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# Multilinear Fractional Integrals and Commutators on Generalized Herz Spaces

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**Abstract.** Suppose $\vec{b}=(b_1,\cdots,b_m)\in(BMO)^m$, $I^{\Pi b}_{\alpha,m}$ is the iterated commutator of $\vec{b}$ and the $m$-linear multilinear fractional integral operator $I_{\alpha,m}$. The purpose of this paper is to discuss the boundedness properties of $I_{\alpha,m}$ and $I^{\Pi b}_{\alpha,m}$ on generalized Herz spaces with general Muckenhoupt weights.

**Key Words**: Multilinear fractional integral, generalized Herz space, commutator, Muckenhoupt weight.

**AMS Subject Classifications**: 42B20, 42B35

## 1 Introduction

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space, $(\mathbb{R}^n)^m=\mathbb{R}^n\times\cdots\times\mathbb{R}^n$ be the $m$-fold product space ($m\in\mathbb{N}$), and let $\vec{f}=(f_1,\cdots,f_m)$ be a collection of $m$ functions on $\mathbb{R}^n$. Given $\alpha\in(0,mn)$ and $(b_1,\cdots,b_m)\in(BMO)^m$. We consider the following multilinear fractional integral operators $I_{\alpha,m}$ defined by

$$I_{\alpha,m}(\vec{f})(x)=\int_{(\mathbb{R}^n)^m}\frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}dy_1\cdots dy_m. \tag{1.1}$$

The corresponding iterated commutators $I^{\Pi b}_{\alpha,m}$ defined by

$$I^{\Pi b}_{\alpha,m}(\vec{f})(x)=\int_{(\mathbb{R}^n)^m}\frac{\prod_{i=1}^m(b_i(x)-b_i(y_i)f_i(y_i)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}dy_1\cdots dy_m. \tag{1.2}$$

As is well known, multilinear fractional integral operator was first studied by Grafakos [1], subsequently, by Kenig and Stein [2], Grafakos and Kalton [3]. In 2009, Moen [4] introduced weight function $A_{\vec{p},q}$ and gave weighted inequalities for multilinear

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fractional integral operators; In 2013, Chen and Wu [5] obtained the weighted norm inequalities for the iterated commutators $I_{\alpha,m}^{\Pi b}$. More results of the weighted inequalities for multilinear fractional integral and commutators can be found in [6–9].

We list some results mentioned above.

**Theorem 1.1** (see [4]). *Let $m\geq 2$ and $0<\alpha<mn$. Suppose $1/p=1/p_1+\cdots+1/p_m$, $1/q=1/p-\alpha/n$, $\vec{\omega}=(\omega_1,\cdots,\omega_m)$ satisfies the $A_{\vec{p},q}$ condition, and $\nu_{\vec{\omega}}=\prod_{i=1}^m\omega_i$. If $p_1,\cdots,p_m\in(1,\infty)$, then there exists a constant $C$ independent of $\vec{f}=(f_1,\cdots,f_m)$ such that*

$$\|I_{\alpha,m}\vec{f}\|_{L^q(\nu_{\vec{\omega}}^q)}\leq C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i})}. \tag{1.3}$$

**Theorem 1.2** (see [5]). *Let $0<\alpha<mn$ and $(b_1,\cdots,b_m)\in(BMO)^m$. For $1<p_1,\cdots,p_m<\infty$, $1/p=1/p_1+\cdots+1/p_m$, and $1/q=1/p-\alpha/n$, if $\vec{\omega}\in A_{\vec{p},q}$, then there exists a constant $C>0$ such that*

$$\|I_{\alpha,m}^{\Pi b}(\vec{f})\|_{L^q(\nu_{\vec{\omega}}^q)}\leq C\prod_{i=1}^m\|b_i\|_*\|f_i\|_{L^{p_i}(\omega_i^{p_i})}, \tag{1.4}$$

*where $\nu_{\vec{\omega}}=\prod_{i=1}^m\omega_i$.*

Let $B_k=\{x\in\mathbb{R}^n:\ |x|\leq 2^k\}$ and $C_k=B_k\setminus B_{k-1}$ for any $k\in\mathbb{Z}$. Denote $\chi_k=\chi_{C_k}$ for $k\in\mathbb{Z}$, where $\chi_{C_k}$ is the characteristic function of the set $C_k$. The following weighted Herz space was introduced by Lu and Yang in [10].

Let $\sigma\in\mathbb{R}$, $0<p,q<\infty$ and $\omega_1$, $\omega_2$ be two weight functions on $\mathbb{R}^n$. The homogeneous weighted Herz space $\dot{K}_q^{\sigma,p}(\omega_1,\omega_2)$ is defined by

$$\dot{K}_q^{\sigma,p}(\omega_1,\omega_2)=\{f\in L_{loc}^q(\mathbb{R}^n\setminus\{0\},\omega_2):\ \|f\|_{\dot{K}_q^{\sigma,p}(\omega_1,\omega_2)}<\infty\},$$

where

$$\|f\|_{\dot{K}_q^{\sigma,p}(\omega_1,\omega_2)}=\Big(\sum_{k=-\infty}^{\infty}\omega_1(B_k)^{\sigma p/n}\|f\chi_k\|_{L_{\omega_2}^q}^p\Big)^{1/p}.$$

In 2000, Lu, Yabuta and Yang in [11] obtained boundedness results for sublinear operators on weighted Herz spaces with general Muckenhoupt weights. Recently, many authors considered the boundedness of operators and their commutators on weighted Herz type spaces. Wang in [12] proved that the intrinsic square functions are bounded on weighted Herz type Hardy spaces. In [13], Hu and Wang considered parametric Marcinkiewicz integral and its commutator on Weighted Herz spaces.

For a sequence $\varphi=\{\varphi(k)\}_{-\infty}^{\infty}$, $\varphi(k)>0$. We suppose that $\varphi$ satisfies doubling condition of order $(a,b)$ and write $\varphi\in D(a,b)$ if there exists $C\geq 1$ such that

$$C^{-1}2^{a(k-j)}\leq\frac{\varphi(k)}{\varphi(j)}\leq C2^{b(k-j)} \tag{1.5}$$

for $k>j$.

In [14], Komori and Matsuoka introduced generalized Herz spaces.

Suppose that $\omega$ is a weight function on $\mathbb{R}^n$. For $1<p<\infty$, $0<q<\infty$, the generalized Herz space is defined by

$$K_p^q(\varphi,\omega)=\{f:\ f \text{ is a measurable function on } \mathbb{R}^n,\ \|f\|_{K_p^q(\varphi,\omega)}<\infty\},$$

where

$$\|f\|_{K_p^q(\varphi,\omega)}=\Big(\sum_{k=-\infty}^{\infty}\varphi(k)^q\|f\chi_k\|_{L^p(\omega)}^q\Big)^{1/q}. \tag{1.6}$$

Let $\varphi(k)=\omega_1(B_k)^{\alpha/n}$, and $\omega=\omega_2$, then $K_p^q(\varphi,\omega)$ is the weighted Herz spaces $\dot{K}_p^{\alpha,q}(\omega_1,\omega_2)$.

Let $\delta>0$, we say $\omega\in RD(\delta)$ (centered reverse doubling) if there is a positive numbers $C$ such that

$$\frac{\omega(B_k)}{\omega(B_j)}\geq C2^{\delta(k-j)} \quad \text{for}\ \ k>j. \tag{1.7}$$

Komori and Matsuoka considered the boundedness of singular integral operators and fractional integral operators on generalized Herz spaces in [14], and as corollaries of their general theory, they obtained the boundedness of these operators on weighted Herz spaces. Hu, He and Wang in [15] considered the boundedness properties of commutator operators generated by $BMO$ function and fractional function $I_\alpha$ on the generalized Herz spaces.

The aim of the present paper is to investigate the boundedness of multilinear fractional integral operator and its iterated commutator on the generalized Herz spaces. Our results can be formulated as follows.

**Theorem 1.3.** *Let $0<q_i<\infty$, $0<\alpha_i<n$, let $1<p_{1i},p_{2i}<\infty$, and let*

$$\frac{1}{p_{2i}}=\frac{1}{p_{1i}}-\frac{\alpha_i}{n}$$

*for $i=1,\cdots,m$. Set*

$$\frac{1}{q}=\sum_{i=1}^m\frac{1}{q_i},\quad \alpha=\sum_{i=1}^m\alpha_i \quad \text{and} \quad \frac{1}{p_2}=\sum_{i=1}^m\frac{1}{p_{2i}}.$$

*Suppose*

*(1) $\varphi_i\in D(a_i,b_i)$, where $-\frac{\delta_i}{p_{2i}}<a_i\leq b_i<n(1-\frac{1}{p_{1i}})$,*

*(2) $\omega_i^{p_{2i}}\in A_{r_i}$, where $r_i=\min\{1+\frac{p_{2i}}{p'_{1i}},1+\frac{p_{2i}}{p'_{1i}}-\frac{b_ip_{2i}}{n}\}$,*

*(3) $\omega_i^{p_{2i}}\in RD(\delta_i)$,*

*hold for* $i=1,\cdots,m$. *Then* $I_{\alpha,m}$ *is bounded from* $K^{q_1}_{p_{11}}(\varphi_1,\omega_1^{p_{11}})\times\cdots\times K^{q_m}_{p_{1m}}(\varphi_m,\omega_m^{p_{1m}})$ *to* $K^{q}_{p_2}(\varphi,\nu_{\vec{\omega}}^{p_2})$, *where*

$$\nu_{\vec{\omega}}=\prod_{i=1}^m\omega_i \quad \textit{and} \quad \varphi=\prod_{i=1}^m\varphi_i.$$

**Theorem 1.4.** *Under the hypothesis of Theorem 1.3, if* $(b_1,\cdots,b_m)\in(BMO)^m$, *then* $I^{\Pi b}_{\alpha,m}$ *is bounded from* $K^{q_1}_{p_{11}}(\varphi_1,\omega_1^{p_{11}})\times\cdots\times K^{q_m}_{p_{1m}}(\varphi_m,\omega_m^{p_{1m}})$ *to* $K^{q}_{p_2}(\varphi,\nu_{\vec{\omega}}^{p_2})$, *where*

$$\nu_{\vec{\omega}}=\prod_{i=1}^m\omega_i \quad \textit{and} \quad \varphi=\prod_{i=1}^m\varphi_i.$$

**Remark 1.1.** In the case $m=1$, when $b_1\le 0$, the condition (2) in Theorem 1.3 is equivalent to the condition that $\omega\in A_{p_{11},p_{21}}$, but when $b_1>0$, (2) is stronger than $A_{p_{11},p_{21}}$- condition. Komori and Matsuoka in [14] showed that the condition (2) is the best possible by a counterexample.

Let $\varphi_i\in D(0,\sigma_i)$ in Theorem 1.3 and Theorem 1.4, we have

**Corollary 1.1.** Let $0<q_i<\infty$, $0<\alpha_i<n$, let $0\le\sigma_i<n\ (1-1/p_{1i})$, let $1<p_{1i},p_{2i}<\infty$, and let

$$\frac{1}{p_{2i}}=\frac{1}{p_{1i}}-\frac{\alpha_i}{n}.$$

Set

$$\frac{1}{q}=\sum_{i=1}^m\frac{1}{q_i},\quad \sigma=\sum_{i=1}^m\sigma_i,\quad \alpha=\sum_{i=1}^m\alpha_i \quad\text{and}\quad \frac{1}{p_2}=\sum_{i=1}^m\frac{1}{p_{2i}}.$$

If $\omega_i^{p_{2i}}\in A_{1+p_{2i}/p'_{1i}-\sigma_i p_{2i}/n}$, and $(b_1,\cdots,b_m)\in(BMO)^m$, then $I_{\alpha,m}$ and $I^{\Pi b}_{\alpha,m}$ are bounded from $\dot{K}^{\sigma_1,q_1}_{p_{11}}(1,\omega_1^{p_{11}})\times\cdots\times\dot{K}^{\sigma_m,q_m}_{p_{1m}}(1,\omega_m^{p_{1m}})$ to $\dot{K}^{\sigma,q}_{p_2}(1,\nu_{\vec{\omega}}^{p_2})$.

## 2 Definitions and preliminaries

We begin with some properties of $A_p$ weights which play a great role in the proofs of our main results.

A weight $\omega$ is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B=B(x_0,r_B)$ denote the ball with the center $x_0$ and radius $r_B$. For any $\lambda>0$, let $\lambda B=B(x_0,\lambda r_B)$. For a given weight function $\omega$ and a measurable set $E$, we also denote the Lebesgue measure of $E$ by $|E|$ and set weighted measure $\omega(E)=\int_E\omega(x)dx$.

A weight $\omega$ is said to belong to $A_p$ for $1<p<\infty$, if there exists a constant $C$ such that for every ball $B\subset\mathbb{R}^n$,

$$\Big(\frac{1}{|B|}\int_B\omega(x)dx\Big)\Big(\frac{1}{|B|}\int_B\omega(x)^{1-p'}dx\Big)^{p-1}\le C, \tag{2.1}$$

where $s'$ is the dual of $s$ such that $1/s+1/s'=1$. The class $A_1$ is defined by replacing the above inequality with

$$\frac{1}{|B|}\int_B w(y)dy \le C\cdot \operatorname*{ess\,inf}_{x\in B} w(x) \quad \text{for every ball } B\subset\mathbb{R}^n. \tag{2.2}$$

A weight $\omega$ is said to belong to $A_\infty$ if there are positive numbers $C$ and $\delta$ so that

$$\frac{\omega(E)}{\omega(B)} \le C\Big(\frac{|E|}{|B|}\Big)^\delta \tag{2.3}$$

for all balls $B$ and all measurable $E\subset B$. It is well known that

$$A_\infty = \bigcup_{1\le p<\infty} A_p. \tag{2.4}$$

The classical $A_p$ weight theory was first introduced by Muckenhoupt in the study of weighted $L^p$-boundedness of Hardy-Littlewood maximal function in [16].

**Lemma 2.1.** *Suppose $\omega\in A_p$ and the following statements hold.*
*(i) For any $1\le p<\infty$, there exists a positive numbers C such that*

$$\frac{\omega(B_j)}{\omega(B_k)} \le C2^{np(j-k)} \quad \textit{for } j>k, \tag{2.5}$$

*(ii) For some $\delta>0$, $\omega\in RD(\delta)$, that is*

$$\frac{\omega(B_j)}{\omega(B_k)} \ge C2^{\delta(j-k)} \quad \textit{for } j>k, \tag{2.6}$$

*(iii) For any $1<p<\infty$, there is some $\bar{r}$, $1<\bar{r}<p$ such that $\omega\in A_{\bar{r}}$.*

We also need another weight class $A_{p,q}$ introduced by Muckenhoupt and Wheeden in [17] to studied weighted boundedness of fractional integral operators.

Given $1\le p\le q<\infty$. We say that $\omega\in A_{p,q}$ if there exists a constant $C$ such that for every ball $B\subset\mathbb{R}^n$, the inequality

$$\Big(\frac{1}{|B|}\int_B \omega(y)^{-p'}dy\Big)^{1/p'}\Big(\frac{1}{|B|}\int_B \omega(y)^q dy\Big)^{1/q} \le C \tag{2.7}$$

holds when $1<p<\infty$, and the inequality

$$\Big(\frac{1}{|B|}\int_B \omega(y)^q dy\Big)^{1/q} \le C\cdot \operatorname*{ess\,inf}_{x\in B} w(x) \quad \text{for every ball } B\subset\mathbb{R}^n \tag{2.8}$$

holds when $p=1$.

By (2.7), we have

$$\Big(\int_B \omega(y)^{-p'}dy\Big)^{1/p'}\Big(\int_B \omega(y)^q dy\Big)^{1/q} \le C|B|^{1/p'+1/q}. \tag{2.9}$$

We summarize some properties about weights $A_{p,q}$ (see [17,18]).

**Lemma 2.2.** *Given* $1\le p\le q<\infty$,

*(i)* $\omega\in A_{p,q}$ *if and only if* $\omega^q\in A_{1+q/p'}$,

*(ii)* $\omega\in A_{p,q}$ *if and only if* $\omega^{-p'}\in A_{1+p'/q}$,

*(iii) If* $p_1<p_2$ *and* $q_2>q_1$, *then* $A_{p_1,q_1}\subset A_{p_2,q_2}$.

Let us recall the definition of multiple weights. For $m$ exponents $p_1,\cdots,p_m$, we write $\vec{p}=(p_1,\cdots,p_m)$. Let $p_1,\cdots,p_m\in[1,\infty)$, $1/p=\sum_{i=1}^m 1/p_i$, and let $q>0$. Given $\vec{\omega}=(\omega_1,\cdots,\omega_m)$, set $\nu_{\vec{\omega}}=\prod_{i=1}^m\omega_i$. We say that $\vec{\omega}$ satisfies the $A_{\vec{p},q}$ condition if it satisfies

$$\sup_B\Big(\frac{1}{|B|}\int_B\nu_{\vec{\omega}}(x)^q dx\Big)^{1/q}\prod_{i=1}^m\Big(\frac{1}{|B|}\int_B\omega_i(x)^{-p_i'}dx\Big)^{1/p_i'}\le C. \tag{2.10}$$

By Remark 3.3 in [4], we have

**Lemma 2.3.** *Let* $1/q=1/q_1+\cdots+1/p_m$. *If* $1\le p_i\le q_i$, $\omega_i\in A_{p_i,q_i}$ *for* $i=1,\cdots,m$, *then*

$$\nu_{\vec{\omega}}=\prod_{i=1}^m\omega_i\in A_{\vec{p},q}.$$

**Lemma 2.4** (see [4]). *Let* $0<\alpha<mn$, *and* $p_1,\cdots,p_m\in[1,\infty)$, *let* $1/p=\sum_{i=1}^m 1/p_i$, *and let* $1/q=1/p-\alpha/n$. *If* $\vec{\omega}\in A_{\vec{p},q}$ *then*

$$\nu_{\vec{\omega}}^q\in A_{mq}\quad\text{and}\quad\omega_i^{-p_i'}\in A_{mp_i'}\quad\text{for }\ i=1,\cdots,m. \tag{2.11}$$

Following [19], a locally integrable function $b$ is said to be in $BMO$ if

$$\sup_{B\subset\mathbb{R}^n}\frac{1}{|B|}\int_B|b(x)-b_B|dx=\|b\|_*<\infty,$$

where $b_B=|B|^{-1}\int_B b(y)dy$.

**Lemma 2.5** (see [9]). *Suppose* $\omega\in A_\infty$ *and* $b\in BMO$. *Then for any* $1\le q<\infty$ *and* $r_1,r_2>0$, *we have*

$$\Big(\frac{1}{\omega(B(x_0,r_1))}\int_{B(x_0,r_1)}|b(x)-b_{B(x_0,r_2)}|^q\omega(x)dx\Big)^{1/q}\le C\|b\|_*\Big(1+\Big|\ln\frac{r_1}{r_2}\Big|\Big). \tag{2.12}$$

## 3 Proof of Theorem 1.3

Without loss of generality, we only prove Theorem 1.3 for the case $m=2$. The method can be extended for any $m$-linear case without any essential difficulty.

Suppose $\omega_l^{p_{2l}}\in A_{r_l}$, $l=1,2$, by (iii) of Lemma 2.1, there exist $1<\bar{r}_l<r_l$ such that $\omega_l^{p_{2l}}\in A_{\bar{r}_l}$. If $k>i$, $k>j$, by (2.5), we have

$$\left(\frac{\omega_1^{p_{21}}(B_k)}{\omega_1^{p_{21}}(B_i)}\right)^{\frac{1}{p_{21}}}\le C2^{n\bar{r}_1(k-i)/p_{21}},\qquad \left(\frac{\omega_2^{p_{22}}(B_k)}{\omega_2^{p_{22}}(B_j)}\right)^{\frac{1}{p_{22}}}\le C2^{n\bar{r}_2(k-j)/p_{22}}. \tag{3.1}$$

Let $\omega_l^{p_{2l}}\in RD(\delta_l)$, when $k<i$, $k<j$, we have

$$\left(\frac{\omega_1^{p_{21}}(B_k)}{\omega_1^{p_{21}}(B_i)}\right)^{\frac{1}{p_{21}}}\le C2^{\delta_1(k-i)/p_{21}},\qquad \left(\frac{\omega_2^{p_{22}}(B_k)}{\omega_2^{p_{22}}(B_j)}\right)^{\frac{1}{p_{22}}}\le C2^{\delta_2(k-j)/p_{22}}. \tag{3.2}$$

On the other hand, if $\varphi_l\in D(a_l,b_l)$, $l=1,2$, we have

$$C^{-1}2^{a_1(k-i)}\le\frac{\varphi_1(k)}{\varphi_1(i)}\le C2^{b_1(k-i)}\quad\text{for }\ k>i, \tag{3.3}$$

and

$$C^{-1}2^{a_2(k-j)}\le\frac{\varphi_2(k)}{\varphi_2(j)}\le C2^{b_2(k-j)}\quad\text{for }\ k>j. \tag{3.4}$$

We have the following conclusions.

**Theorem 3.1.** *Let $0<\alpha_l<n$, $1<p_{1l},p_{2l}<\infty$ and*

$$\frac{1}{p_{2l}}=\frac{1}{p_{1l}}-\frac{\alpha_l}{n}$$

*for $l=1,2$. Suppose*

$$\frac{1}{p_2}=\frac{1}{p_{21}}+\frac{1}{p_{22}},\quad \alpha=\alpha_1+\alpha_2\quad and\quad \varphi=\varphi_1\varphi_2.$$

*We assume that, for $l=1,2$,*
*(1) $\varphi_l\in D(a_l,b_l)$, where $-\frac{\delta_l}{p_{2l}}<a_l\le b_l<n(1-\frac{1}{p_{1l}})$,*
*(2) $\omega_l^{p_{2l}}\in A_{r_l}$, where $r_l=\min\{1+\frac{p_{2l}}{p'_{1l}},1+\frac{p_{2l}}{p'_{1l}}-\frac{b_lp_{2l}}{n}\}$,*
*(3) $\omega_l^{p_{2l}}\in RD(\delta_l)$.*
*Then*

$$\begin{aligned}&\varphi(k)\big\|I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\big\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\ \le\ & CD_1(k,i)\varphi_1(i)\big\|f\chi_i\big\|_{L^{p_{11}}(\omega_1^{p_{11}})}\times D_2(k,j)\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})},\end{aligned} \tag{3.5}$$

*where* $\nu_{\vec{\omega}}=\omega_1\omega_2$, *and*

$$D_l(k,i)=\begin{cases} 2^{(k-i)(b_l+n\bar{r}_l/p_{2l}-n+\alpha_l)}, & \textit{if } i\le k-2,\\ 1, & \textit{if } k-1\le i\le k-1,\\ 2^{(k-i)(a_l+\delta_l/p_{2l})}, & \textit{if } i\ge k+2,\end{cases} \tag{3.6}$$

*for* $l=1,2$.

*Proof.* From Lemma 2.3 and Theorem 1.1, we know that $I_{\alpha,2}$ is bounded from $L^{p_{11}}(\omega_1^{p_{11}})\times L^{p_{12}}(\omega_2^{p_{12}})$ to $L^{p_2}(\nu_{\vec{\omega}}^{p_2})$. When $k-1\le i,j\le k+1$, then by $\varphi_1(k)\sim\varphi_1(i)$, $\varphi_2(k)\sim\varphi_2(j)$, we obtain

$$\begin{aligned}&\varphi(k)\big\|I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\big\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\ \le& C\varphi_1(i)\big\|f\chi_i\big\|_{L^{p_{11}}(\omega_1^{p_{11}})}\times\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}.\end{aligned} \tag{3.7}$$

This means $D_1(k,i)=D_2(k,j)=1$ for $k-1\le i,j\le k+1$.

In the other case, we see that

$$|x-y_1|+|x-y_2|\sim 2^{\max\{i,j,k\}} \tag{3.8}$$

for $x\in C_k$, $y_1\in C_i$, and $y_2\in C_j$. Then

$$\begin{aligned}&\varphi(k)\big\|I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\big\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\ \le& C\varphi(k)2^{-\max\{i,j,k\}(2n-\alpha)}\|f\chi_i\|_{L^1}\|g\chi_j\|_{L^1}(\nu_{\vec{\omega}}^{p_2}(B_k))^{1/p_2}.\end{aligned} \tag{3.9}$$

By Hölder's inequality, we get

$$\|f\chi_i\|_{L^1}\le\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\Big(\omega_1^{-p_{11}'}(B_i)\Big)^{1/p_{11}'}, \tag{3.10a}$$

$$\|g\chi_j\|_{L^1}\le\|g\chi_j\|_{L^{p_{12}}(\omega_2^{p_{12}})}\Big(\omega_2^{-p_{12}'}(B_j)\Big)^{1/p_{12}'}, \tag{3.10b}$$

and

$$\Big(\nu_{\vec{\omega}}^{p_2}(B_k)\Big)^{1/p_2}\le\Big(\omega_1^{p_{21}}(B_k)\Big)^{1/p_{21}}\Big(\omega_2^{p_{22}}(B_k)\Big)^{1/p_{22}}. \tag{3.11}$$

Since $\omega_l^{p_{2l}}\in A_{r_l}$, $r_l\le 1+\frac{p_{2l}}{p_{1l}'}$, we know $\omega_l\in A_{p_{l1,l2}}$ for $l=1,2$. Then by (2.9) and $\frac{1}{p_{2l}}=\frac{1}{p_{2l}}-\frac{\alpha_l}{n}$, we get

$$\Big(\omega_1^{p_{21}}(B_i)\Big)^{1/p_{21}}\Big(\omega_1^{-p_{11}'}(B_i)\Big)^{1/p_{11}'}\le C|B_i|^{1/p_{21}+1/p_{11}'}=C2^{i(n-\alpha_1)}, \tag{3.12}$$

and

$$\Big(\omega_2^{p_{22}}(B_i)\Big)^{1/p_{22}}\Big(\omega_1^{-p_{12}'}(B_i)\Big)^{1/p_{12}'}\le C|B_j|^{1/p_{22}+1/p_{12}'}=C2^{j(n-\alpha_2)}. \tag{3.13}$$

Then, by (3.9)-(3.13),

$$\begin{aligned}
&\varphi(k)\big\|I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\big\|_{L^{p_2}(\nu_{\vec\omega}^{p_2})}\\
\le& C2^{-\max\{i,j,k\}(2n-\alpha)}\varphi(k)\|f\chi_i\|_{L^1}\|g\chi_j\|_{L^1}\left(\nu_{\vec\omega}^{p_2}(B_k)\right)^{1/p_2}\\
\le& C2^{-\max\{i,k\}(n-\alpha_1)}\varphi_1(k)\|f\chi_i\|_{L^1}\left(\omega_1^{p_{21}}(B_k)\right)^{1/p_{21}}\\
&\times 2^{-\max\{j,k\}(n-\alpha_2)}\varphi_2(k)\|g\chi_j\|_{L^1}\left(\omega_2^{-p'_{12}}(B_j)\right)^{1/p'_{12}}\\
=& C2^{-\max\{i,k\}(n-\alpha_1)}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\\
&\times\left(\omega_1^{p_{21}}(B_i)\right)^{1/p_{21}}\left(\omega_1^{-p'_{11}}(B_i)\right)^{1/p'_{11}}\frac{\varphi_1(k)}{\varphi_1(i)}\left(\frac{\omega_1^{p_{21}}(B_k)}{\omega_1^{p_{21}}(B_i)}\right)^{\frac{1}{p_{21}}}\\
&\times 2^{-\max\{j,k\}(n-\alpha_2)}\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}\\
&\times\left(\omega_2^{p_{22}}(B_j)\right)^{1/p_{22}}\left(\omega_1^{-p'_{12}}(B_j)\right)^{1/p'_{12}}\frac{\varphi_2(k)}{\varphi_2(j)}\left(\frac{\omega_2^{p_{22}}(B_k)}{\omega_2^{p_{22}}(B_j)}\right)^{\frac{1}{p_{22}}}\\
\le& C2^{i(n-\alpha_1)-\max\{i,k\}(n-\alpha_1)}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\frac{\varphi_1(k)}{\varphi_1(i)}\left(\frac{\omega_1^{p_{21}}(B_k)}{\omega_1^{p_{21}}(B_i)}\right)^{\frac{1}{p_{21}}}\\
&\times 2^{j(n-\alpha_2)-\max\{j,k\}(n-\alpha_2)}\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}\frac{\varphi_2(k)}{\varphi_2(j)}\left(\frac{\omega_2^{p_{22}}(B_k)}{\omega_2^{p_{22}}(B_j)}\right)^{\frac{1}{p_{22}}}.
\end{aligned}\tag{3.14}$$

When $i\le k-2$, $j\le k-2$, by (3.1), (3.3), (3.4) and (3.14), we have

$$\begin{aligned}
&\varphi(k)\big\|I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\big\|_{L^{p_2}(\nu_{\vec\omega}^{p_2})}\\
\le& C2^{(k-i)(b_1+n\bar r_1/p_{21}-n+\alpha_1)}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\\
&\times 2^{(k-j)(b_2+n\bar r_2/p_{22}-n+\alpha_2)}\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}.
\end{aligned}\tag{3.15}$$

When $i\le k-2$, $j\ge k+2$, by (3.1)-(3.4), we have

$$\begin{aligned}
&\varphi(k)\big\|I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\big\|_{L^{p_2}(\nu_{\vec\omega}^{p_2})}\\
\le& C2^{(k-i)(b_1+n\bar r_1/p_{21}-n+\alpha_1)}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\\
&\times 2^{(k-j)(a_2+\delta_2/p_{22})}\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}.
\end{aligned}\tag{3.16}$$

By symmetry, when $i\ge k+2$, $j\le k-2$, we have

$$\begin{aligned}
&\varphi(k)\big\|I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\big\|_{L^{p_2}(\nu_{\vec\omega}^{p_2})}\\
\le& C2^{(k-i)(a_1+\delta_1/p_{21})}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\\
&\times 2^{(k-j)(b_2+n\bar r_2/p_{22}-n+\alpha_2)}\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}.
\end{aligned}\tag{3.17}$$

When $i\geq k+2$, $j\geq k+2$, by (3.2)-(3.4), we get

$$\begin{aligned}&\varphi(k)\big\|I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\big\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\ \leq& C2^{(k-i)(a_1+\delta_1/p_{21})}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\\ &\times 2^{(k-j)(a_2+\delta_2/p_{22})}\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}. \qquad (3.18)\end{aligned}$$

When $i\leq k-2$, $k-1\leq j\leq k+1$. Note that $\varphi_2(k)\sim\varphi_2(j)$, and $\omega_2^{p_{22}}(B_k)\sim\omega_2^{p_{22}}(B_j)$, then

$$\begin{aligned}&\varphi(k)\big\|I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\big\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\ \leq& C2^{(k-i)(b_1+n\bar{r}_1/p_{21}-n+\alpha_1)}\varphi_1(i)\big\|f\chi_i\big\|_{L^{p_{11}}(\omega_1^{p_{11}})}\times\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}. \qquad (3.19)\end{aligned}$$

Similar to the estimates (3.19), we can verify (3.5), (3.6) in the following case: $k-1\leq i\leq k+1$, $j\leq k-2$, $k-1\leq i\leq k+1$, $j\geq k+2$ and $i\geq k+2$, $k-1\leq j\leq k+1$. Thus we obtain

$$\begin{aligned}&\varphi(k)\big\|I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\big\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\ \leq& CD_1(k,i)\varphi_1(i)\big\|f\chi_i\big\|_{L^{p_{11}}(\omega_1^{p_{11}})}\times D_2(k,j)\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}. \qquad (3.20)\end{aligned}$$

Thus, we complete the proof. □

Now, we give the proof of Theorem 1.3.

*Proof of* Theorem 1.3. For $l=1,2$, since

$$r_l\leq 1+\frac{p_{2l}}{p'_{1l}}-\frac{b_l p_{2l}}{n}\quad\text{and}\quad\frac{1}{p_{2l}}=\frac{1}{p_{1l}}-\frac{\alpha_l}{n},$$

we get

$$-n+\alpha_l+\frac{nr_l}{p_{2l}}+b_l\leq 0.$$

Thus

$$-n+\alpha_l+\frac{n\bar{r}_l}{p_{2l}}+b_l<0. \qquad (3.21)$$

If $p_2>1$, by Minkowski's inequality and Theorem 3.1, we have

$$\begin{aligned}&\varphi(k)\|I_{\alpha,2}(f,g)\chi_k\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\ \leq& C\varphi(k)\Big\|\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\Big\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\ \leq& C\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\varphi(k)\big\|I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\big\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\end{aligned}$$

$$\le C\sum_{i=-\infty}^{\infty} D_1(k,i)\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\times\sum_{j=-\infty}^{\infty} D_2(k,j)\varphi_2(j)\|g\chi_j\|_{L^{p_{12}}(\omega_2^{p_{12}})}$$
$$\le C\sum_{i=-\infty}^{\infty} D_1(k,i)^{1-\epsilon}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\times\sum_{j=-\infty}^{\infty} D_2(k,j)^{1-\epsilon}\varphi_2(j)\|g\chi_j\|_{L^{p_{12}}(\omega_2^{p_{12}})}$$

for any $0<\epsilon<1$, since $D_1(k,i)+D_2(k,j)\to 0$ whenever $i,j\to\pm\infty$.

If $0<p_2\le 1$, note the fact

$$\sum_{i=-\infty}^{\infty} D_1(k,i)^{\frac{\epsilon p_2}{1-p_2}}<\infty,\qquad \sum_{j=-\infty}^{\infty} D_2(k,j)^{\frac{\epsilon p_2}{1-p_2}}<\infty,$$

then by Theorem 3.1, the inequality $(\sum|a_i|)^{p_2}\le\sum|a_i|^{p_2}$, and Hölder's inequality, we have

$$\begin{aligned}
&\varphi(k)\|I_{\alpha,2}(f,g)\chi_k\|_{L^{p_2}(\nu_{\vec\omega}^{p_2})}\\
\le& C\Big\{\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\Big[\varphi(k)\|I_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\|_{L^{p_2}(\nu_{\vec\omega}^{p_2})}\Big]^{p_2}\Big\}^{\frac{1}{p_2}}\\
\le& C\Big\{\sum_{i=-\infty}^{\infty} D_1(k,i)^{p_2}\varphi_1(i)^{p_2}\|f\chi_i\|^{p_2}_{L^{p_{11}}(\omega_1^{p_{11}})}\times\sum_{j=-\infty}^{\infty} D_2(k,j)^{p_2}\varphi_2(j)^{p_2}\|g\chi_j\|^{p_2}_{L^{p_{12}}(\omega_2^{p_{12}})}\Big\}^{\frac{1}{p_2}}.\\
\le& C\sum_{i=-\infty}^{\infty} D_1(k,i)^{1-\epsilon}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\Big\{\sum_{i=-\infty}^{\infty} D_1(k,i)^{\frac{p_2\epsilon}{1-p_2}}\Big\}^{\frac{1-p_2}{p_2}}\\
&\times\sum_{j=-\infty}^{\infty} D_2(k,j)^{1-\epsilon}\varphi_2(j)\|g\chi_j\|_{L^{p_{12}}(\omega_2^{p_{12}})}\Big\{\sum_{j=-\infty}^{\infty} D_2(k,j)^{\frac{p_2\epsilon}{1-p_2}}\Big\}^{\frac{1-p_2}{p_2}}\\
\le& C\sum_{i=-\infty}^{\infty} D_1(k,i)^{1-\epsilon}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\times\sum_{j=-\infty}^{\infty} D_2(k,j)^{1-\epsilon}\varphi_2(j)\|g\chi_j\|_{L^{p_{12}}(\omega_2^{p_{12}})}.
\end{aligned}$$

Since $0<q<\infty$, then by Hölder's inequality,

$$\begin{aligned}
&\Big\|I_{\alpha,2}(f,g)\Big\|_{K^q_{p_2}(\varphi,\nu_{\vec\omega}^{p_2})}\\
\le&\Big\{\sum_{k=-\infty}^{\infty}\varphi(k)^q\|I_{\alpha,2}(f,g)\chi_k\|^q_{L^{p_2}(\nu_{\vec\omega}^{p_2})}\Big\}^{\frac{1}{q}}\\
\le& C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{i=-\infty}^{\infty} D_1(k,i)^{1-\epsilon}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\times\sum_{j=-\infty}^{\infty} D_2(k,j)^{1-\epsilon}\varphi_2(j)\|g\chi_j\|_{L^{p_{12}}(\omega_2^{p_{12}})}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{i=-\infty}^{\infty} D_1(k,i)^{1-\epsilon}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\Big)^{q_1}\Big\}^{\frac{1}{q_1}}\\
&\times\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{\infty} D_2(k,j)^{1-\epsilon}\varphi_2(j)\|g\chi_j\|_{L^{p_{12}}(\omega_2^{p_{12}})}\Big)^{q_2}\Big\}^{\frac{1}{q_2}}\\
=& CN_1\times N_2.
\end{aligned}$$

It's enough to show that

$$N_1 \le C\|f\|_{K^{q_1}_{p_{11}}(\varphi_1,\omega_1^{p_{11}})} \quad \text{and} \quad N_2 \le C\|g\|_{K^{q_2}_{p_{12}}(\varphi_2,\omega_2^{p_{12}})}.$$

By the symmetry, we only give the estimate for $N_1$.

$$\begin{aligned}
N_1^{q_1} \le & \sum_{k=-\infty}^{\infty}\Big(\sum_{i=-\infty}^{k-2} D_1(k,i)^{1-\epsilon}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\Big)^{q_1} \\
& + \sum_{k=-\infty}^{\infty}\Big(\sum_{i=k-1}^{k+1} D_1(k,i)^{1-\epsilon}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\Big)^{q_1} \\
& + \sum_{k=-\infty}^{\infty}\Big(\sum_{i=k+2}^{\infty} D_1(k,i)^{1-\epsilon}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\Big)^{q_1} \\
= & N_{11}+N_{12}+N_{13}.
\end{aligned}$$

By (3.21) we get

$$\sum_{k=i+2}^{\infty} D_1(k,i)^{(1-\epsilon)q_1} = \sum_{k=i+2}^{\infty} 2^{(1-\epsilon)q_1(k-i)(b_1+n\bar{r}_1/p_{21}-n+\alpha_1)} < \infty.$$

Then, if $0<q_1\le 1$, from the inequality

$$\Big(\sum|a_i|\Big)^{q_1} \le \sum|a_i|^{q_1},$$

we obtain

$$\begin{aligned}
N_{11} \le & \sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{k-2} D_1(k,i)^{(1-\epsilon)q_1}\varphi_1(i)^{q_1}\|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})} \\
\le & \sum_{i=-\infty}^{\infty}\varphi_1(i)^{q_1}\|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})}\sum_{k=i+2}^{\infty} D_1(k,i)^{(1-\epsilon)q_1} \\
\le & C\sum_{i=-\infty}^{\infty}\varphi_1(i)^{q_1}\|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})} \\
\le & C\|f\|^{q_1}_{K^{q_1}_{p_{11}}(\varphi_1,\omega_1^{p_{11}})}.
\end{aligned} \tag{3.22}$$

Note that

$$\begin{aligned}
& \sum_{i=-\infty}^{k-2} D_1(k,i)^{1-\epsilon} = \sum_{k=i+2}^{\infty} D_1(k,i)^{1-\epsilon} \\
\le & C\sum_{k=i+2}^{\infty} 2^{(1-\epsilon)(k-i)(b_1+n\bar{r}_1/p_{21}-n+\alpha_1)} < \infty.
\end{aligned} \tag{3.23}$$

Then for $q_1>1$, we have

$$\begin{aligned}
N_{11}\le& C\sum_{k=-\infty}^{\infty}\Big(\sum_{i=-\infty}^{k-2}D_1(k,i)^{1-\epsilon}\varphi_1(i)^{q_1}\|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})}\Big)\Big(\sum_{i=-\infty}^{k-2}D_1(k,i)^{1-\epsilon}\Big)^{\frac{q_1}{q_1'}}\\
\le& C\sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{k-2}D_1(k,i)^{1-\epsilon}\varphi_1(i)^{q_1}\|f\chi_i\|^{p_1}_{L^{q_1}(\omega_1)}\\
\le& C\sum_{i=-\infty}^{\infty}\varphi_1(i)^{q_1}\|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})}\sum_{k=i+2}^{\infty}D_1(k,i)^{1-\epsilon}\\
\le& C\|f\|^{q_1}_{K^{q_1}_{p_{11}}(\varphi_1,\omega_1^{p_{11}})}.
\end{aligned}\tag{3.24}$$

Since $D_1(k,i)=1$ for $i=k-1,k,k+1$, then

$$\begin{aligned}
N_{12}\le& C\sum_{k=-\infty}^{\infty}\sum_{i=k-1}^{k+1}\varphi_1(i)^{q_1}\|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})}\\
\le& C\sum_{i=-\infty}^{\infty}\sum_{k=i-1}^{i+1}\varphi_1(i)^{q_1}\|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})}\\
\le& C\sum_{i=-\infty}^{\infty}\varphi_1(i)^{q_1}\|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})}\\
\le& C\|f\|^{q_1}_{K^{q_1}_{p_{11}}(\varphi_1,\omega_1^{p_{11}})}.
\end{aligned}\tag{3.25}$$

Thus, we have obtained

$$N_{12}\le C\|f\|^{q_1}_{K^{q_1}_{p_{11}}(\varphi_1,\omega_1^{p_{11}})}$$

holds for any $0<q_1<\infty$.

Let us now turn to estimate the last term $N_{13}$. If $0<q_1\le 1$, then

$$\begin{aligned}
N_{13}\le& C\sum_{k=-\infty}^{\infty}\sum_{i=k+2}^{\infty}D_1(k,i)^{(1-\epsilon)q_1}\varphi_1(i)^{q_1}\|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})}\\
\le& C\sum_{i=-\infty}^{\infty}\varphi_1(i)^{q_1}\|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})}\sum_{k=-\infty}^{i-2}D_1(k,i)^{(1-\epsilon)q_1}.
\end{aligned}$$

Since $a_1+\delta_1/p_{21}>0$, we have

$$\sum_{k=-\infty}^{i-2}D_1(k,i)^{(1-\epsilon)q_1}=\sum_{k=-\infty}^{i-2}2^{(1-\epsilon)q_1(k-i)(a_1+\delta_1/p_{21})}<\infty.$$

Thus

$$N_{13}\le C\sum_{i=-\infty}^{\infty}\varphi_1(i)^{q_1}\|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})}\le C\|f\|^{q_1}_{K^{q_1}_{p_{11}}(\varphi_1,\omega_1^{p_{11}})}.\tag{3.26}$$

Note that

$$\sum_{k=-\infty}^{i-2} D_1(k,i)^{1-\epsilon} = \sum_{i=k+2}^{\infty} D_1(k,i)^{1-\epsilon} = \sum_{i=k+2}^{\infty} 2^{(1-\epsilon)(k-i)(a_1+\delta_1/p_{21})} < \infty.$$

Then for $q_1 > 1$, we have

$$\begin{aligned} N_{13} \le & C \sum_{k=-\infty}^{\infty} \Big( \sum_{i=k+2}^{\infty} D_1(k,i)^{1-\epsilon} \varphi_1(i)^{q_1} \|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})} \Big) \Big( \sum_{i=k+2}^{\infty} D_1(k,i)^{1-\epsilon} \Big)^{\frac{q_1}{q_1'}} \\ \le & C \sum_{k=-\infty}^{\infty} \sum_{i=k+2}^{\infty} D_1(k,i)^{1-\epsilon} \varphi_1(i)^{q_1} \|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})} \\ \le & C \sum_{i=-\infty}^{\infty} \varphi_1(i)^{q_1} \|f\chi_i\|^{q_1}_{L^{p_{11}}(\omega_1^{p_{11}})} \sum_{k=-\infty}^{i-2} D_1(k,i)^{1-\epsilon} \\ \le & C \|f\|^{q_1}_{K^{q_1}_{p_{11}}(\varphi_1, \omega_1^{p_{11}})}. \end{aligned} \tag{3.27}$$

This completes the proof of Theorem 1.3. □

# 4 Proof of Theorem 1.4

According to the proof of Theorem 1.3, we just need to prove the following result.

**Theorem 4.1.** *Under the hypothesis of Theorem 3.1, if* $(b_1,b_2) \in (BMO)^2$, *then*

$$\begin{aligned} & \varphi(k) \big\| I^{\Pi b}_{\alpha,2}(f\chi_i, g\chi_j)\chi_k \big\|_{L^{p_2}(\nu_{\vec\omega}^{p_2})} \\ \le & C \|b_1\|_* E_1(k,i) \varphi_1(i) \|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})} \times \|b_2\|_* E_2(k,j) \varphi_2(j) \|g\chi_j\|_{L^{p_{12}}(\omega_2^{p_{12}})}, \end{aligned} \tag{4.1}$$

*where* $\nu_{\vec\omega} = \omega_1\omega_2$, *and*

$$E_l(k,i) = \begin{cases} (k-i) 2^{(k-i)(b_l + n\bar{r}_l/p_{2l} - n + \alpha_l)}, & \text{if } i \le k-2, \\ 1, & \text{if } k-1 \le i \le k-1, \\ (i-k) 2^{(k-i)(a_l+\delta_l/p_{2l})}, & \text{if } i \ge k+2, \end{cases} \tag{4.2}$$

*for* $l = 1,2$.

*Proof.* When $k-1 \le i,j \le k+1$, by the boundedness of $I^{\Pi b}_{\alpha,2}$ from $L^{p_{11}}(\omega_1^{p_{11}}) \times L^{p_{12}}(\omega_2^{p_{12}})$ to $L^{p_2}(\nu_{\vec\omega}^{p_2})$ and $\varphi_1(k) \sim \varphi_1(i)$, $\varphi_2(k) \sim \varphi_2(j)$, we obtain

$$\begin{aligned} & \varphi(k) \big\| I^{\Pi b}_{\alpha,2}(f\chi_i, g\chi_j)\chi_k \big\|_{L^{p_2}(\nu_{\vec\omega}^{p_2})} \\ \le\ & C \|b_1\|_* \varphi_1(i) \|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})} \times \|b_2\|_* \varphi_2(j) \|g\chi_j\|_{L^{p_{12}}(\omega_2^{p_{12}})}. \end{aligned} \tag{4.3}$$

This means $E_1(k,i)=E_2(k,j)=1$ if $k-1\le i,j\le k+1$.

Taking

$$\lambda_{1i}=(b_1)_{B_i}=\frac{1}{|B_i|}\int_{B_i}b_1(x)dx,\quad \lambda_{2j}=(b_2)_{B_j}=\frac{1}{|B_j|}\int_{B_j}b_2(x)dx.$$

Then

$$\begin{aligned}
&I^{\Pi b}_{\alpha,2}(f\chi_i,g\chi_j)(x)\\
=&(b_1(x)-\lambda_{1i})(b_2(x)-\lambda_{2j})I_{\alpha,2}(f\chi_i,g\chi_j)-(b_1(x)-\lambda_{1i})I_{\alpha,2}(f\chi_i,(b_2-\lambda_{2j})g\chi_j)\\
&-(b_2(x)-\lambda_{2j})I_{\alpha,2}((b_1-\lambda_{1i})f\chi_i,g\chi_j)(x)+I_{\alpha,2}((b_1-\lambda_{1i})f\chi_i,(b_2-\lambda_{2j})g\chi_j)(x)\\
=&L_1(x)+L_2(x)+L_3(x)+L_4(x).
\end{aligned}\tag{4.4}$$

Thus

$$\varphi(k)\big\|I^{\Pi b}_{\alpha,2}(f\chi_i,g\chi_j)\chi_k\big\|_{L^{p_2}(\nu^{p_2}_{\vec\omega})}\le\sum_{m=1}^{4}\varphi(k)\big\|(L_m)\chi_k\big\|_{L^{p_2}(\nu^{p_2}_{\vec\omega})}=\sum_{m=1}^{4}J_m.\tag{4.5}$$

Now, we will estimate each $J_m$ $(m=1,2,3,4)$, separately.

Applying (3.8)-(3.10b),

$$\begin{aligned}
|L_1(x)|=&\big|(b_1(x)-\lambda_{1i})(b_2(x)-\lambda_{2j})I_{\alpha,2}(f\chi_i,g\chi_j)(x)\big|\\
\le&C2^{-\max\{i,j,k\}(2n-\alpha)}\big|(b_1(x)-\lambda_{1i})(b_2(x)-\lambda_{2j})\big|\big\|f\chi_i\big\|_{L^1}\big\|g\chi_j\big\|_{L^1}\\
\le&C2^{-\max\{i,j,k\}(2n-\alpha)}\big|(b_1(x)-\lambda_{1i})(b_2(x)-\lambda_{2j})\big|\\
&\times\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\Big(\omega_1^{-p'_{11}}(B_i)\Big)^{1/p'_{11}}\|g\chi_j\|_{L^{p_{12}}(\omega_2^{p_{12}})}\Big(\omega_2^{-p'_{12}}(B_j)\Big)^{1/p'_{12}}.
\end{aligned}\tag{4.6}$$

By Lemma 2.4, we know $\nu^{p_2}_{\vec\omega}\in A_\infty$. If $p_2\ge 1/2$, then by Hölder's inequality, Lemma 2.5, and (3.11), we have

$$\begin{aligned}
&\Big(\int_{B_k}\big|(b_1(x)-\lambda_{1i})(b_2(x)-\lambda_{2j})\big|^{p_2}\nu^{p_2}_{\vec\omega}(x)dx\Big)^{\frac{1}{p_2}}\\
\le&\Big(\int_{B_k}\big|b_1(x)-\lambda_{1i}\big|^{2p_2}\nu^{p_2}_{\vec\omega}(x)dx\Big)^{\frac{1}{2p_2}}\Big(\int_{B_k}\big|b_2(x)-\lambda_{1j}\big|^{2p_2}\nu^{p_2}_{\vec\omega}(x)dx\Big)^{\frac{1}{2p_2}}\\
\le&C\|b_1\|_*\|b_2\|_*|k-i||k-j|\big(\nu^{p_2}_{\vec\omega}(B_k)\big)^{1/p_2}\\
\le&C\|b_1\|_*\|b_2\|_*|k-i||k-j|\Big(\omega_1^{p_{21}}(B_k)\Big)^{1/p_{21}}\Big(\omega_2^{p_{22}}(B_k)\Big)^{1/p_{22}}.
\end{aligned}\tag{4.7}$$

If $0<p_2<1/2$, then by Hölder's inequality and Lemma 2.5 we have

$$\Big(\int_{B_k}\big|(b_1(x)-\lambda_{1i})(b_2(x)-\lambda_{2j})\big|^{p_2}\nu^{p_2}_{\vec\omega}(x)dx\Big)^{\frac{1}{p_2}}$$

$$
\begin{aligned}
&\leq\left(\int_{B_k}\left|b_1(x)-\lambda_{1i}\right|\nu_{\vec{\omega}}^{p_2}(x)dx\right)\left(\int_{B_k}\left|b_2(x)-\lambda_{1j}\right|^{\frac{p_2}{1-p_2}}\nu_{\vec{\omega}}^{p_2}(x)dx\right)^{\frac{1}{p_2}-1}\\
&\leq C\|b_1\|_*|k-i|\nu_{\vec{\omega}}^{p_2}(B_k)\left(\int_{B_k}\left|b_2(x)-\lambda_{2j}\right|\nu_{\vec{\omega}}^{p_2}(x)dx\right)\nu_{\vec{\omega}}^{p_2}(B_k)^{\frac{1}{p_2}-2}\\
&\leq C\|b_1\|_*\|b_2\|_*|k-i||k-j|\left(\omega_1^{p_{21}}(B_k)\right)^{1/p_{21}}\left(\omega_2^{p_{22}}(B_k)\right)^{1/p_{22}}.
\end{aligned}
\tag{4.8}
$$

Then by (3.12), (3.13),

$$
\begin{aligned}
&\varphi(k)\left\|(L_1)\chi_k\right\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\
\leq& C\|b_1\|_*|k-i|2^{i(n-\alpha_1)-\max\{i,k\}(n-\alpha_1)}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\frac{\varphi_1(k)}{\varphi_1(i)}\left(\frac{\omega_1^{p_{21}}(B_k)}{\omega_1^{p_{21}}(B_i)}\right)^{\frac{1}{p_{21}}}\\
&\times\|b_2\|_*|k-j|2^{j(n-\alpha_2)-\max\{j,k\}(n-\alpha_2)}\varphi_2(j)\left\|g\chi_j\right\|_{L^{p_{12}}(\omega_2^{p_{12}})}\frac{\varphi_2(k)}{\varphi_2(j)}\left(\frac{\omega_2^{p_{22}}(B_k)}{\omega_2^{p_{22}}(B_j)}\right)^{\frac{1}{p_{22}}}.
\end{aligned}
\tag{4.9}
$$

Similar to the estimates in (3.20), we get

$$
\begin{aligned}
&\varphi(k)\left\|(L_1)\chi_k\right\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\
\leq& C\|b_1\|_*E_1(k,i)\varphi_1(i)\left\|f\chi_i\right\|_{L^{p_{11}}(\omega_1^{p_{11}})}\times\|b_2\|_*E_2(k,j)\varphi_2(j)\left\|g\chi_j\right\|_{L^{p_{12}}(\omega_2^{p_{12}})},
\end{aligned}
\tag{4.10}
$$

where $E_1(k,i)$ and $E_2(k,j)$ satisfy (4.2).

Applying (3.8) and Hölder's inequality,

$$
\begin{aligned}
|L_2(x)|=&\left|(b_1(x)-\lambda_{1i})I_{\alpha,2}(f\chi_i,(b_2-\lambda_{2j})g\chi_j)\right|\\
\leq& C2^{-\max\{i,j,k\}(2n-\alpha)}\left|b_1(x)-\lambda_{1i}\right|\|f\chi_i\|_{L^1}\|(b_2-\lambda_{2j})g\chi_j)\|_{L^1}\\
\leq& C2^{-\max\{i,k\}(n-\alpha_1)}\left|b_1(x)-\lambda_{1i}\right|\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\left(\omega_1^{-p_{11}'}(B_i)\right)^{1/p_{11}'}\\
&\times C2^{-\max\{j,k\}(n-\alpha_2)}\|(b_2-\lambda_{2j})g\chi_j)\|_{L^1}.
\end{aligned}
\tag{4.11}
$$

Since $\omega_l^{p_{2l}}\in A_{r_l}$, $r_l\leq 1+p_{2l}/p_{1l}'$, we know $\omega_l\in A_{p_{1l,2l}}$ for $l=1,2$. Then by (ii) of Lemma 2.2 we know $\omega_1^{-p_{11}'},\omega_2^{-p_{12}'}\in A_\infty$. By $\omega_2^{-p_{12}'}\in A_\infty$, Hölder's inequality and Lemma 2.5 we get

$$
\begin{aligned}
&\|(b_2-\lambda_{2j})g\chi_j)\|_{L^1}\\
\leq&\left(\int_{B_j}\left|b_2(x)-\lambda_{2j}\right|^{p_{12}'}(\omega_2(x))^{-p_{12}'}dx\right)^{1/p_{12}'}\left\|g\chi_j\right\|_{L^{p_{12}}(\omega_2^{p_{12}})}\\
\leq& C\|b_2\|_*\left(\omega_2^{-p_{12}'}(B_j)\right)^{1/p_{12}'}\left\|g\chi_j\right\|_{L^{p_{12}}(\omega_2^{p_{12}})},
\end{aligned}
\tag{4.12}
$$

and applying Lemma 2.5 again,

$$\Big(\int_{B_k}\big|b_1(x)-\lambda_{1i}\big|(\nu_{\vec{\omega}}(x))^{p_2}dx\Big)^{1/p_2}\le C|k-i|\|b_1\|_*(\nu_{\vec{\omega}}^{p_2}(B_k))^{1/p_2}. \tag{4.13}$$

Then by (3.10b)-(3.12)

$$\begin{aligned}&\varphi(k)\big\|(L_2)\chi_k\big\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\ \le& C\|b_1\|_*|k-i|2^{i(n-\alpha_1)-\max\{i,k\}(n-\alpha_1)}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\frac{\varphi_1(k)}{\varphi_1(i)}\left(\frac{\omega_1^{p_{21}}(B_k)}{\omega_1^{p_{21}}(B_i)}\right)^{\frac{1}{p_{21}}}\\ &\times\|b_2\|_*2^{j(n-\alpha_2)-\max\{j,k\}(n-\alpha_2)}\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}\frac{\varphi_2(k)}{\varphi_2(j)}\left(\frac{\omega_2^{p_{22}}(B_k)}{\omega_2^{p_{22}}(B_j)}\right)^{\frac{1}{p_{22}}}\\ \le& C\|b_1\|_*E_1(k,i)\varphi_1(i)\big\|f\chi_i\big\|_{L^{p_{11}}(\omega_1^{p_{11}})}\times\|b_2\|_*E_2(k,j)\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}.\end{aligned}$$

By symmetry, we also get

$$\begin{aligned}&\varphi(k)\big\|(L_3)\chi_k\big\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\ \le& C\|b_1\|_*E_1(k,i)\varphi_1(i)\big\|f\chi_i\big\|_{L^{p_{11}}(\omega_1^{p_{11}})}\times\|b_2\|_*E_2(k,j)\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}.\end{aligned}$$

Finally, it remain to estimate $J_4$. By $\omega_1^{-p'_{11}},\omega_2^{-p'_{12}}\in A_\infty$ and the estimate for (4.13) we get

$$\|(b_1-\lambda_{1i})f\chi_i)\|_{L^1}\le C\|b_1\|_*\Big(\omega_1^{-p'_{11}}(B_i)\Big)^{1/p'_{11}}\big\|f\chi_i\big\|_{L^{p_{11}}(\omega_1^{p_{11}})} \tag{4.14}$$

and

$$\|(b_2-\lambda_{2j})g\chi_j)\|_{L^1}\le C\|b_2\|_*(\omega_2^{-p'_{12}}(B_j))^{1/p'_{12}}\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}. \tag{4.15}$$

Then

$$\begin{aligned}|L_4(x)|=&\big|I_{\alpha,2}((b_1-\lambda_{1i})f\chi_i,(b_2-\lambda_{2j})g\chi_j)\big|\\ \le& C2^{-\max\{i,j,k\}(2n-\alpha)}\|(b_1-\lambda_{1i})f\chi_i\|_{L^1}\|(b_2-\lambda_{2j})g\chi_j)\|_{L^1}\\ \le& C\|b_1\|_*2^{-\max\{i,,k\}(n-\alpha_1)}\big\|f\chi_i\big\|_{L^{p_{11}}(\omega_1^{p_{11}})}\Big(\omega_1^{-p'_{11}}(B_i)\Big)^{1/p'_{11}}\\ &\times\|b_2\|_*2^{-\max\{j,k\}(n-\alpha_2)}\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}\Big(\omega_2^{-p'_{12}}(B_j)\Big)^{1/p'_{12}}.\end{aligned}$$

Thus

$$
\begin{aligned}
&\varphi(k)\big\|(L_4)\chi_k\big\|_{L^{p_2}(\nu_{\vec{\omega}}^{p_2})}\\
\leq& C\|b_1\|_* 2^{i(n-\alpha_1)-\max\{i,k\}(n-\alpha_1)}\varphi_1(i)\|f\chi_i\|_{L^{p_{11}}(\omega_1^{p_{11}})}\frac{\varphi_1(k)}{\varphi_1(i)}\left(\frac{\omega_1^{p_{21}}(B_k)}{\omega_1^{p_{21}}(B_i)}\right)^{\frac{1}{p_{21}}}\\
&\times\|b_2\|_* 2^{j(n-\alpha_2)-\max\{j,k\}(n-\alpha_2)}\varphi_2(j)\|g\chi_j\|_{L^{p_{12}}(\omega_2^{p_{12}})}\frac{\varphi_2(k)}{\varphi_2(j)}\left(\frac{\omega_2^{p_{22}}(B_k)}{\omega_2^{p_{22}}(B_j)}\right)^{\frac{1}{p_{22}}}\\
\leq& C\|b_1\|_* E_1(k,i)\varphi_1(i)\big\|f\chi_i\big\|_{L^{p_{11}}(\omega_1^{p_{11}})}\times\|b_2\|_* E_2(k,j)\varphi_2(j)\big\|g\chi_j\big\|_{L^{p_{12}}(\omega_2^{p_{12}})}.
\end{aligned}
$$

Then, the proof of Theorem 1.4. is completed □

## References


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