad x\in\mathbb{R},$$

where

$$\Pi(x):=\Pi(x,Y):=\prod_{i=1}^{2s}\sin\frac{x-y_i}{2}, \qquad (\Pi(x)>0, \quad x\in(y_1,y_0)).$$

Theorem 1.1. *If a function $f\in\Delta^{(1)}(Y)$, then there exists a constant $N(Y)$ depending only on $\min\limits_{i=1,\cdots,2s}\{y_i-y_{i+1}\}$ such that for each $n\geq N(Y)$ there is a polynomial $P_n\in\mathbb{T}_{cn}$ satisfying*

$$P_n'(x)\Pi(x)\geq 0, \quad x\in\mathbb{R}\setminus\cup_{i\in\mathbb{Z}}(y_i-\pi/n,y_i+\pi/n), \tag{1.2a}$$

$$\|f-P_n\|\leq c(s)\omega_3(f,\pi/n), \tag{1.2b}$$

where c, $c(s)$ are constant depending only on s.

The following Theorem 1.2 is a simple corollary of Theorem 1.1 and Whitney's inequality [14]

$$\|f-f(0)\|\le 3\omega_3(f,4\pi).$$

Theorem 1.2. *If a function $f\in\Delta^{(1)}(Y)$, then for each $n\in\mathbb{N}$ there is a polynomial $P_n\in\mathbb{T}_n$ such that*

$$P_n'(x)\Pi(x)\ge 0,\quad x\in\mathbb{R}\setminus\cup_{i\in\mathbb{Z}}(y_i-c/n,y_i+c/n), \tag{1.3a}$$

$$\|f-P_n\|\le C(Y)\omega_3(f,\pi/n), \tag{1.3b}$$

where c is a constant depending only on s, and $C(Y)$ is a constant depending only on $\min\limits_{i=1,\cdots,2s}\{y_i-y_{i+1}\}$.

Remark 1.1. We believe that ω_3 in (1.2b) and (1.3b) cannot be replaced by ω_k with $k>3$. Also we believe that the constants $N(Y)$ and $C(Y)$ in Theorems 1.1 and 1.2 cannot be replaced by constants independent of $\min\limits_{i=1,\cdots,2s}\{y_i-y_{i+1}\}$ (and depending, say, on s). These both assumptions are not proven in this paper. Also in the paper we do not pay attention to the constant c in the both theorems, i.e., we did not try to replace it by an absolute constant or/and by a smallest possible one.

2 Auxiliary facts I

For each $n\in\mathbb{N}$ denote

$$h:=h_n:=\frac{\pi}{n},\quad x_j:=x_{j,n}:=-jh,\quad I_j:=I_{j,n}:=[x_j,x_{j-1}],\quad j\in\mathbb{Z}.$$

Let

$$m=30,20,10,4,3.$$

For a fixed $Y=\{y_i\}_{i\in\mathbb{Z}}$ and a fixed n denote

$$O_{i,m}:=O_i(Y,n,m):=(x_{j+m},x_{j-m})\quad\text{if }\ y_i\in[x_j,x_{j-1}).$$

Set

$$O_m:=O(Y,n,m):=\bigcup_{i\in\mathbb{Z}}O_{i,m}.$$

We will write

$$j\in H(Y,n,m)\quad\text{if }\ I_j\subset\mathbb{R}\setminus O_m.$$

Let

$$H_m:=\{j:j\in H(Y,n,m),\ |j|\le n\}.$$

Choose $N(Y):=N(Y,30)\in\mathbb{N}$ sufficiently large so that

$$O_{i,30}\cap O_{i-1,30}=\emptyset \tag{2.1}$$

for all $n\ge N(Y)$ and all $i=1,\cdots,2s$ (thus, $N(Y)$ depends only on $\min\limits_{i=1,\cdots,2s}\{y_i-y_{i+1}\}$). In what follows $n>N(Y)$.

Denote

$$\chi(x,a):=\begin{cases}0, & \text{if } x\le a,\\ 1, & \text{if } x>a,\end{cases}\quad a\in\mathbb{R},\quad \chi_j(x):=\chi(x,x_j),\quad (x-x_j)_+:=(x-x_j)\chi_j(x),$$

$$\Gamma_j(x):=\Gamma_{j,n}(x):=\min\left\{1,\frac{1}{n\left|\sin\frac{x-(x_j+h/2)}{2}\right|}\right\},\quad j\in\mathbb{Z},\quad n\in\mathbb{N},$$

and note that

$$\left\|\sum_{j=1-n}^{n}\Gamma_j^2\right\|<6, \tag{2.2}$$

for details, see [9].

For each $j\in\mathbb{Z}$ and $b\in\mathbb{N}$ we set the positive polynomial $J_j\in\mathbb{T}_{(n-1)b}$, $n\in\mathbb{N}$,

$$J_j(x):=J_{j,n}(x):=\left(\frac{\sin\frac{n(x-x_j)}{2}}{\sin\frac{x-x_j}{2}}\right)^{2b}+\left(\frac{\sin\frac{n(x-x_{j-1})}{2}}{\sin\frac{x-x_{j-1}}{2}}\right)^{2b} \tag{2.3}$$

(i.e., the sum of two "adjacent" kernels of Jackson type).

For each $j\in H_{10}$ denote

$$t_j(x):=t_{j,n}(x,b,Y):=\frac{\int_{x_j-\pi}^{x}J_j(u)\Pi(u)du}{\int_{x_j-\pi}^{x_j+\pi}J_j(u)\Pi(u)du}. \tag{2.4}$$

In what follows $c_i=c_i(b)=c_i(s,b)$, $i=1,\cdots,8$, stand for positive constants which may depend only on s and b.

Lemma 2.1 (see [4]). *If $j\in H_{10}$ and $b\ge s+2$, then*

$$t_j'(x)\Pi(x)\Pi(x_j)\ge 0,\quad x\in\mathbb{R}, \tag{2.5a}$$

$$\left|\chi_j(x)-t_j(x)\right|\le c_1\left(\Gamma_j(x)\right)^{2b-2s-1},\quad x\in[x_j-\pi,x_j+\pi], \tag{2.5b}$$

$$\left|t_j'(x)\right|\le c_2\frac{1}{h}\left(\Gamma_j(x)\right)^{2b-2s},\quad x\in\mathbb{R}, \tag{2.5c}$$

$$\left|t_j'(x)\right|\ge c_3\frac{1}{h}\left(\Gamma_j(x)\right)^{2b+2s},\quad x\in\mathbb{R}\setminus O_{10}, \tag{2.5d}$$

$$\left|t_j'(x)\right|\ge c_3\frac{1}{h}\left(\Gamma_j(x)\right)^{2b+2s}\left|\frac{x-y_i}{x_j-y_i}\right|,\quad x\in O_{i,10},\quad i\in\mathbb{Z}. \tag{2.5e}$$

Note that Lemma 2.1 is proved by using the inequalities

$$\frac{1}{c_4 h}\Gamma_j^{2b}(x)\left|\frac{\Pi(x)}{\Pi(x_j)}\right| \le \left|t_j'(x)\right| \le \frac{c_4}{h}\Gamma_j^{2b}(x)\left|\frac{\Pi(x)}{\Pi(x_j)}\right|, \tag{2.6a}$$

$$\left|\frac{\Pi(x)}{\Pi(x_j)}\right| \le 2^{2s}\Gamma_j^{-2s}(x), \quad x\in\mathbb{R}, \quad j\in H_m, \quad m\ge 10, \tag{2.6b}$$

$$\left|\int_x^{x_j+\pi}\Gamma_j^b(u)du\right| \le c_5 h\Gamma_j^{b-1}(x), \quad b\in\mathbb{N}, \quad x\in[x_j,x_j+2\pi], \tag{2.6c}$$

$$\left|\int_x^{x_j-\pi}\Gamma_j^b(u)du\right| \le c_5 h\Gamma_j^{b-1}(x), \quad b\in\mathbb{N}, \quad x\in[x_j-2\pi,x_j], \tag{2.6d}$$

for details, see [9].

For each $j\in H_{20}$ set the function

$$\tau_j(x):=\tau_{j,n}(x,b,t_j):=\alpha\int_{x_j-\pi}^x t_{j+10}(u)du+(1-\alpha)\int_{x_j-\pi}^x t_{j-10}(u)du, \tag{2.7}$$

where the number $\alpha\in[0,1]$ is chosen from the condition

$$\tau_j(x_j+\pi)=\pi$$

(note that the inequalities $0\le\alpha\le 1$ follow from the estimate (2.5b) and the choice of the indices $j\pm 10$ if $b\ge s+2$, for details, see [10, pp. 923]).

Note that the functions t_j and τ_j can be expressed on $\mathbb{R}$ as

$$t_j(x)=\frac{1}{2\pi}x+\hat{R}_j(x), \qquad j\in H_{10}, \tag{2.8a}$$

$$\tau_j(x)=\frac{1}{4\pi}x^2+\frac{\pi-x_j}{2\pi}x+\tilde{R}_j(x), \qquad j\in H_{20}, \tag{2.8b}$$

where $\hat{R}_j$ and $\tilde{R}_j$ are polynomials from $\mathbb{T}_{c_6 n}$ (see similar cases in [9] and [10], respectively).

In what follows $c>0$ denote different absolute constants or constants depending only on s. They can be different even if they are in the same line.

Let $j\in H_{10}$. Denote

$$\mathring{t}_j(x):=\mathring{t}_{j,n}(x,b):=\bar{t}_j(x)+\sum_{i=1}^{2s}\frac{\chi_j(y_i)-\bar{t}_j(y_i)}{\hat{t}_{j_i}(y_i)}\hat{t}_{j_i}(x),$$

where $\bar{t}_j(x):=t_{j,n}(x,\bar{b},\emptyset)$ is the function defined by (2.4) with $\Pi(x):\equiv 1$ and $\bar{b}=b+3$, and

$$\hat{t}_{j_i}(x):=\left(\bar{t}_{j_i+10}(x)-\breve{t}_{j_i-10}(x)\right)\frac{\Pi(x,Y_i)}{\Pi(x_{j_i},Y_i)}$$

is the polynomial, where j_i is an index j such that $y_i\in[x_j,x_{j-1})$, $i=1,\cdots,2s$, $\breve{t_j}(x):=t_{j,n}(x,\bar{b},\breve{Y}_i$ is the function (2.4) with $\breve{Y}_i:=\{y_i-\pi\nu\}_{\nu\in\mathbb{Z}}$, and

$$Y_i:=(Y\setminus\{y_i+2\pi\nu\}_{\nu\in\mathbb{Z}})\cup\{y_i^*+2\pi\nu\}_{\nu\in\mathbb{Z}},$$

where y_i^* is the left endpoint of the interval $O_{i,20}$, if i is odd, and–the right one, if i is even.

Lemma 2.2. *For each $j\in H_{10}$ and $b\geq 3s+2$ the function $\overset{\circ}{t}_j(x)$ satisfies the relations* (2.5b), (2.8a), *and in addition,*

$$\left|\chi_j(x)-\overset{\circ}{t}_j(x)\right|\leq c_7\left(\Gamma_j(x)\right)^{2b-2s-1}\left|\frac{x-y_i}{x_j-y_i}\right|,\quad x\in O_{i,10},\quad i=1,\cdots,2s, \tag{2.9}$$

(in particular, $\chi_j(y_i)-\overset{\circ}{t}_j(y_i)=0$).

Proof. Denote $\Lambda_j(x):=\chi_j(x)-\overset{\circ}{t}_j(x)$. It follows from (2.5b), (2.5d), the second line in (2.6b) and direct calculations that

$$\begin{aligned}
\left|\Lambda_j(x)\right|\leq&\left|\chi_j(x)-\bar{t}_j(x)\right|+\sum_{i=1}^{2s}\frac{|\chi_j(y_i)-\bar{t}_j(y_i)|}{|\hat{t}_{j_i}(y_i)|}|\hat{t}_{j_i}(x)|\\
\leq& c_1\left(\Gamma_j(x)\right)^{2\bar{b}-1}+\sum_{i=1}^{2s}\frac{c_1\left(\Gamma_j(y_i)\right)^{2\bar{b}-1}}{c_3}\\
&\times\left|\bar{t}_{j_i+10}(x)-\chi_{j_i+10}(x)+\chi_{j_i+10}(x)-\chi_{j_i-10}(x)+\chi_{j_i-10}(x)-\check{t}_{j_i-10}(x)\right|\left|\frac{\Pi(x,Y_i)}{\Pi(x_{j_i},Y_i)}\right|\\
\leq& c_1\left(\Gamma_j(x)\right)^{2\bar{b}-1}+c\sum_{i=1}^{2s}\left(\Gamma_j(y_i)\right)^{2\bar{b}-1}\left[c_1\left(\Gamma_{j_i+10}(x)\right)^{2\bar{b}-1}+c\left(\Gamma_{j_i}(x)\right)^{2\bar{b}-3}\right.\\
&\left.+c_1\left(\Gamma_{j_i-10}(x)\right)^{2\bar{b}-3}\right]\Gamma_{j_i}^{-2s}(x)\\
\leq& c_1\left(\Gamma_j(x)\right)^{2\bar{b}-1}+c\sum_{i=1}^{2s}\left(\Gamma_j(y_i)\right)^{2\bar{b}-1}\left(\Gamma_{j_i}(x)\right)^{2\bar{b}-3-2s}\\
\leq& c\left(\Gamma_j(x)\right)^{2b-2s-1},\quad x\in[x_j-\pi,x_j+\pi],
\end{aligned}$$

where, knowing from (2.5a) that $\bar{t}'_{j_i+10}(u)>0$ everywhere and $\check{t}'_{j_i-10}(u)<0$ on $(x_{j_i-10}-\pi,y_i)$, we used the inequality

$$\begin{aligned}
\hat{t}_{j_i}(y_i)=&\int_{x_{j_i+10}-\pi}^{y_i}\bar{t}'_{j_i+10}(u)du-\int_{x_{j_i-10}-\pi}^{y_i}\check{t}'_{j_i-10}(u)du\\
>&\int_{x_{j_i+10}-\pi}^{y_i}c_3\frac{1}{h}\Gamma_{j_i+10}^{2\bar{b}}(u)du\\
>&c_3\frac{1}{h}\int_{x_{j_i+10}}^{x_{j_i+9}}\Big(\chi_{x_{j_i+10}}(u)-\chi_{x_{j_i+9}}(u)\Big)du\\
=&c_3.
\end{aligned}$$

So, for $\overset{\circ}{t}_j$ (2.5b) holds and (2.8a) is obvious. Prove (2.9). Having, by definition, for a fixed i, $\hat{t}_{j_i}(y_i)>0$ (in fact $>c_3$), whereas $\hat{t}_{j_i}(y_k)=0$, $k\neq i$, $1\leq k\leq 2s$, and hence for all $i=1,\cdots,2s$,

$\Lambda_j(y_i)=0$, we, for some $\theta\in O_{i,10}$ and all $x\in O_{i,10}$, analogously write

$$
\begin{aligned}
|\Lambda_j(x)| &= \left|\int_{y_i}^{x}\Lambda_j'(u)du\right| \\
&\le |x-y_i|\left(\left|\bar{t}_j'(\theta)\right|+\sum_{i=1}^{2s}\frac{|\chi_j(y_i)-\bar{t}_j(y_i)|}{|\hat{t}_{j_i}(y_i)|}|\hat{t}_{j_i}'(\theta)|\right) \\
&\le |x-y_i|\Big(c\frac{1}{h}\Gamma_j^{2\bar{b}}(x)+\sum_{i=1}^{2s}\frac{c_1}{c_3}\big(\Gamma_j(y_i)\big)^{2\bar{b}-1}\Big(c\frac{1}{h}\big(\Gamma_{j_i}(\theta)\big)^{2\bar{b}-2-2s} \\
&\quad +c\big(\Gamma_{j_i}(\theta)\big)^{2\bar{b}-3}\left|\frac{\Pi'(\theta,Y_i)}{\Pi(x_{j_i},Y_i)}\right|\Big)\Big) \\
&\le |x-y_i|\left(c\frac{1}{h}\Gamma_j^{2\bar{b}}(x)+c\sum_{i=1}^{2s}\big(\Gamma_j(y_i)\big)^{2\bar{b}-1}\frac{1}{h}\big(\Gamma_{j_i}(\theta)\big)^{2\bar{b}-2-2s}\right) \\
&\le c|x-y_i|\frac{1}{h}\Gamma_j(x)\big(\Gamma_j(x)\big)^{2b-2s-1},
\end{aligned}
$$

where we also used (2.5c) and the inequality

$$
\left|\frac{\Pi'(x,Y_i)}{\Pi(x_{j_i},Y_i)}\right|=\left|\sum_{k=1}^{2s}\frac{\cos((x-y_k)/2)}{\sin((x_{j_i}-y_k)/2)}\prod_{\nu=1,\nu\neq k}^{2s}\frac{\sin((x-y_\nu)/2)}{\sin((x_{j_i}-y_\nu)/2)}\right|\le c\frac{1}{h}\Gamma_{j_i}^{-2s+1}(x),
$$

which holds for any $x\in\mathbb{R}$ ($|x_{j_i}-y_{k\,\mathrm{or}\,\nu}|\ge 20h$). If now $|x_j-y_i|\ge 20h$, then

$$
\frac{1}{h}\Gamma_j(x)=\frac{1}{\pi\left|\sin((x-(x_j+h/2))/2)\right|}\le\frac{1}{\left|x-x_j-h/2\right|}<\frac{3}{\left|x_j-y_i\right|},\quad x\in O_{i,10},
$$

otherwise

$$
\frac{1}{h}\Gamma_j(x)<\frac{20}{\left|x_j-y_i\right|}\Gamma_j(x).
$$

Thus, (2.9) is proved. Lemma 2.2 is proved. $\square$

Remark 2.1. a) Instead of the polynomial $\hat{t}_{j_i}(x)$ one can use $t'_{j_i,n}(x,\bar{b},Y_i)$ as more "natural" in such a function but it makes the proof longer. b) In some other form but with the analogous property (2.9) an algebraic polynomial first was used by Gilewicz and Shevchuk (for details, see [4]).

The following Lemma 2.3 is proved using the same arguments as Lemma 2.2.

Lemma 2.3. *For each $j\in H_{20}$ and $b\ge 3s+2$ the function*

$$
\mathring{\tau}_j(x):=\mathring{\tau}_{j,n}(x,b):=\tau_{j,n}(x,b,\bar{t}_j)+\sum_{i=1}^{2s}\frac{(y_i-x_j)_+-\tau_{j,n}(y_i,b,\bar{t}_j)}{\hat{t}_{j_i}(y_i)}\hat{t}_{j_i}(x)
$$

satisfies relation (2.8b)*, and in addition,*

$$\left|(x-x_j)_+ - \overset{\circ}{\tau}_j(x)\right| \le c_8 h\left(\Gamma_j(x)\right)^{2(b-s-1)}, \quad x\in[x_j-\pi,x_j+\pi], \tag{2.10a}$$

$$\left|(x-x_j)_+ - \overset{\circ}{\tau}_j(x)\right| \le c_8 h\left(\Gamma_j(x)\right)^{2(b-s-1)}\left|\frac{x-y_i}{x_j-y_i}\right|, \quad x\in O_{i,10}, \quad i=1,\cdots,2s, \tag{2.10b}$$

(in particular, $(y_i-x_j)_+ - \overset{\circ}{\tau}_j(y_i)=0$*).*

Indeed, having, by (2.7), the equalities $\tau_j(x_j-\pi)=0$ and $\tau_j(x_j+\pi)=\pi$, we can use for the difference $(x-x_j)_+ - \tau_j(x) =: \Lambda(x)$, two representations $\Lambda(x)=\int_{x_j-\pi}^{x}\Lambda'(u)du$, for $x\in[x_j-\pi,x_j]$, and $\Lambda(x)=-\int_{x}^{x_j+\pi}\Lambda'(u)du$, for $x\in[x_j,x_j+\pi]$, that together with (2.5b), (2.6c) and (2.6d) implies the inequality

$$\begin{aligned}|\Lambda(x)|&=|\Lambda(x)-\alpha\chi_{j+10}(x)+\alpha\chi_{j+10}(x)-(1-\alpha)\chi_{j-10}(x)+(1-\alpha)\chi_{j-10}(x)|\\&\le ch\left(\Gamma_j(x)\right)^{2(b-s-1)}, \quad x\in[x_j-\pi,x_j+\pi],\end{aligned}$$

by virtue of which we get (2.10a) and (2.10b) analogously to the proof of Lemma 2.2.

3 Auxiliary facts II

We prove Theorem 1.1 using the intermediate approximation by a spline, i.e., the inequality

$$||f-S+S-P_n|| \le ||f-S|| + ||S-P_n|| \ (*).$$

The spline S is a sum of parabolas ψ_j, truncated at points x_j, or x_{j-1}, or of its linear combinations in depending on relations between differences of f, so that S is nearly comonotone to f. We can approximate ψ_j only by functions φ_j consisting from trigonometric polynomials and, alas, some algebraic addends. Therefore we have to choose φ_j so, to eliminate these addends when the sum of all φ_j forming P_n (over the partition "j" covering the period) is evaluated, and, simultaneously, to preserve the monotonicity changing of S in the P_n. For this we replace the continuous ψ_j by "technical" discontinuous Ψ_j (forming spline S_0) which are the same parabolas but truncated at 3 other points near x_j and x_{j-1}, so that the functions φ_j are constructed "identically" to Ψ_j. Since we from the beginning do not care about the behavior of S and P_n in neighborhoods of points Y, we replace f here with interpolating parabolas (denoting it by f_0). This helps us in arithmetics with the algebraic parts of φ_j. Thus, instead of $(*)$ in fact we have

$$||f-f_0+f_0-S+S-S_0+S_0-P_n|| \le ||f-f_0|| + ||f_0-S|| + ||S-S_0|| + ||S_0-P_n||.$$

Let

$$(\underline{y}_i,\overline{y}_i) := O_{i,4}.$$

Set

$$f_0(x):=\begin{cases} f(x), & \text{if } x\in\mathbb{R}\setminus O_4,\\ L_2(x,\underline{y}_i,y_i,\overline{y}_i,f), & \text{if } x\in O_{i,4},\ i\in\mathbb{Z},\end{cases}$$

where L_2 is the parabola interpolating f at three listed points of $\overline{O}_{i,4}$ (the closure of $O_{i,4}$). Note that Whitney's inequality readily implies

$$\|f-f_0\|\le c\omega_3(f,|O_{i,4}|)\le c\omega_3(f,h), \tag{3.1}$$

and f_0 is only *nearly comonotone* to f, i.e., on O_4 points x may be found such that $f_0'(x)\Pi(x)<0$. Without loss of generality suppose that $-\pi=y_{2s}$ (so, $\pi=y_0$, $\Pi(\overline{y}_{2s})<0$, $\Pi(\underline{y}_0)>0$). For $j\in\mathbb{Z}$ set

$$\begin{aligned}&\Delta_j:=-f_0(x_j)+f_0(x_{j-1}),\\ &\lambda_j:=f_0(x_j)-2f_0(x_{j-1})+f_0(x_{j-2}),\\ &\delta_j:=-f_0(x_j)+3f_0(x_{j-1})-3f_0(x_{j-2})+f_0(x_{j-3}).\end{aligned}$$

Note that

$$\Delta_j\Pi(x_j)\ge 0 \qquad \text{if } (x_j,x_{j-1})\cap O_4=\emptyset, \tag{3.2a}$$

$$\delta_j=-\lambda_j+\lambda_{j-1}=0 \qquad \text{if } (x_j,x_{j-3})\subset O_4. \tag{3.2b}$$

Moreover, the inequality (3.1) readily implies

$$|\delta_j|\le\omega_3(f_0-f+f,h)\le 8\|f_0-f\|+\omega_3(f,h)\le c\omega_3(f,h). \tag{3.3}$$

On each interval $[x_j,x_{j-1}]$, $j=1-n,\cdots,n$, for the function f_0 we define an algebraic polynomial p_j of degree 2 or 1 as follows. If $\operatorname{sgn}\lambda_{j+1}=\operatorname{sgn}\lambda_j$ then set

$$p_j(x):=\begin{cases} L_2(x,x_j,x_{j-1},x_{j-2},f_0), & \text{if } |\lambda_{j+1}|>|\lambda_j|,\\ L_2(x,x_{j+1},x_j,x_{j-1},f_0), & \text{otherwise},\end{cases}$$

otherwise (i.e., if $\operatorname{sgn}\lambda_{j+1}\ne\operatorname{sgn}\lambda_j$), put

$$p_j(x):=L_1(x,x_j,x_{j-1},f_0),$$

where L_1 is the linear function interpolating f_0 at x_j and x_{j-1}. Set

$$S|_{[x_j,x_{j-1}]}:=p_j,\quad j=1-n,\cdots,n.$$

Note that in the case $p_j=L_1$ the line $L_1(x,x_j,x_{j-1},f_0)$ is placed between two parabolas $L_2(x,x_{j+1},x_j,x_{j-1},f_0)$ and $L_2(x,x_j,x_{j-1},x_{j-2},f_0)$ for each of which Whitney's inequality holds on $[x_j,x_{j-1}]$. Hence, together with (3.1), this yields

$$\|f-S\|_{[-\pi,\pi]}\le\|f-f_0\|+\|f_0-S\|_{[-\pi,\pi]}\le c\omega_3(f,h). \tag{3.4}$$

Moreover, it is easy to verify that $S'(x)\Pi(x)\geq 0$, $x\in(x_j,x_{j-1})$ for each $j\in H_4$. So,

$$S'(x)\Pi(x)\geq 0,\quad x\in[-\pi,\pi]\setminus(O_4\cup\{x_j\}_{j\in H_4}),\tag{3.5}$$

holds.

On $[-\pi,\pi]$ we represent the continuous on $[-\pi,\pi]$ spline S as follows. If $\operatorname{sgn}\lambda_{j+1}=\operatorname{sgn}\lambda_j$ then we set

$$\psi_j(x):=\begin{cases}\underline{\psi}_j(x):=(x-x_j)_+(x-x_{j-1}), & \text{if } |\lambda_{j+1}|>|\lambda_j|,\\ \overline{\psi}_j(x):=(x-x_j)(x-x_{j-1})_+, & \text{otherwise,}\end{cases}$$

otherwise (i.e., if $\operatorname{sgn}\lambda_{j+1}\neq\operatorname{sgn}\lambda_j$), set

$$\psi_j(x):=\alpha_j\underline{\psi}_j(x)+(1-\alpha_j)\overline{\psi}_j(x),\quad \alpha_j:=\frac{|\lambda_{j+1}|}{|\lambda_{j+1}|+|\lambda_j|}\in[0,1].$$

Put $\psi_n(x):=(x-x_n)(x-x_{n-1})$, $\psi_{1-n}(x):\equiv 0$.

So, we have

$$S(x)=f(x_n)+\frac{\Delta_n}{h}(x-x_n)+\frac{1}{2h^2}\sum_{j=2-n}^{n}\lambda_j\big(\psi_j(x)-\psi_{j-1}(x)\big),\tag{3.6}$$

or, equivalently,

$$S(x)=f(x_n)+\frac{\Delta_n}{h}(x-x_n)+\frac{\lambda_n}{2h^2}\psi_n(x)+\frac{1}{2h^2}\sum_{j=2-n}^{n-1}\delta_{j+1}\psi_j(x)\tag{3.7}$$

(it is convenient to look at the sums in (3.6) and (3.7) starting from the last addend, for details of such kind of representations, see [3, Proposition 1]).

To approximate the spline S by a required polynomial introduce a technical spline S_0 which is a discontinuous modification of S on $[-\pi,\pi]$. Set

$$a_j:=x_j-\frac{h}{2},\quad v_j:=\frac{x_j+x_{j-1}}{2},\quad d_j:=x_{j-1}+\frac{h}{2}.$$

Fix $j=2-n,\cdots,n-1$. If

$$\delta_{j+1}\Pi(x_j)\geq 0,\tag{3.8}$$

then set

$$\Psi_j(x):=\begin{cases}\Psi_{j,1}(x):=(x-x_j)(x-x_{j-1})\chi(x,a_j), & \text{if } \psi_j\equiv\underline{\psi}_j,\\ \Psi_{j,3}(x):=(x-x_j)(x-x_{j-1})\chi(x,d_j), & \text{if } \psi_j\equiv\overline{\psi}_j,\\ \alpha_j\Psi_{j,1}(x)+(1-\alpha_j)\Psi_{j,3}(x), & \text{if } \psi_j\equiv\alpha_j\underline{\psi}_j+(1-\alpha_j)\overline{\psi}_j,\end{cases}$$

otherwise (i.e., if $\delta_{j+1}\Pi(x_j)<0$), set

$$\Psi_j(x):=\Psi_{j,2}(x):=(x-x_j)(x-x_{j-1})\chi(x,v_j).$$

Put $\Psi_n(x):=(x-x_n)(x-x_{n-1})$, $\Psi_{1-n}(x):\equiv 0$.

Now, let S_0 be the spline defined by (3.7) (or (3.6)) with Ψ_j instead of ψ_j. Note that for S_0 the following inequalities hold

$$\|f-S_0\|_{[-\pi,\pi]}\le c\omega_3(f,h), \tag{3.9a}$$
$$(S_0(v_j+0)-S_0(v_j-0))\Pi(v_j)\ge 0,\quad j=2-n,\cdots,n-1, \tag{3.9b}$$
$$S_0'(x)\Pi(x)\ge 0,\quad x\in[-\pi,\pi]\setminus\big(O_4\cup\{v_j\}_{j\in H_4}\big). \tag{3.9c}$$

Indeed, using (3.7), (3.7) for S_0, and (3.3) we write

$$\begin{aligned}
\|S-S_0\|_{I_\nu}&=\frac{1}{2h^2}\left\|\sum_{j=2-n}^{n-1}\delta_{j+1}(\psi_j(\cdot)-\Psi_j(\cdot))\right\|_{I_\nu}\\
&\le\frac{1}{2h^2}\sum_{j=2-n}^{n-1}|\delta_{j+1}|\big\|\psi_j-\Psi_j\big\|_{I_\nu}\\
&=\frac{1}{2h^2}\sum_{\substack{j=2-n\\ j\in\{\nu+1,\nu,\nu-1\}}}^{n-1}|\delta_{j+1}|\big\|\psi_j-\Psi_j\big\|_{I_\nu}\\
&\le\frac{1}{2h^2}\sum_{j=\nu-1}^{\nu+1}|\delta_{j+1}|\frac{3h^2}{4}\\
&\le c\omega_3(f,h),\quad \nu=1-n,\cdots,n,
\end{aligned}$$

that together with (3.4) yields (3.9a). The definition of S_0 (namely, by means of the condition (3.8)) and its representation in the form (3.7) readily imply (3.9b). Taking into account the definition of S, one can see that the inequality (3.9c) follows from the fact that the differences $\Psi_j'-\Psi_{j-1}'=\Psi_{j,\nu}'-\Psi_{j-1,\mu}'$, $\nu,\mu=1\vee 2\vee 3$, (in the representation (3.6) for S_0) are nonnegative on $[\max\{\nu_j,\mu_{j-1}\},\infty)$, $\nu_j,\mu_j=a_j\vee v_j\vee d_j$, always even in the possible case when $d_j=a_{j-1}+h$ since, $\Psi_{j-1}'(x)=\Psi_{j-1,1}'(x)\ge 0$ for $x\in[d_j,\infty)$. Whereas a possible negative part of Ψ_j' on $[a_j,v_j]$ (in the cases $\Psi_j=\Psi_{j,1}\vee\alpha_j\Psi_{j,1}$) is compensated (on the next addend) by the inequality $|\lambda_{j+1}|>|\lambda_j|$ (in the case $\psi_j=\underline{\psi}_j$) or by the value of α_j (in the case $\psi_j=\alpha_j\underline{\psi}_j$) such that $S_0'(x)\Pi(x)\ge 0$, $x\in[a_j,v_j]$. (If $\Psi_j=\Psi_{j,3}$ then the mirror situation with $|\lambda_{j+1}|<|\lambda_j|$ and $(1-\alpha_j)$ takes place).

Note that

$$\Psi_{j,1}(x)=2\int_{v_j-\pi}^{x}\big((u-a_j)_+-h\chi(u,a_j)\big)du+\frac{3}{4}h^2\chi(x,a_j), \tag{3.10a}$$

$$\Psi_{j,3}(x)=2\int_{v_j-\pi}^{x}\big((u-d_j)_++h\chi(u,d_j)\big)du+\frac{3}{4}h^2\chi(x,d_j), \tag{3.10b}$$

$$\Psi_{j,2}(x)=2\int_{v_j-\pi}^{x}(u-v_j)_+du-\frac{1}{4}h^2\chi(x,v_j). \tag{3.10c}$$

Denote the numbers

$$b_1:=s+2,\quad b_2:=3(s+1),$$

$$c_9:=\max\left\{\frac{2((2\pi)^{2b_2}\max\{c_1(b_2),c_7(b_2)\}+c_8(b_2)+2)}{3c_3(b_1)},2\right\},\quad n_1:=2[c_9+1]n,\quad h_1:=h_{n_1},$$

$$c_{10}:=\max\left\{c_5(b_2)\left(\frac{c_8(b_2)}{2c_9}+c_1(b_2)\right),10\right\},\quad n_2:=2[c_{10}+1]n_1,\quad h_2:=h_{n_2},$$

where $[\cdot]$ stands for the integer part.

For each $j=2-n,\cdots,n-1$ and each $\nu=1,2,3$ let j_ν denotes the index such that $x_{j_\nu}:=x_{j_\nu,n_1}=\nu_j$, $\nu_j=a_j,v_j,d_j$, whereas j_ν^* denotes the index such that $x_{j_\nu^*}:=x_{j_\nu^*,n_2}=x_{j_\nu}(=x_{j_\nu,n_1})$.

Let $j\in H_3$. For each j_ν, $\nu=1,2,3$, we take

$$\overset{\circ}{\tau}_{j_\nu^*}(x)=\overset{\circ}{\tau}_{j_\nu^*,n_2}(x,b_2),\quad \overset{\circ}{t}_{j_\nu^*}(x)=\overset{\circ}{t}_{j_\nu^*,n_2}(x,b_2),\quad t_{j_\nu}(x)=t_{j_\nu,n_1}(x,b_1,Y).$$

Now, put

$$\varphi_{j,1}(x):=2\int_{v_j-\pi}^{x}\left(\overset{\circ}{\tau}_{j_1^*}(u)-h\left(\alpha\overset{\circ}{t}_{(j_1+1)^*}(u)+(1-\alpha)\overset{\circ}{t}_{(j_1-1)^*}(u)\right)\right)du+\frac{3}{4}h^2t_{j_1}(x),$$

$$\varphi_{j,3}(x):=2\int_{v_j-\pi}^{x}\left(\overset{\circ}{\tau}_{j_3^*}(u)+h\left(\beta\overset{\circ}{t}_{(j_3+1)^*}(u)+(1-\beta)\overset{\circ}{t}_{(j_3-1)^*}(u)\right)\right)du+\frac{3}{4}h^2t_{j_3}(x),$$

$$\varphi_{j,2}(x):=2\int_{v_j-\pi}^{x}\left(\overset{\circ}{\tau}_{j_2^*}(u)-\frac{1}{16}h^2\left(\gamma t'_{(j_2+5)^*}(u)+(1-\gamma)t'_{(j_2-5)^*}(u)\right)\right)du-\frac{1}{8}h^2t_{j_2}(x),$$

and denote

$$\varphi_{j,\nu}(x)=:A_\nu(x)+B_\nu(x),\quad \nu=1,2,3.$$

Lemma 3.1. *If $j\in H_3$ then $\alpha,\beta,\gamma\in[0,1]$ can be chosen such that*

$$A_\nu(v_j+\pi)=\pi^2-h^2,\quad \nu=1,3,\quad A_2(v_j+\pi)=\pi^2-\pi h^2/8, \tag{3.11}$$

and then

$$\Big(\varphi'_{j,\nu}(x)-\Psi'_{j,\nu}(x)\Big)\Pi(x_j)\Pi(x)\ge 0,\quad \nu=1,3,\quad x\in[-\pi,\pi], \tag{3.12a}$$

$$\Big(\varphi'_{j,2}(x)-\Psi'_{j,2}(x)\Big)\Pi(x_j)\Pi(x)\le 0,\quad x\in[-\pi,\pi], \tag{3.12b}$$

$$\big|\Psi_{j,\nu}(x)-\varphi_{j,\nu}(x)\big|\le ch^2\Gamma_j^3(x),\quad \nu=1,2,3,\quad x\in[-\pi,\pi]. \tag{3.12c}$$

In addition,

$$\varphi_{j,\nu}(x)=\frac{1}{6\pi}x^3+\frac{\pi-v_j}{2\pi}x^2+\left(\frac{x_jx_{j-1}}{2\pi}-v_j+\frac{\pi}{3}\right)x+Q_{j,\nu}(x),\quad \nu=1,2,3, \tag{3.13}$$

where $Q_{j,\nu}\in\mathbb{T}_{cn}$.

Proof. Across the proof we will use the choice of n_1 and n_2 without special references as well as the inequalities

$$\Gamma_{(j_\nu\pm1)^*,n_2}(x)<\Gamma_{j_\nu\pm1,n_1}(x)<2\pi\Gamma_{j_\nu,n_1}(x)<2\pi\Gamma_{j,n}(x),\quad x\in\mathbb{R}.$$

Using (2.10a), (2.5b) for $\overset{\circ}{t}_{j_\nu^*}$, and (2.6c) with (2.6d) we verify the existence of $\beta\in[0,1]$ for (3.11) with $\nu=3$. If $\beta=1$ then

$$\begin{aligned}
A_3(v_j+\pi)=&2\int_{v_j-\pi}^{v_j+\pi}\Big[\overset{\circ}{\tau}_{j_3^*}(u)-(u-d_j)_++h\Big(\overset{\circ}{t}_{(j_3+1)^*}(u)-\chi(u,x_{j_3+1})\Big)\\
&+h\big(\chi(u,x_{j_3+1})-\chi(u,d_j)\big)\Big]du+2\int_{v_j-\pi}^{v_j+\pi}\big((u-d_j)_++h\chi(u,d_j)\big)du\\
\ge&\pi^2-h^2+2hh_1-2\left|\int_{v_j-\pi}^{v_j+\pi}\Big[\overset{\circ}{\tau}_{j_3^*}(u)-(u-d_j)_++h\Big(\overset{\circ}{t}_{(j_3+1)^*}(u)-\chi(u,x_{j_3+1})\Big)\Big]du\right|\\
\ge&\pi^2-h^2+2hh_1-2c_8(b_2)h_2\int_{v_j-\pi}^{v_j+\pi}\Gamma^{2(b_2-s-1)}_{j_3^*,n_2}(u)du\\
&-2c_1(b_2)h\int_{v_j-\pi}^{v_j+\pi}\Gamma^{2b_2-2s-1}_{(j_3+1)^*,n_2}(u)du\\
\ge&\pi^2-h^2+2hh_1-2c_5(b_2)\big(c_8(b_2)h_2^2+c_1(b_2)hh_2\big)\\
>&\pi^2-h^2,
\end{aligned}$$

whereas for $\beta=0$ we analogously have the opposite inequality $A_3(v_j+\pi)<\pi^2-h^2$. So, for $\nu=3$ the equality (3.11) is proved. For $\nu=1,2$ it is proved by analogy.

From three analogous inequalities (3.12) we verify only one of them with $\nu=3$. Take (3.10) into account, and having (2.10), (2.5b) for $\overset{\circ}{t}_j$, and (2.9), use (2.5d) and (2.5e). Namely, for $j\in H_3$ denote

$$K_{i,j}(x,n_1):=\begin{cases}1, & \text{if } x\in[x_j-\pi,x_j+\pi]\setminus O_i(Y,n_1,10),\\ \left|\dfrac{x-y_i}{x_j-y_i}\right|, & \text{if } x\in O_i(Y,n_1,10),\end{cases}\qquad i=1,\cdots,2s,$$

and, due to (2.5a), see that the inequality

$$\begin{aligned}&\Big(\varphi'_{j,3}(x)-\Psi'_{j,3}(x)\Big)\Pi(x_j)\Pi(x)\\ =&\Big(2\Big(\mathring{\tau}_{j_3^*}(x)-(x-d_j)_+\Big)+2h\Big[\beta\Big(\mathring{t}_{(j_3+1)^*}(x)-\chi(x,x_{j_3+1})\Big)\\ &+(1-\beta)\Big(\mathring{t}_{(j_3-1)^*}(x)-\chi(x,x_{j_3-1})\Big)\\ &+\beta\chi(x,x_{j_3+1})+(1-\beta)\chi(x,x_{j_3-1})-\chi(x,d_j)\Big]+\frac{3}{4}h^2t'_{j_3}(x)\Big)\Pi(x_j)\Pi(x)\\ \ge&0,\end{aligned}$$

holds if the following value is non negative

$$\begin{aligned}&-c_8(b_2)h_2\Gamma^{2(b_2-s-1)}_{j_3^*,n_2}(x)K_{i,j_3}(x,n_1)\\ &-2h\Big[\max\{c_1(b_2),c_7(b_2)\}2\Gamma^{2b_2-2s-1}_{(j_3\pm1)^*,n_2}(x)K_{i,j_3\pm1}(x,n_1)+\chi(x,d_j)-\chi(x,x_{j_3-1})\Big]\\ &+\frac{3}{4}h^2c_3(b_1)\frac{1}{h_1}\Gamma^{2b_1+2s}_{j_3,n_1}(x)K_{i,j_3}(x,n_1)\\ \ge&-c_8(b_2)h_2\Gamma^{2b_1+2s}_{j_3,n_1}(x)K_{i,j_3}(x,n_1)-2h\Big[\max\{c_1(b_2),c_7(b_2)\}2\Gamma^{2b_1+2s}_{j_3\pm1,n_1}(x)K_{i,j_3\pm1}(x,n_1)\\ &+\Gamma^{2b_1+2s}_{j_3,n_1}(x)K_{i,j_3}(x,n_1)\Big]+\frac{3}{4}h^2c_3(b_1)\frac{1}{h_1}\Gamma^{2b_1+2s}_{j_3,n_1}(x)K_{i,j_3}(x,n_1)\\ \ge&\frac{3}{4}hc_3(b_1)\frac{n_1}{n}-c_8(b_2)h_2-2h\big[\max\{c_1(b_2),c_7(b_2)\}2(2\pi)^{2b_1+2s}+1\big]\\ \ge&0,\end{aligned}$$

that is true. Thus, the inequalities (3.12) are proved.

Prove (3.12c). By (3.10), (2.10a) and (2.5b) if $x<v_j$ then

$$\begin{aligned}&\big|\Psi_{j,1}(x)-\varphi_{j,1}(x)\big|\\ =&\Bigg|2\int_{v_j-\pi}^{x}\Big((u-a_j)_+-\mathring{\tau}_{j_1^*}(u)-h\Big(\chi(u,a_j)-\alpha\chi(u,x_{j_1+1})-(1-\alpha)\chi(u,x_{j_1-1})\Big)\\ &-h\Big[\alpha\Big(\chi(u,x_{j_1+1})-\mathring{t}_{(j_1+1)^*}(u)\Big)+(1-\alpha)\Big(\chi(u,x_{j_1-1})-\mathring{t}_{(j_1-1)^*}(u)\Big)\Big]\Big)du\\ &+\frac{3}{4}h^2\Big(\chi(x,a_j)-t_{j_1}(x)\Big)\Bigg|\\ \le&2c_8(b_2)h_2\int_{v_j-\pi}^{x}\Gamma^{2(b_2-s-1)}_{j_1^*,n_2}(u)du+2h\int_{v_j-\pi}^{x}\big|\chi(u,a_j)-\chi(u,x_{j_1+1})\big|du\\ &+2c_1(b_2)h2\int_{v_j-\pi}^{x}\Gamma^{2(b_2-s-1)}_{(j_1\pm1)^*,n_2}(u)du+\frac{3}{4}h^2c_1(b_1)\Gamma^{2b_1-s-1}_{j_1,n_1}(x)\\ \le&2c_8(b_2)c_5(b_2)h_2^2\Gamma^3_{j_1^*,n_2}(x)+2hh_1\Gamma^3_{j_1+1,n_1}(x)\end{aligned}$$

$$
\begin{aligned}
&+2c_1(b_2)c_5(b_2)((4\pi)^3+1)h2h_2\Gamma^3_{(j_1-1)^*,n_2}(x)+(4\pi)^3h^2c_1(b_1)\Gamma^3_{j,n}(x)\\
\le& ch^2\Gamma^3_j(x),
\end{aligned}
$$

otherwise, if $x>v_j$ then we use (3.11) and, denoting $\Psi_{j,1}(x)=:C_1(x)+D_1(x)$, analogously write

$$
\begin{aligned}
\big|\Psi_{j,1}(x)-\varphi_{j,1}(x)\big|&=\big|C_1(x)-A_1(x)-\big(C_1(v_j+\pi)-A_1(v_j+\pi)\big)+D_1(x)-B_1(x)\big|\\
&\le\left|\int_{v_j+\pi}^x\big(C_1'(u)-A_1'(u)\big)du\right|+\big|D_1(x)-B_1(x)\big|\\
&=\left|2\int_{v_j+\pi}^x\cdots du\right|+\big|D_1(x)-B_1(x)\big|\\
&\le ch^2\Gamma^3_j(x).
\end{aligned}
$$

So, the estimate (3.12c) is proved for $\nu=1$ and by analogy for $\nu=2,3$ as well.

Finally, prove (3.13) with $\nu=1$ for definiteness. By (2.8) write

$$
\begin{aligned}
&\mathring{t}_{j_1^*}(x)=\frac{1}{2\pi}x+\hat{R}_{j_1^*}(x),\quad \mathring{\tau}_{j_1^*}(x)=\frac{1}{4\pi}x^2+\frac{\pi-x_j}{2\pi}x+\tilde{R}_{j_1^*}(x),\\
&\hat{r}_{j_1^*}(x):=\hat{R}_{j_1^*}(x)-\hat{R}_{j_1^*,0},\quad \tilde{r}_{j_1^*}(x):=\tilde{R}_{j_1^*}(x)-\tilde{R}_{j_1^*,0},
\end{aligned}
$$

where $\hat{R}_{j_1^*,0}$ and $\tilde{R}_{j_1^*,0}$ are free terms of polynomials $\hat{R}_{j_1^*},\tilde{R}_{j_1^*}\in\mathbb{T}_{cn}$, respectively. Then

$$
\begin{aligned}
A_1(x)=&\left(\frac{1}{6\pi}x^3+\frac{\pi-a_j}{2\pi}x^2+2\tilde{R}_{j_1^*,0}x\right)-\Big(\cdots(v_j-\pi)\Big)\\
&-2h\left(\frac{1}{4\pi}x^2+\Big(\alpha\hat{R}_{(j_1+1)^*,0}+(1-\alpha)\hat{R}_{(j_1-1)^*,0}\Big)x\right)+2h\Big(\cdots(v_j-\pi)\Big)\\
&+2\int_{v_j-\pi}^x\Big(\tilde{r}_{j_1^*}(u)-h\big(\alpha\hat{r}_{(j_1+1)^*}(u)+(1-\alpha)\hat{r}_{(j_1-1)^*}(u)\big)\Big)du\\
=&\frac{1}{6\pi}x^3+\frac{\pi-v_j}{2\pi}x^2+2Cx-\left(\frac{1}{6\pi}(v_j-\pi)^3+\frac{\pi-v_j}{2\pi}(v_j-\pi)^2+2C(v_j-\pi)\right)\\
&+q_{j_1}(x),
\end{aligned}
$$

where

$$
C:=\tilde{R}_{j_1^*,0}-h\Big(\alpha\hat{R}_{(j_1+1)^*,0}+(1-\alpha)\hat{R}_{(j_1-1)^*,0}\Big),
$$

and $q_{j_1}\in\mathbb{T}_{cn}$ does not have a free term. Taking this and (3.11) into account we calculate the value of C, namely

$$
\begin{aligned}
\pi^2-h^2=&\frac{1}{6\pi}\big((v_j+\pi)^3-(v_j-\pi)^3\big)+\frac{\pi-v_j}{2\pi}\big((v_j+\pi)^2-(v_j-\pi)^2\big)+4\pi C\\
\Rightarrow C=&\frac{\frac{1}{3}\pi^2+\frac{1}{2}v_j^2-\frac{1}{2}h^2-v_j\pi}{2\pi},
\end{aligned}
$$

that together with (2.8a) (for B_1) yields (3.13). Two other equalities (3.13) are proved analogously. Lemma 3.1 is proved. $\square$

4 Proof of Theorem 1.1

Set

$$P_n(x):=f(x_n)+\frac{\Delta_n}{h}(x-x_n)+\frac{\lambda_n}{2h^2}\Psi_n(x)+\frac{1}{2h^2}\sum_{j\in H_3}\delta_{j+1}\varphi_j(x), \tag{4.1}$$

where for $\delta_{j+1}\Pi(x_j)\ge 0$

$$\varphi_j(x):=\begin{cases}\varphi_{j,1}(x), & \text{if } \psi_j\equiv\underline{\psi}_j,\\ \varphi_{j,3}(x), & \text{if } \psi_j\equiv\overline{\psi}_j,\\ \alpha_j\varphi_{j,1}(x)+(1-\alpha_j)\varphi_{j,3}(x), & \text{if } \psi_j\equiv\alpha_j\underline{\psi}_j+(1-\alpha_j)\overline{\psi}_j,\end{cases}$$

and for $\delta_{j+1}\Pi(x_j)<0$

$$\varphi_j(x):=\varphi_{j,2}(x).$$

Show that P_n is a required polynomial in Theorem 1.1. First, using (3.13) and (3.2b), show that $P_n\in\mathbb{T}_{cn}$. Namely,

$$\begin{aligned}P_n(x)=&\frac{A}{6\pi}x^3+\frac{\lambda_n}{2h^2}x^2+\frac{1}{2h^2}x^2\sum_{j\in H_3}\delta_{j+1}\frac{\pi-v_j}{2\pi}\\&+\frac{\Delta_n}{h}x-\frac{\lambda_n}{2h^2}(x_n+x_{n-1})x+\frac{1}{2h^2}x\sum_{j\in H_3}\delta_{j+1}\left(\frac{x_jx_{j-1}}{2\pi}-v_j+\frac{\pi}{3}\right)+Q_n(x),\end{aligned}$$

where

$$A:=\frac{1}{2h^2}\sum_{j\in H_3}\delta_{j+1}=\frac{1}{2h^2}\sum_{j=2-n}^{n-1}\delta_{j+1}=\frac{1}{2h^2}(\lambda_{2-n}-\lambda_n)=\frac{1}{2h^2}(\lambda_{n+2}-\lambda_n)=0$$

(due to the periodicity of f_0 and the equality $\lambda_\nu=\lambda_\mu$ for a parabola), and

$$Q_n(x):=f(x_n)-\frac{\Delta_n}{h}x_n+\frac{\lambda_n}{2h^2}x_nx_{n-1}+\frac{1}{2h^2}\sum_{j\in H_3,\nu=1\vee2\vee3}\delta_{j+1}Q_{j,\nu}(x)\in\mathbb{T}_{cn}.$$

Thus,

$$
\begin{aligned}
P_n(x) =& Q_n(x) + x^2\left[\frac{\lambda_n}{2h^2} + \frac{1}{8\pi h^2}\sum_{j\in H_3}\delta_{j+1}(2\pi - x_j - x_{j-1})\right] \\
&+ x\left[\frac{\Delta_n}{h} - \frac{\lambda_n}{2h^2}(x_n + x_{n-1}) + \frac{1}{12\pi h^2}\sum_{j\in H_3}\delta_{j+1}(3x_jx_{j-1} - 3\pi(x_j + x_{j-1}) + 2\pi^2)\right] \\
=& Q_n(x) + x^2\left[\frac{\lambda_n}{2h^2} + \frac{1}{8nh^2}\sum_{j=2-n}^{n-1}\delta_{j+1}(2j + 2n - 1)\right] \\
&+ x\left[\frac{2\Delta_n n - \lambda_n n(1-2n)}{2\pi} + \frac{1}{12\pi}\sum_{j=2-n}^{n-1}\delta_{j+1}\big(3j^2 + (6n-3)j + 2n^2 - 3n\big)\right] \\
=& Q_n(x) + x^2\left[\frac{\lambda_n}{2h^2} + \frac{1}{4nh^2}\sum_{j=2-n}^{n-1}\delta_{j+1}j\right] \\
&+ x\left[\frac{2\Delta_n n - \lambda_n n(1-2n)}{2\pi} + \frac{1}{12\pi}\left(3\sum_{j=2-n}^{n-1}\delta_{j+1}j^2 - 2(6n-3)n\lambda_n\right)\right] \\
=& Q_n(x) + x^2\frac{1}{4h^2}\left[2\lambda_n + \frac{1}{n}\left(-(n-1)\lambda_n + (1-n)\lambda_{2-n} + \sum_{j=2-n}^{n-1}\lambda_j\right)\right] \\
&+ x\left[\frac{2\Delta_n n - \lambda_n n(1-2n)}{2\pi} + \frac{1}{12\pi}\Big(12n(\lambda_n - \lambda_{n-1}) - 2(6n-3)n\lambda_n\Big)\right] \\
=& Q_n(x) + x^2\frac{1}{4h^2}(\lambda_n - \lambda_{n+2}) \\
&+ x\frac{1}{12\pi}\big(12n\Delta_n + 12n^2\lambda_n - 6n\lambda_n - 12n^2\lambda_n + 6n\lambda_n + 12n\lambda_n - 12n\Delta_{n-1}\big) \\
=& Q_n(x),
\end{aligned}
$$

where we again used the periodicity of f_0 and the equalities

$$
\begin{aligned}
&\sum_{j=2-n}^{n-1}\lambda_j = -\lambda_{n+1} - \lambda_n = -2\lambda_n, \quad \frac{1}{n}\sum_{j=2-n}^{n-1}\delta_{j+1}j = -\lambda_n - \lambda_{n+2} = -2\lambda_n, \\
&\sum_{j=2-n}^{n-1}\delta_{j+1}j^2 = -(n-1)^2\lambda_n + \sum_{j=2-n}^{n-1}\lambda_j(2j-1) + (1-n)^2\lambda_{n+2} \\
&\qquad\qquad = 2\sum_{j=2-n}^{n-1}\lambda_j j + 2\lambda_n = 4n(\lambda_n - \Delta_{n-1}).
\end{aligned}
$$

So, $P_n \in \mathbb{T}_{cn}$.

Now, prove (1.2a). Taking into account (3.2b), write (3.7) for S_0 in the form

$$S_0(x)=f(x_n)+\frac{\Delta_n}{h}(x-x_n)+\frac{\lambda_n}{2h^2}\Psi_n(x)+\frac{1}{2h^2}\sum_{j\in H_3}\delta_{j+1}\Psi_j(x). \tag{4.2}$$

Using this with (4.1), (3.12) with (3.8), and (3.9c), write

$$\begin{aligned}P_n'(x)\Pi(x)&=\big(P_n'(x)-S_0'(x)+S_0'(x)\big)\Pi(x)\\&=\frac{1}{2h^2}\sum_{j\in H_3}\frac{1}{\Pi^2(x_j)}\delta_{j+1}\Pi(x_j)\big(\Psi_j'(x)-\varphi_j'(x)\big)\Pi(x_j)\Pi(x)+S_0'(x)\Pi(x)\\&\geq 0,\end{aligned}$$

where $x\in[-\pi,\pi]\setminus\big(O_4\cup\{v_j\}_{j\in H_4}\big)$, i.e., by the continuity of P_n', $P_n'(x)\Pi(x)\geq 0$, for $x\in[-\pi,\pi]\setminus O_4$, that implies (1.2a).

The estimate (1.2b) readily follows from the periodicity of f and P_n, (4.2), (4.1), (3.12c), (3.3) and (2.2). Namely,

$$\begin{aligned}\|f-P_n\|&=\|f-S_0+S_0-P_n\|_{[-\pi,\pi]}\leq\|f-S_0\|_{[-\pi,\pi]}+\left\|\frac{1}{2h^2}\sum_{j\in H_3}\delta_{j+1}\big(\Psi_j-\varphi_j\big)\right\|_{[-\pi,\pi]}\\&\leq c\omega_3(f,h)+c\omega_3(f,h)\left\|\frac{1}{2h^2}\sum_{j=1-n}^{n}h^2\Gamma_j^3\right\|\leq c\omega_3(f,h).\end{aligned}$$

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