Numer. Math. Theor. Meth. Appl. Vol. **6**, No. 4, pp. 657-684
doi: 10.4208/nmtma.2013.1208nm November 2013

Extremal Eigenvalues of the Sturm-Liouville Problems with Discontinuous Coefficients

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Received 25 November 2011; Accepted (in revised version) 28 September 2012

Available online 2 October 2013

Abstract. In this paper, an extremal eigenvalue problem to the Sturm-Liouville equations with discontinuous coefficients and volume constraint is investigated. Liouville transformation is applied to change the problem into an equivalent minimization problem. Finite element method is proposed and the convergence for the finite element solution is established. A monotonic decreasing algorithm is presented to solve the extremal eigenvalue problem. A global convergence for the algorithm in the continuous case is proved. A few numerical results are given to depict the efficiency of the method.

AMS subject classifications: 65L60, 65L15, 34B09

Key words: Extremal eigenvalue problem, Sturm-Liouville problem, finite element method, convergence analysis.

1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^n$ $(n = 1,2,3)$, consider the following eigenvalue problem:

$$\begin{cases} -\mathrm{div}(\sigma(x)\nabla u) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\sigma(x)$ is a positive piecewise constant function. For any given such function σ, it is known (cf. [11,16,22]) that the Eq. (1.1) admits a sequence of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty,$$

and its smallest eigenvalue λ_1 is denoted by $\lambda_1(\sigma)$. We are interested in the minimization of the first eigenvalue $\lambda_1(\sigma)$ among all possible choices of function $\sigma(x)$. This extremal eigenvalue problem arises from a lot of structural engineering and optimal design problems (cf. [1,5]). For example, if we consider the non-homogenous heat conductor with different

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conductivities, the first eigenvalue and its corresponding eigenfunction represents the first mode of heat diffusion pattern. In particular, we assume that the heat conductor is made by two materials with conductivities α and β, $0 < \alpha < \beta$. Let the materials with the conductivity α occupies a measurable set $D \subset \Omega$, then

$$\sigma(x) = \alpha\chi_D + \beta\chi_{\Omega\setminus D},$$

where χ_D is the characteristic function of D. The material with conductivity α are assumed to have a fixed volume, which leads to the following optimization problem:

Problem 1.1. For

$$\min_{\sigma\in\mathscr{A}} \lambda_1(\sigma),$$

where $\mathscr{A}$ is the admissible set for all possible choices of the conductivity function which is defined as:

$$\mathscr{A} = \left\{\sigma : \sigma = \alpha\chi_D + \beta\chi_{\Omega\setminus D},\ \fint_\Omega \sigma = c\right\},$$

where $\fint$ is the average of integral function on the domain and c is a constant which satisfies: $\alpha \le c \le \beta$.

The condition to the existence for the minimizer of this problem remains an open question. From the work of Murat and Tartar on a control problem involving immiscible fluids [26], it is known that the Problem 1.1 may not always possess a solution, and in general one should consider the framework of homogenization theory. Existence of a solution and optimality conditions in the class of relaxed designs has been discussed in Cox and Lipton [12].

If the domain Ω is an interval in $\mathbb{R}^1$ or a ball in $\mathbb{R}^n$, the existence of the minimizer has been studied in various papers. The one dimensional problem was solved by Krein [23] by exploiting the equivalence between this problem and a similar extremal eigenvalue problem for a composite membrane with variable densities. The technique is so-called Liouville transformation, to transfer the variable conductivity σ into the lower-order term, then one can use the results of extremal eigenvalue problem for a composite membrane, see Cox, Mclaughlin [14, 15]. The Liouville transformation can be found in many papers in the context of Sturm-Liouville problems with discontinuous coefficients such as [6, 18, 19, 25]. When the domain is a ball, the existence of a radially symmetric minimizer has been proved in [2] by using rearrangement technique.

On the other hand, the numerical treatment to the extremal eigenvalue problem of a variable density membrane have been studied in [13–15, 17, 29], but there are only few works for the extremal eigenvalue problem with variable conductivity (cf. [9, 10, 12]). The finite element method for the eigenvalue problem have been studied extensively, see [1, 7, 8, 30] for constant conductivity function case and [3] for discontinuous conductivity. The computational result can also be found in Nemat-Nasser et al. [27,28]. Recently, Liang et al. [24] study the convergence of the finite element method for the extremal eigenvalue problem with variable density function. Inspired by the previous works, we exploit the finite element method in the extremal eigenvalue problems with variable conductivities in the one dimension interval. The idea is to transfer the extremal eigenvalue problem with variable conductivities to the extremal eigenvalue problem with variable densities by Liouville transformation, then applying finite element method to solve the later equivalent extremal eigenvalue problem and using discrete inverse transformation to obtain the numerical solution of original extremal eigenvalue problem.

The outline of this paper is as follows. In last part of this section, we give some notations of function space. In Section 2, Liouville transformation, and a monotonic decreasing algorithm are introduced. The property of σ-problem and corresponding ρ-problem are given and the convergence for monotonic decreasing algorithm for ρ-problem is proved. In Section 3, the extremal eigenvalue problem is discretized by finite element method and the convergence analysis is given. In Section 4, generalization to the extremal eigenvalue to the Sturm-Liouville problems with nonlinear potential term is studied, a modified monotonic decreasing algorithm combining with Dinkelbach's iterative algorithm for linear fractional optimization is provided. In Section 5, some numerical examples are given to depict the efficiency of our method. In the last Section 6, conclusion as well as the possible further research are discussed.

The standard Sobolev space $W^{m,p}$ and H^m on Ω with the norm $\|\cdot\|_{W^{m,p}}$ and $\|\cdot\|_m$. The L^2 norm is simplified as $\|\cdot\|$, and $(\cdot,\cdot)$ is inner product in L^2 space.

2. Continuous extremal eigenvalue problem

2.1. Monotonic decreasing algorithm for σ-problem

Define Rayleigh's quotients $\mathscr{R} : \mathscr{A} \times H_0^1(\Omega)\backslash\{0\} \to R$ by:

$$\mathscr{R}(\sigma,u) = \frac{\int_\Omega \sigma|\nabla u|^2}{\int_\Omega u^2},$$

then, by Rayleigh's principle (cf. [18]),

$$\lambda_1(\sigma) = \min_{u\in H_0^1(\Omega)\backslash\{0\}} \mathscr{R}(\sigma,u).$$

Therefore, Problem 1.1 can be equivalently presented as

$$\min_{\sigma\in\mathscr{A},u\in H_0^1(\Omega),u\neq 0} \mathscr{R}(\sigma,u).$$

Similar as in [24], we may propose a monotonic decreasing algorithm for Problem 1.1, see Algorithm 2.1. It can be shown that the above algorithm produces a monotonic decreasing eigenvalue sequence, and it converges to a fixed point of the composite map $\sigma \to \sigma \circ u$. But unfortunately, the fixed point of this composite map is not unique. More importantly, even in the one dimensional case, the limit point of conductivity σ is very sensitive to the initial guess. Our numerical tests in Section 5 verify this phenomena. On the other hand, Algorithm 2.1: Continuous monotonic decreasing algorithm for σ-problem.

1. initialize σ^0,
2. for $k = 1, 2, \cdots$, solve

$$u^{k-1} = u(\sigma^{k-1}) \triangleq \underset{u\in H_0^1,\ \int_0^1 (u)^2=1,\ \int_0^1 u>0}{\text{argmin}} \mathscr{R}(\sigma^{k-1}, u) \quad \text{and} \quad \sigma^k = \sigma(u^{k-1}) \triangleq \underset{\sigma\in\mathscr{A}}{\text{argmin}}\, \mathscr{R}(\sigma, u^{k-1}),$$

3. if $\sigma^k = \sigma^{k-1}$, then stop.

if the domain is specified as an interval in $\mathbb{R}^1$, one may apply Liouville transformation to transfer this σ-problem to an equivalent ρ-problem (see the details in next subsection), and we can prove the stability of the monotonic decreasing algorithm for ρ-problem. In this paper, we will focus on the one dimensional case:

$$\begin{cases} -(\sigma(x)u_x)_x = \lambda u & \text{in } (0, L_x), \\ u(0) = u(L_x) = 0. \end{cases} \tag{2.1}$$

The corresponding extremal eigenvalue problem reads:

Problem 2.1. For

$$\min_{\sigma\in\mathscr{A}_x} \lambda_1(\sigma),$$

where

$$\mathscr{A}_x = \left\{ \sigma(x) \in L^\infty(0, L_x) : \alpha \le \sigma(x) \le \beta,\ \fint_0^{L_x} \sigma = c \right\}. \tag{2.2}$$

Different from Problem 1.1, we replace the nonconvex admissible set $\mathscr{A}$ by its L^∞ weak star closure $\mathscr{A}_x$. Cox and Jouron [12, 20] has shown that Problem 1.1 and Problem 2.1 admit the same minimizer for the one dimensional case. It means that the minimizers of $\lambda_1(\sigma)$ for Problem 2.1 in $\mathscr{A}_x$ has "bang-bang" property.

Remark 2.1. We employ the concept "bang-bang" from the optimal control theory. A function $\sigma \in \mathscr{A}_x$ has "bang-bang" property, if $\sigma(x) = \alpha$ or β for a.e. $x \in (0, L_x)$.

For the convenience we give the solution to the σ-problem in one dimensional case.

Lemma 2.1 (see [18]). *There exists a unique minimizer $\sigma^*(x) \in \mathscr{A}_x$ to the Problem* 2.1. *To be more specific, the conductivity function $\sigma^*(x)$ is defined by:*

$$\sigma^*(x) = \begin{cases} \alpha, & \textit{for } x \in (0, z_1^\sigma), \\ \beta, & \textit{for } x \in (z_1^\sigma, z_2^\sigma), \\ \alpha, & \textit{for } x \in (z_2^\sigma, L_x), \end{cases}$$

where

$$z_1^\sigma = \frac{L_x}{2}\frac{\beta - c}{\beta - \alpha}, \quad z_2^\sigma = L_x - z_1^\sigma = \frac{L_x}{2}\frac{\beta + c - 2\alpha}{\beta - \alpha}.$$

2.2. Liouville transformation

As we mentioned in last subsection, to avoid the instability of the algorithm, we use Liouville transformation to reformulate σ-problem into an equivalent ρ-problem. For Eq. (2.1), we introduce the change of variable as:

$$y = \int_0^x \frac{dt}{\sigma(t)},$$

and define new functions $v(y) = u(x)$, $\rho(y) = \sigma(x)$. Then the Eq. (2.1) can be rewrote as the system of the density $\rho(y)$:

$$\begin{cases} -v_{yy} = \lambda \rho v \quad \text{in } (0, L_y), \\ v(0) = v(L_y) = 0, \end{cases} \tag{2.3}$$

where $L_y = \int_0^{L_x} \sigma^{-1}(t)dt$. For any "bang-bang" function $\sigma \in \mathscr{A}_x$, in particular, for the optimal solution of Problem 2.1, we have

$$L_y = \Big(\frac{\alpha + \beta - c}{\alpha\beta}\Big) L_x.$$

One can find that function $\rho(y)$ satisfies $\alpha \le \rho(y) \le \beta$ and the integral constraint

$$\int_0^{L_y} \rho(y)dy = \int_0^{L_x} \frac{\sigma(x)dx}{\sigma(x)} = L_x.$$

The admissible set for the function ρ is:

$$\mathscr{A}_y = \Big\{\rho(y) \in L^\infty(0, L_y) : \alpha \le \rho(y) \le \beta, \int_0^{L_y} \rho(y)dy = L_x\Big\}. \tag{2.4}$$

If the first eigenvalue of Eq. (2.3) is denoted by $\lambda_1(\rho)$, then another extremal eigenvalue problem

Problem 2.2. For

$$\min_{\rho \in \mathscr{A}_y} \lambda_1(\rho)$$

admits the same minimum value as Problem 2.1 and the corresponding minimizer follows $\sigma(x) = \rho(y)$. The inverse Liouville transformation from Eqs. (2.3) to (2.1) can also be defined by

$$x = \int_0^y \rho(s)ds, \quad \sigma(x) = \rho(y).$$

Similarly we can define Rayleigh's quotient $\mathscr{R} : \mathscr{A}_y \times H_0^1(0,L_y)\setminus\{0\} \to R$ as:

$$\mathscr{R}(\rho,v)=\frac{\int_0^{L_y}|v_y|^2dy}{\int_0^{L_y}\rho v^2dy}.$$

Then for any given $\rho(y)\in\mathscr{A}_y$,

$$\lambda_1(\rho)=\min_{v\in H_0^1(0,L_y),v\neq 0}\mathscr{R}(\rho,v),$$

and hence Problem 2.2 can be equivalently presented as

$$\min_{\rho\in\mathscr{A}_y,v\in H_0^1(0,L_y),\ v\neq 0}\mathscr{R}(\rho,v).$$

In the next section, we will introduce monotonic decreasing algorithm to the ρ-problem. Below the solution to Problem 2.2 is provided.

Lemma 2.2 (see [18])**.** *There exists a unique minimizer $\rho^*(y)\in\mathscr{A}_y$ to the Problem* 2.2*. To be more specific, the density function $\rho^*(y)$ is defined by:*

$$\rho^*(y)=\begin{cases}\alpha, & \textit{for}\ \ y\in(0,z_1^\rho),\\ \beta, & \textit{for}\ \ y\in(z_1^\rho,z_2^\rho),\\ \alpha, & \textit{for}\ \ y\in(z_2^\rho,L_y),\end{cases}$$

where

$$z_1^\rho=L_y\frac{\beta(\beta-c)}{2(\beta-\alpha)(\beta+\alpha-c)},\quad z_2^\rho=L_y-z_1^\rho=L_y\frac{\beta(\beta-c)+2\alpha(c-\alpha)}{2(\beta-\alpha)(\beta+\alpha-c)}.$$

2.3. Continuous monotonic decreasing algorithm for ρ-problem

In this subsection, without loss of generality, we assume

$$(0,L_y)=(0,1),\qquad \int_0^{L_y}\rho(y)dy=\frac{\alpha+\beta}{2},$$

for simplicity. Consider the eigenvalue problem

$$\begin{cases}-v''=\lambda\rho v \quad \text{in}\ \ (0,1),\\ v(0)=v(1)=0,\end{cases}\tag{2.5}$$

and the admissible set

$$\mathscr{A}_y=\left\{\rho\in L^\infty(0,1):\alpha\le\rho(y)\le\beta,\ a.e.\ y\in(0,1),\ \int_0^1\rho(y)=\frac{\alpha+\beta}{2}\right\}.$$

Algorithm 2.2: Continuous version of monotonic decreasing algorithm for ρ-problem.

1. initialize ρ^0,
2. for $k = 1, 2, \cdots$, solve
$$v^{k-1} = v(\rho^{k-1}) \triangleq \underset{v\in H_0^1,\ \int_0^1 (v_y)^2=1,\ \int_0^1 v>0}{\operatorname{argmin}} \mathscr{R}(\rho^{k-1}, v) \quad \text{and} \quad \rho^k = \rho(v^{k-1}) \triangleq \underset{\rho\in\mathscr{A}_y}{\operatorname{argmin}} \mathscr{R}(\rho, v^{k-1}),$$
3. if $\rho^k = \rho^{k-1}$, then stop.

A monotonic decreasing algorithm is used to solve Problem 2.2, see the details in Algorithm 2.2. It can be observed that there exists a unique minimizer $\min_{v\in H_0^1,\ \int_0^1 v>0,\ \int_0^1 (v)_y^2 dy=1} \mathscr{R}(\rho, v)$ for any given $\rho \in \mathscr{A}_y$. On the other hand, for any v be eigenfunction associated to the smallest eigenvalue of Eq. (2.5), there exists a unique minimizer for $\min_{\rho\in\mathscr{A}_y} \mathscr{R}(\rho, v)$, denoted by $\rho(v)$. Therefore, Algorithm 2.2 is a type of fixed point iteration, and we have its convergence as follows.

Lemma 2.3. *ρ^k in Algorithm 2.2 converges to the solution of Problem 2.2.*

Proof. The proof is divided into two parts. Firstly we show ρ^k converges to a fixed point of composite map $\rho \circ v$, and secondly we prove the only fixed point of $\rho \circ v$ is the solution of Problem 2.2.

Step 1. If algorithm terminate in finite step, i.e., $\rho^k = \rho^{k+1}$, then clearly ρ^k is a fixed point of $\rho \circ v$. Otherwise, we can observe that

$$\mathscr{R}(\rho^k, v^k) \le \mathscr{R}(\rho^k, v^{k-1}) \le \mathscr{R}(\rho^{k-1}, v^{k-1}),$$

hence $\lambda^k = \mathscr{R}(\rho^k, v^k)$ is a monotonic decreasing sequence, and let $\lambda^k \to \lambda$. It is known that triples (λ^k, ρ^k, v^k) satisfy

$$((v^k)_y, (w)_y) = \lambda^k(\rho^k v^k, w), \quad \forall w \in H_0^1, \tag{2.6}$$

and $\int_0^1 (v^k)_y^2 dy = 1$, $v^k(y) > 0$ for $y \in (0,1)$. Since $\|v^k\|_{H^1}$ is bounded, passage to a subsequence and still denote it by v^k, we have

$$\begin{aligned} &\rho^k \to \rho && \text{weakly in } L^2(0,1), \\ &v^k \to v && \text{weakly in } H_0^1(0,1), \\ &\lambda^k \to \lambda. \end{aligned}$$

Since $H_0^1(0,1) \hookrightarrow C_0(0,1)$ compactly, then we have $v^k \to v$ in $C_0(0,1)$ and $v \ge 0$. The limit triple (λ, ρ, v) satisfies

$$(v_y, w_y) = \lambda(\rho v, w), \quad \forall w \in H_0^1. \tag{2.7}$$

Form Eq. (2.6), we can deduce the following:

$$\int_0^1 (v_y)^2 dy = \lambda \int_0^1 \rho(v)^2 = \lim_{k\to\infty} \lambda^k \int_0^1 \rho^k (v^k)^2 = \lim_{k\to\infty} \int_0^1 (v_y^k)^2 = 1.$$

Together with $v \geq 0$ for y in [0,1], it implies that

$$v = \underset{\substack{v\in H_0^1,\ \int_0^1 v>0 \\ \int_0^1 (v^k)_y^2 dy=1}}{\operatorname{argmin}} \mathscr{R}(\rho, v)$$

and hence $v = v(\rho)$. It remains to show that $\rho = \operatorname{argmin}_{\rho\in\mathscr{A}_y} \mathscr{R}(\rho, v)$. By $\rho^k = \rho(v^{k-1}) \triangleq$ argmin $_{\rho\in\mathscr{A}_y} \mathscr{R}(\rho, v^{k-1})$ in Algorithm 2.1, we know that the density function ρ^k satisfies

$$\rho^k = \alpha + (\beta - \alpha)\chi_{[a_k,b_k]}, \quad b_k = a_k + \frac{1}{2},$$

where a_k is the unique point in $[0, 1/2]$ which satisfies (uniqueness is from the strictly convexity of v^k):

$$v^{k-1}(a_k) = v^{k-1}\Big(a_k + \frac{1}{2}\Big).$$

Now let a is the unique point in $[0, 1/2]$, which satisfies

$$v(a) = v\Big(a + \frac{1}{2}\Big),$$

we will show that $a_k \to a$. Let $w(y) = v(y) - v(y + 1/2)$ for $0 \leq y \leq 1/2$, then we have $w(a) = 0$, which implies a is the unique root of w. Since $w(y)$ is continuous, if $w(a_k) \to 0$, then $a_k \to a$ (otherwise $\exists a_{n_k} \to \overline{a} \neq a \Rightarrow w(\overline{a}) = 0$, which is a contradiction). We notice that $|w(a_k)| = |(v(a_k) - v^k(a_k)) - (v(b_k) - v^k(b_k))| \leq 2\|v - v^k\|_{L^\infty}$, hence $w(a_k) \to 0 \Rightarrow a_k \to a$.

Let $\hat{\rho} = \rho(v) = \alpha + (\beta - \alpha)\chi_{[a,b]}$, then

$$\|\rho_k - \hat{\rho}\| = \sqrt{2}|a_k - a|(\beta - \alpha) \to 0.$$

This implies that $\rho = \hat{\rho} = \rho(v)$. Now we have proved there exists a subsequence of ρ^k which converges to a fixed point of $\rho \circ v$. In the following, we should show that ρ is the only one fixed point of the systems. By the uniqueness of the fixed point, the whole sequence converges to ρ.

Step 2. Clearly the solution of Problem 2.2 is the fixed point of $\rho \circ v$. Next assume ρ be one fixed point of iteration, and (λ, v) be the corresponding eigenvalue and eigenfunction. Since v is strictly convex function in $(0, 1)$, there exists a unique $s \in (0, 1/2)$, such that $v(s) = v(s + 1/2)$, moreover, $v(y) > v(s)$ for $y \in (s, s + 1/2)$ and $v(y) < v(s)$ for $x \in (0,s)\cup(s+1/2, 1)$. Then $\rho = \alpha+(\beta-\alpha)\xi_{[s,s+1/2]}$. We will show that the only possible s be

$1/4$ and obtain the uniqueness of fixed point. Let $v(s)=v(s+1/2)=b$, then $-v''=\lambda\rho v$ in $(0,s)$ with boundary condition $v(0)=0$, $v(s)=b$. We can solve this boundary value problem and find

$$v(y)=\frac{\sin[\sqrt{\alpha\lambda}y]}{\sin[\sqrt{\alpha\lambda}s]}b, \quad \forall y\in[0,s].$$

Similarly we also have

$$v(y)=\frac{\sin[\sqrt{\alpha\lambda}(1-y)]}{\sin\left[\sqrt{\alpha\lambda}\left(\frac{1}{2}-s\right)\right]}b, \quad \forall y\in\left[s+\frac{1}{2},1\right],$$

and

$$v(y)=c\sin(\sqrt{\beta\lambda}y)+d\cos(\sqrt{\beta\lambda}y), \quad \forall y\in\left[s,s+\frac{1}{2}\right],$$

where

$$c=\frac{\cos(\sqrt{\beta\lambda}s)-\cos\left[\sqrt{\beta\lambda}\left(s+\frac{1}{2}\right)\right]}{\sin\left(\frac{\sqrt{\beta\lambda}}{2}\right)}b, \quad d=\frac{\sin\left[\sqrt{\beta\lambda}\left(s+\frac{1}{2}\right)\right]-\sin(\sqrt{\beta\lambda}s)}{\sin\left(\frac{\sqrt{\beta\lambda}}{2}\right)}b.$$

From above formula, we can calculate the left derivative and right derivative at point s and $s+1/2$. By Embedding Theorem, $v\in H^2(0,1)\hookrightarrow C^1(0,1)$, $v'(y)$ should be exist and continuous. Therefore

$$\begin{aligned}
&\sqrt{\alpha\lambda}\frac{\cos(\sqrt{\alpha\lambda}s)}{\sin(\sqrt{\alpha\lambda}s)}b=v'_-(s)=v'_+(s)=\frac{\sqrt{\beta\lambda}}{\sin\left(\frac{\sqrt{\beta\lambda}}{2}\right)}\left(1-\cos\left(\frac{\sqrt{\beta\lambda}}{2}\right)\right)b,\\
&-\sqrt{\alpha\lambda}\frac{\cos\left[\sqrt{\alpha\lambda}\left(\frac{1}{2}-s\right)\right]}{\sin\left[\sqrt{\alpha\lambda}\left(\frac{1}{2}-s\right)\right]}b=v'_+\left(s+\frac{1}{2}\right)=v'_-\left(s+\frac{1}{2}\right)\\
&\qquad\qquad\qquad\qquad=-\frac{\sqrt{\beta\lambda}}{\sin\left(\frac{\sqrt{\beta\lambda}}{2}\right)}\left(1-\cos\left(\frac{\sqrt{\beta\lambda}}{2}\right)\right)b.
\end{aligned}$$

Then we have

$$\cot(\sqrt{\alpha\lambda}s)=\cot\left[\sqrt{\alpha\lambda}\left(\frac{1}{2}-s\right)\right].$$

Due to the strictly monotonic property for cot function, we obtain $s=1/2-s$ and hence $\rho=\beta\chi_{(1/4,3/4)}+\alpha\chi_{(0,1/4)\cap(3/4,1)}$ be the only fixed point of composite map $\rho\circ v$. $\square$

Remark 2.2. The uniqueness of the fixed point to the composite map $\rho\circ v$ plays the important role for the stability of this algorithm. This global convergence result can be verified numerically in the Section 5.

3. Finite element approximation

In the Subsection 3.1, we consider the finite element method for Problem 2.2 and establish some properties for finite element solution. A discrete inverse Liouville transformation is introduced in Subsection 3.2 to obtain the discrete solution for Problem 2.1 from the finite element solution of Problem 2.2. The convergence of finite element method is given in Subsection 3.3. Finally, we will list the discrete version of monotone decrease algorithm in Subsection 3.4.

3.1. Finite element method for Problem 2.2

Now we consider a partition on the interval $[0, L_y]$, where

$$L_y = \int_0^{L_x} \frac{1}{\sigma^*(x)} dx = \frac{\beta+\alpha-c}{\alpha\beta} L_x. \tag{3.1}$$

Let $0 = y_0 < y_1 < \cdots < y_N = L_y$, $\Delta y_i = |y_i - y_{i-1}|$, for $i = 1, \cdots, N$, and $h = \max_{i=1}^N \{\Delta y_i\}$. The finite element space are defined by:

$$\mathscr{V}_{h,y} = \{v_h \in C_0(0, L_y),\ \ v_h|_{[y_{i-1}, y_i]} \text{ be linear function},\ \ i = 1, \cdots, N\},$$

and

$$\mathscr{A}_{h,y} = \Big\{\rho_h \in L^\infty(0, L_y),\ \ \rho_h|_{(y_{i-1}, y_i)} \text{ be constant } \rho_{h,i},\ \ i = 1, \cdots, N,\ \ \sum_{i=1}^N \rho_{h,i} \Delta y_i = L_x\Big\}.$$

Let $\{\psi_i\}_{i=1}^{N-1}$ be the basis of the function space $\mathscr{V}_{h,y}$ with $\psi_i(y_j) = \delta_{i,j}$, then any function $v_h \in \mathscr{V}_{h,y}$ can be represented as:

$$v_h = \sum_{j=1}^{N-1} v_h(y_j)\psi_j.$$

Given $v_h \in \mathscr{V}_{h,y}$ and $\rho_h \in \mathscr{A}_{h,y}$, the Rayleigh's quotient:

$$\mathscr{R}(\rho_h, v_h) = \frac{\int_0^{L_y} |(v_h)_y|^2 dy}{\int_0^{L_y} \rho_h v_h^2 dy} = \frac{\displaystyle\int_0^{L_y} \Big[\sum_{j=1}^{N-1} [v_h(y_j)(\psi_j)_y\Big]^2 dy}{\displaystyle\int_0^{L_y} \rho_h \Big[\sum_{j=1}^{N-1} v_h(y_j)\psi_j\Big]^2 dy} = \frac{V_h^T \mathscr{K} V_h}{V_h^T \mathscr{M} V_h}, \tag{3.2}$$

where $\mathscr{K}$ and $\mathscr{M}$ are two matrices with entries:

$$\mathscr{K}_{i,j} = \int_0^{L_y} (\psi_i)_y (\psi_j)_y dy, \quad \mathscr{M}_{i,j} = \int_0^{L_y} \rho_h \psi_i \psi_j dy,$$

and V_h be the vector with $V_h = (v_h(y_1), v_h(y_2), \cdots, v_h(y_{N-1}))^T$. Then the discretization form of Problem 2.2 reads:

Problem 3.1. For

$$\min_{\rho_h \in \mathscr{A}_{h,y}} \lambda_{1,h}(\rho_h),$$

where $\lambda_{1,h}(\rho_h)$ be the smallest generalized eigenvalue of $\mathscr{K}$ respect to $\mathscr{M}$. Let $V_h = (q_1, \cdots, q_{N-1})^T$ be the corresponding generalized eigenvector, we call $v_h = \sum_{j=1}^{N-1} q_j \psi_j$ the corresponding eigenfunction. It is known that Problem 3.1 admits a solution.

Lemma 3.1. *For any given $\rho_h \in \mathscr{A}_{h,y}$, the eigenfunction corresponding to the smallest generalized eigenvalue of $\mathscr{K}$ respect to $\mathscr{M}$ is strict positive in $(0, L_y)$ (up to a multiplicative constant).*

Proof. let $\lambda_{1,h}$ be the smallest generalized eigenvalue and $V_h = (q_1, \cdots, q_{N-1})$ be the corresponding eigenvector. From Rayleigh's principle, we know that the function $v_h = \sum_{i=1}^{N-1} q_i \psi_i$ is the minimizer of $\mathscr{R}(\rho_h, u_h)$ for $u_h \in \mathscr{V}_{h,y}$. Now we define $\widetilde{v}_h = \sum_{i=1}^{N-1} \left| q_i \right| \psi_i$, one can easily verify (let $q_0 = q_N = 0$):

$$\begin{aligned} \int_0^{L_y} (\widetilde{v}_h)_y^2 dy &= \sum_1^N \int_{y_{i-1}}^{y_i} (\widetilde{v}_h)_y^2 dy = \sum_1^N \int_{y_{i-1}}^{y_i} \Big(\frac{1}{\Delta y_i} (|q_i| - |q_{i-1}|) \Big)^2 dy \\ &\leq \sum_1^N \int_{y_{i-1}}^{y_i} \Big(\frac{1}{\Delta y_i} (q_i - q_{i-1}) \Big)^2 dy \\ &= \sum_1^N \int_{y_{i-1}}^{y_i} (v_h)_y^2 dy = \int_0^{L_y} (v_h)_y^2 dy \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \int_0^{L_y} \rho_h (\widetilde{v}_h)^2 dy &= \sum_1^N \rho_{h,i} \int_{y_{i-1}}^{y_i} (\widetilde{v}_h)^2 dy = \sum_{i=1}^N \frac{\rho_{h,i} \Delta y_i}{3} (|q_{i-1}|^2 + |q_{i-1} q_i| + |q_i|^2) \\ &\geq \sum_{i=1}^N \frac{\rho_{h,i} \Delta y_i}{3} (q_{i-1}^2 + q_{i-1} q_i + q_i^2) \\ &= \sum_{i=1}^N \rho_{h,i} \int_{y_{i-1}}^{y_i} v_h^2 dy = \int_0^{L_y} \rho_h v_h^2 dy. \end{aligned} \tag{3.4}$$

Then $\mathscr{R}(\rho_h, \widetilde{v}_h) \leq \mathscr{R}(\rho_h, v_h)$ and hence $\widetilde{v}_h$ is also the eigenfunction corresponding to $\lambda_{1,h}$. In the following we will show that $q_i \neq 0$, $i = 1, 2, \cdots, N-1$. Without loss of generality, assume $q_i = 0$ and $|q_{i+1}| > 0$, $1 \leq i \leq N-2$. If $|q_{i-1}| > 0$, then let $q_i^* = \min(|q_{i-1}|, |q_{i+1}|)$ and define $v_h^* = \sum_{j=1}^{i-1} |q_j| \psi_j + q_i^* \psi_i + \sum_{i+1}^{N-1} |q_j| \psi_j = \widetilde{v}_h + q_i^* \psi_i$. One can observe

$$\begin{aligned} &\int_0^{L_y} (v_h^*)_y^2 dy - \int_0^{L_y} (\widetilde{v}_h)_y^2 dy \\ =& \int_{y_{i-1}}^{y_i} [(v_h^*)_y^2 - (\widetilde{v}_h)_y^2] dy + \int_{y_i}^{y_{i+1}} [(v_h^*)_y^2 - (\widetilde{v}_h)_y^2] dy < 0 \end{aligned}$$

and

$$\begin{aligned}&\int_0^{L_y}\rho_h(v_h^*)^2dy-\int_0^{L_y}\rho_h(\widetilde{v}_h)^2dy\\=&\int_{y_{i-1}}^{y_i}\rho_h[(v_h^*)^2-(\widetilde{v}_h)^2]dy+\int_{y_i}^{y_{i+1}}\rho_h[(v_h^*)^2-(\widetilde{v}_h)^2]dy>0.\end{aligned}$$

Hence $\mathscr{R}(\rho_h, v_h^*) < \mathscr{R}(\rho_h, v_h)$, which implies that v_h is not a minimizer of $\mathscr{R}(\rho_h,\cdot)$. On the other hand, if $|q_{i-1}| = 0$ we choose

$$0<q_i^*<\frac{2\Delta y_i|q_{i+1}|}{\Delta y_i+\Delta y_{i+1}}$$

and let $v_h^* = \widetilde{v}_h + q_i^*\psi_i$, then we have

$$\begin{aligned}&\int_0^{L_y}(v_h^*)_y^2dy-\int_0^{L_y}(\widetilde{v}_h)_y^2dy\\=&\int_{y_{i-1}}^{y_i}[(v_h^*)_y^2-(\widetilde{v}_h)_y^2]dy+\int_{y_i}^{y_{i+1}}[(v_h^*)_y^2-(\widetilde{v}_h)_y^2]dy\\=&\frac{(q_i^*)^2}{\Delta y_i}+\frac{(|q_{i+1}|-q_i^*)^2}{\Delta y_{i+1}}-\frac{q_{i+1}^2}{\Delta y_{i+1}}=q_i^*\Big(\frac{q_i^*}{\Delta y_i}+\frac{q_i^*}{\Delta y_{i+1}}-\frac{2|q_{i+1}|}{\Delta y_{i+1}}\Big)<0\end{aligned}$$

and

$$\begin{aligned}&\int_0^{L_y}\rho_h(v_h^*)^2dy-\int_0^{L_y}\rho_h(\widetilde{v}_h)^2dy\\=&\int_{y_{i-1}}^{y_i}\rho_h[(v_h^*)^2-(\widetilde{v}_h)^2]dy+\int_{y_i}^{y_{i+1}}\rho_h[(v_h^*)^2-(\widetilde{v}_h)^2]dy\\=&\frac{q_i^*}{3}[\rho_{h,i}\Delta y_iq_i^*+\rho_{h,i+1}\Delta y_{i+1}(q_i^*+|q_{i+1}|)]>0.\end{aligned}\tag{3.5}$$

Similarly, $\mathscr{R}(\rho_h, v_h^*) < \mathscr{R}(\rho_h, \widetilde{v}_h) \le \mathscr{R}(\rho_h, v_h)$, which is the contradiction. Then we obtain that $q_i \neq 0$, $i = 1, 2, \cdots, N-1$.

Next we will show that $\{q_i\}_1^{N-1}$ have same sign. Assume $\{q_i\}_1^{N-1}$ have both positive and negative components, then there exists i such that q_i, q_{i+1} have different signs. Since the inequalities (3.3) and (3.4) are strict in this case, we obtain $\mathscr{R}(\rho_h, \widetilde{v}_h) < \mathscr{R}(\rho_h, v_h)$. Therefore v_h is strictly positive (up to a multiplicative constant). □

In later of this section, for given $\rho_h \in \mathscr{A}_{h,y}$, $\lambda_{1,h}(\rho_h)$ and $v_h(\rho_h)$ are the smallest generalized eigenvalue of respect to the stiffness matrix $\mathscr{K}$ and corresponding normalized positive eigenfunction ($\int_0^{L_y}(v_h(\rho_h))_y^2dy = 1$), without confusion we denote it by $\lambda_{1,h}$ and v_h for simplicity.

Lemma 3.2. *For any given $\rho_h \in \mathscr{A}_{h,y}$, let $v_h(\rho_h) = \sum_{i=1}^{N-1} q_i \psi_i$, then we have*

$$\frac{q_i - q_{i-1}}{\Delta y_i} > \frac{q_{i+1} - q_i}{\Delta y_{i+1}}, \quad i = 1, 2, \cdots, N-1, \tag{3.6}$$

where $q_0 = q_N = 0$.

Proof. Recall the definition of matrix $\mathscr{K}$ and $\mathscr{M}$, we find

$$\mathscr{K} = \begin{bmatrix} \frac{1}{\Delta y_1} + \frac{1}{\Delta y_2} & -\frac{1}{\Delta y_2} & \cdots & \cdots & \cdots \\ -\frac{1}{\Delta y_2} & \frac{1}{\Delta y_2} + \frac{1}{\Delta y_3} & -\frac{1}{\Delta y_3} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & -\frac{1}{\Delta y_{N-2}} & \frac{1}{\Delta y_{N-2}} + \frac{1}{\Delta y_{N-1}} & -\frac{1}{\Delta y_{N-1}} \\ \cdots & \cdots & \cdots & -\frac{1}{\Delta y_{N-1}} & \frac{1}{\Delta y_{N-1}} + \frac{1}{\Delta y_N} \end{bmatrix}$$

and

$$\mathscr{M} = \frac{1}{6} \begin{bmatrix} 2\rho_{h,1}\Delta y_1 + 2\rho_{h,2}\Delta y_2 & \rho_{h,2}\Delta y_2 & \cdots & \cdots & \cdots \\ \rho_{h,2}\Delta y_2 & 2\rho_{h,2}\Delta y_2 + 2\rho_{h,3}\Delta y_3 & \rho_{h,3}\Delta y_3 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & \rho_{h,N-2}\Delta y_{N-2} & 2\rho_{h,N-2}\Delta y_{N-2} + 2\rho_{h,N-1}\Delta y_{N-1} & \rho_{h,N-1}\Delta y_{N-1} \\ \cdots & \cdots & \cdots & \rho_{h,N-1}\Delta y_{N-1} & 2\rho_{h,N-1}\Delta y_{N-1} + 2\rho_{h,N}\Delta y_N \end{bmatrix}.$$

From $\mathscr{K} V_h = \lambda_{1,h} \mathscr{M} V_h$ with $V_h = (q_1, q_2, \cdots, q_{N-1})^T$, we get:

$$\Big(\frac{1}{\Delta y_1} + \frac{1}{\Delta y_2}\Big) q_1 - \frac{1}{\Delta y_2} q_2 = \frac{\lambda_{1,h}}{6} [2(\rho_{h,1}\Delta y_1 + \rho_{h,2}\Delta y_2) q_1 + \rho_{h,2}\Delta y_2 q_2], \tag{3.7a}$$

$$\begin{aligned} &-\frac{1}{\Delta y_j} q_{j-1} + \Big(\frac{1}{\Delta y_j} + \frac{1}{\Delta y_{j+1}}\Big) q_j - \frac{1}{\Delta y_{j+1}} q_{j+1} \\ &\quad = \frac{\lambda_{1,h}}{6} [\rho_{h,j}\Delta y_j q_{j-1} + 2(\rho_{h,j}\Delta y_j + \rho_{h,j+1}\Delta y_{j+1}) q_j + \rho_{h,j+1}\Delta y_{j+1} q_{j+1}], \\ &\qquad\qquad 2 \le j \le N-2, \end{aligned} \tag{3.7b}$$

$$\begin{aligned} &-\frac{1}{\Delta y_{N-1}} q_{N-2} + \Big(\frac{1}{\Delta y_{N-1}} + \frac{1}{\Delta y_N}\Big) q_{N-1} \\ &\quad = \frac{\lambda_{1,h}}{6} [\rho_{h,N-1}\Delta y_{N-1} q_{N-2} + 2(\rho_{h,N-1}\Delta y_{N-1} + \rho_{h,N}\Delta y_N) q_{N-1}]. \end{aligned} \tag{3.7c}$$

By virtue of (3.7a)-(3.7c), $q_i > 0$ for $i = 1, \cdots, N-1$ and $q_0 = q_N = 0$, we can deduce that the terms $(q_j - q_{j-1})/\Delta y_j$, $(j = 1, 2, \cdots, N)$ decreases strictly. □

Let ρ_h^* be solution of discrete minimization Problem 3.1, $\lambda_{1,h}^*$ and v_h^* be the corresponding eigenvalue ($\lambda_{1,h}^* = \lambda_{1,h}(\rho_h^*)$) and eigenfunction. Clearly

$$\rho_h^* = \underset{\rho_h \in \mathscr{A}_{y,h}}{\operatorname{argmin}} \frac{\int_0^{L_y} (v_h^*)_y^2 dy}{\int_0^{L_y} \rho_h (v_h^*)^2 dy},$$

which implies that ρ_h^* be the maximizer the denominate $\int_0^{L_y} \rho_h (v_h^*)^2 dy$ for the fixed v_h^*. Let $v_h^* = \sum_1^{N-1} q_i \psi_i$, then

$$\int_0^{L_y} \rho_h (v_h^*)^2 dy = \sum_1^N \rho_{h,i} \int_{y_{i-1}}^{y_i} (v_h^*)^2 dy = \sum_1^N \frac{\rho_{h,i} \Delta y_i}{3} (q_{i-1}^2 + q_{i-1} q_i + q_i^2). \tag{3.8}$$

Define $m_i = q_{i-1}^2 + q_{i-1} q_i + q_i^2$, $t_i = \rho_{h,i} \Delta y_i$, $i = 1, 2, \cdots, N$, then t_i satisfies constraint $\sum_{i=1}^N t_i = L_x$.

Lemma 3.3. *There exists $i^* \in \{1, 2, \cdots, N-1\}$, such that:*

$$m_1 < m_2 < \cdots < m_{i^*-1} < m_{i^*}, \quad m_{i^*+1} > m_{i^*+2} > \cdots > m_{N-1} > m_N.$$

Proof. Since $(q_1 - q_0)/h_1 > 0$, $(q_N - q_{N-1})/h_N < 0$ and $\{(q_i - q_{i-1})/h_i\}_{i=1}^N$ is strictly decreasing sequence, there exists i^* with $1 \le i^* \le N-1$ which satisfies:

$$\frac{q_{i^*} - q_{i*-1}}{\Delta y_{i^*}} > 0 \ge \frac{q_{i^*+1} - q_{i*}}{\Delta y_{i^*+1}}.$$

Hence

$$0 = q_0 = q_1 < q_2 < \cdots < q_{i^*-1} < q_{i^*}, \quad q_{i^*} \ge q_{i^*+1} > q_{i^*+2} > \cdots > q_{N-1} > q_N.$$

Therefore

$$m_1 < m_2 < \cdots < m_{i^*-1} < m_{i^*}, \quad m_{i^*+1} > m_{i^*+2} > \cdots > m_{N-1} > m_N.$$

The proof is completed. □

Next we rearrange $\{1, 2, \cdots, N\}$ to $\{\tau_1, \tau_2, \cdots, \tau_N\}$ by the following rule: $m_{\tau_1} \le m_{\tau_2} \le \cdots \le m_{\tau_N}$. Define $I_\alpha = \{\tau_1, \cdots, \tau_{k-1}\}$, $I_\beta = \{\tau_{k+1}, \cdots, \tau_N\}$, $I_\theta = \{\tau_k\}$, s.t.

$$|\Delta y_{\tau_1}| + \cdots + |\Delta y_{\tau_{k-1}}| \le \frac{\beta(\beta - c)}{(\beta - \alpha)(\alpha + \beta - c)} L_y < |\Delta y_{\tau_1}| + \cdots + |\Delta y_{\tau_k}|. \tag{3.9}$$

One can observe: if $j \in I_\alpha$ and $j \le i^*$ then $\{1, \cdots, j\} \subseteq I_\alpha$; if $j \in I_\alpha$ and $j \ge i^* + 1$ then $\{j, \cdots, N\} \subseteq I_\alpha$. The proof is the consequence of Lemma 3.3. Therefore I_α can be identified as $I_\alpha = \{1, \cdots, j_1\} \cup \{j_2, \cdots, N\}$, and $\tau_k = j_1 + 1$ or $j_2 - 1$. Define

$$\rho^*_{h,i} = \begin{cases} \alpha, & i \in I_\alpha, \\ \beta, & i \in I_\beta, \\ \theta, & i = \tau_k, \end{cases} \tag{3.10}$$

where θ is chosen to satisfy constraint

$$\theta |\Delta y_{\tau_k}| + \alpha \sum_{i \in I_\alpha} |\Delta y_i| + \beta \sum_{i \in I_\beta} |\Delta y_i| = L_x.$$

It can be verified $\alpha \le \theta \le \beta$.

Lemma 3.4. *Define $\rho^*_h \in \mathscr{A}_{h,y}$ by $\rho^*_h|_{(y_{i-1}, y_1)} = \rho^*_{h,i}$, then ρ^*_h is the maximizer of $\int_0^{L_y} \rho_h (v^*_h)^2 dy$.*

Proof. For any $\rho_h \in \mathscr{A}_{h,y}$, recall $m_i = q^2_{i-1} + q_{i-1} q_i + q^2_i$, $t_i = \rho_{h,i} \Delta y_i$, $i = 1, 2, \cdots, N$ and $t^*_i = \rho^*_{h,i} \Delta y_i$, $i = 1, 2, \cdots, N$, then it is sufficient to check $\sum_{i=1}^N t_i m_i \le \sum_{i=1}^N t^*_i m_i$.

Let $e_i = t_i - t^*_i$, then we have

$$\sum_{i=1}^N t_i m_i - \sum_{i=1}^N t^*_i m_i = \sum_{i \in I_\alpha} e_i m_i + \sum_{i \in I_\beta} e_i m_i + e_{\tau_k} m_{\tau_k}.$$

Since $\sum_{i=1}^N e_i = 0$, then $e_{\tau_k} = -(\sum_{i \in I_\alpha} e_i + \sum_{i \in I_\beta} e_i)$, and

$$\sum_{i=1}^N t_i m_i - \sum_{i=1}^N t^*_i m_i = \sum_{i \in I_\alpha} e_i (m_i - m_{\tau_k}) + \sum_{i \in I_\beta} e_i (m_i - m_{\tau_k}).$$

By observing

$$\begin{cases} m_i \le m_{\tau_k}, & \forall i \in I_\alpha, \\ m_{\tau_k} \le m_i, & \forall i \in I_\beta, \end{cases}$$

and the fact $\alpha \le \rho_{h,i} \le \beta$, we have

$$\begin{cases} e_i (m_i - m_{\tau_k}) = \Delta y_i (\rho_{h,i} - \alpha)(m_i - m_{\tau_k}) \le 0, & \forall i \in I_\alpha, \\ e_i (m_i - m_{\tau_k}) = \Delta y_i (\rho_{h,i} - \beta)(m_i - m_{\tau_k}) \le 0, & \forall i \in I_\beta. \end{cases}$$

Combining above results, we obtain the desired result. $\square$

From above Lemma, the solution to Problem 3.1 can be represented as:

$$\rho^*_h = \begin{cases} \alpha, & \text{if } 0 \le y < y_{j_1}, \\ \theta, & \text{if } y_{j_1} < y \le y_{j_1+1}, \\ \beta, & \text{if } y_{j_1+1} \le y < y_{j_2}, \\ \alpha, & \text{if } y_{j_2} < y \le L_y, \end{cases} \quad \text{or} \quad \rho^*_h = \begin{cases} \alpha, & \text{if } 0 \le y < y_{j_1}, \\ \beta, & \text{if } y_{j_1} < y \le y_{j_2-1}, \\ \theta, & \text{if } y_{j_2-1} < y \le y_{j_2}, \\ \alpha, & \text{if } y_{j_2} < y \le L_y. \end{cases} \tag{3.11}$$

3.2. The discrete inverse transformation

In the last subsection, we obtain the optimal solution for the discretization of Problem 2.2. Now we introduce a discrete inverse transform to recover the discrete solution of Problem 2.1. Recall the solution of Problem 3.1 is ρ_h^* in (3.11), and define:

$$\Delta x_i = \rho_{h,i}^*\Delta y_i, \qquad \sigma_{h,i}^* = \rho_{h,i}^*, \tag{3.12a}$$
$$x_0 = 0, \qquad x_i = x_{i-1} + \Delta x_i, \quad i = 1,\cdots,N, \tag{3.12b}$$

then $x_N = \sum_i^N \rho_{h,i}^*\Delta y_i = L_x$. Therefore $\{x_i\}_0^N$ forms a partition of interval $[0, L_x]$. The discretization of admissible set $\mathscr{A}_x$ (2.2) can be defined as

$$\mathscr{A}_{h,x} = \left\{\sigma_h \in L^\infty(0, L_x),\ \ \sigma_h|_{x_{i-1}}^{x_i} = \sigma_{h,i},\ \ \fint_0^{L_x} \sigma_h = c\right\}. \tag{3.13}$$

Let $\sigma_h^* = \sum_{i=1}^N \sigma_{h,i}^*\chi_{[x_{i-1},x_i]}$, one should be noticed that in general $\sigma_h^* \notin \mathscr{A}_{h,x}$ since ρ_h^* is not exactly "bang-bang" function. The next Lemma implies σ_h^* is not far away from the admissible set $\mathscr{A}_{h,x}$.

Lemma 3.5. *When $h \to 0$, then $\fint \sigma_h^* dx \to c$.*

Proof. Let $L_{\alpha,y} = \sum_{i\in I_\alpha} |\Delta y_i|$, $L_{\beta,y} = \sum_{i\in I_\beta} |\Delta y_i|$, $L_{\theta,y} = \sum_{i\in I_\theta} |\Delta y_i|$. Then

$$\begin{cases} L_{\alpha,y} + L_{\beta,y} + L_{\theta,y} = L_y, \\ \alpha L_{\alpha,y} + \beta L_{\beta,y} + \theta L_{\theta,y} = L_x. \end{cases}$$

It implies that

$$\begin{aligned}\int_0^{L_x} \sigma_h^* dx =& \sum_{i=1}^N \sigma_{h,i}^*\Delta x_i = \sum_{i=1}^N \rho_{h,i}^*\Delta y_i = \alpha^2 L_{\alpha,y} + \beta^2 L_{\beta,y} + \theta^2 L_{\theta,y} \\ =&(\alpha+\beta)L_x - \alpha\beta L_y + (\theta-\alpha)(\theta-\beta)L_{\theta,y} \\ =&(\alpha+\beta)L_x - (\alpha+\beta-c)L_x + (\theta-\alpha)(\theta-\beta)L_{\theta,y} \\ =&cL_x + (\theta-\alpha)(\theta-\beta)L_\theta.\end{aligned}$$

Therefore

$$\left|\int_0^{L_x} \sigma_h^* dx - cL_x\right| \le |\theta-\alpha||\theta-\beta||L_\theta| \le h\frac{(\alpha+\beta)^2}{4}.$$

So, the lemma is proved. $\square$

From the proof of above Lemma, we immediately have following result:

Corollary 3.1. *For ρ_h^* and ρ^*, the optimal solutions to Problem 2.2 and Problem 2.1 respectively, we have*

$$\fint_0^{L_y} (\rho_h^*)^2(y)dy \to \fint_0^{L_y} (\rho^*)^2(y)dy = \frac{c\alpha\beta}{\alpha+\beta-c}.$$

3.3. Convergence analysis

We first consider the discretization of Problem 2.2.

Theorem 3.1. *Let ρ_h^* be the solution of Problem* 3.1*, $\lambda_{1,h}^*$ and v_h^* be corresponding eigenvalue and positive normalized eigenfunction ($\int_0^{L_y}(v_h^*)_y^2 dy = 1$). For $h \to 0^+$ we have:*

$$\begin{aligned}&\lambda_{1,h}^* \to \lambda_1^*,\\&\rho_h^* \to \rho^* \quad \text{in} \;\; L^2(0,L_y),\\&v_h^* \to v^* \quad \text{in} \;\; H^1(0,L_y),\end{aligned}$$

where ρ^* be the optimal solution to Problem 2.2*, and (λ_1^*, v^*) be the corresponding eigenvalue and eigenfunction.*

Proof. Firstly we claim that

$$\begin{aligned}&\lambda_{1,h}^* \to \lambda_1^*,\\&\rho_h^* \to \rho^* \quad \text{weak star in} \;\; L^\infty(0,L_y),\\&v_h^* \to v^* \quad \text{in} \;\; H^1(0,L_y).\end{aligned}$$

The proof of this claim is similar to Theorem 3.8 in the paper [24] and Lemma 2.3. Firstly passage to a subsequence we have $\rho_h^* \to \rho^*$ weak star in $L^\infty(0,L_y)$, $\lambda_{1,h}^*(\rho_h^*) \to \lambda^*$ and $v_h^*(\rho_h^*) \to v^*$ weakly in $H^1(0,L_y)$. Then we notice that $(\lambda_{1,h}^*(\rho_h^*), \rho_h^*, v_h^*(\rho_h^*))$ satisfies

$$\int_0^{L_y} (v_h^*(\rho_h^*))' v_h' = \lambda_{1,h}^*(\rho_h^*) \int_0^{L_y} \rho_h^* v_h^*(\rho_h^*) v_h, \quad \forall v_h \in \mathscr{V}_{h,y}.$$

Sobolev embedding theorem ($H_0^1 \hookrightarrow L^2$ compactly) and dense property for finite element space ($\overline{\cup_h \mathscr{V}_{h,y}}^{H^1} = H_0^1$) implies that the limit pair $(\lambda_1^*, \rho^*, v^*)$ satisfies

$$\int_0^{L_y} v^{*\prime} v' = \lambda^* \int_0^{L_y} \rho^* v^* v, \quad \forall v \in H_0^1.$$

It is not difficult to see that the limit function v^* is a positive eigenfunction, which implies that $\lambda^* = \lambda_1^*$. The uniqueness of solution gives the convergence for the whole sequence. Lastly norm convergence with weak convergence in H^1 leads the strong convergence for eigenfunction, which complete this claim.

Then by the uniqueness of weak limit we have $\rho_h^* \to \rho^*$ weakly $L^2(0,L_y)$. Then together with Corollary 3.1, we can deduce that $\rho_h^* \to \rho^*$ in $L^2(0,L_y)$. □

Now we move to Problem 2.1. By discrete inverse transformation and the formula of ρ_h^* (3.11), we can get

$$\sigma_h^* = \begin{cases} \alpha, & \text{if } 0 \le x < x_{j_1},\\ \theta, & \text{if } x_{j_1} < x \le x_{j_1+1},\\ \beta, & \text{if } x_{j_1+1} \le x < x_{j_2},\\ \alpha, & \text{if } x_{j_2} < x \le L_x,\end{cases} \quad \text{or} \quad \sigma_h^* = \begin{cases} \alpha, & \text{if } 0 \le x < x_{j_1},\\ \beta, & \text{if } x_{j_1} < x \le x_{j_2-1},\\ \theta, & \text{if } x_{j_2-1} < x \le x_{j_2},\\ \alpha, & \text{if } x_{j_2} < x \le L_x.\end{cases} \tag{3.14}$$

Without loss of generality, assume σ_h^* has the first representation in later of this subsection, we will prove $\int_0^{L_x} |\sigma_h^* - \sigma^*| dx \to 0$.

Lemma 3.6. *Recall the formula* (3.11) *and Lemma* 2.2 *for the discrete optimal density* ρ_h^* *and the continuous optimal density* ρ^* *respectively. Let* $h \to 0$*, then we have* $y_{j_1} \to z_1^\rho$ *and* $y_{j_2} \to z_2^\rho$.

Proof. By Theorem 3.1, $\int_0^{L_y} |\rho_h^* - \rho^*|^2 dy \to 0$, then $\forall \epsilon > 0$, $\exists \overline{h} > 0$, s.t. $\forall h < \min(\overline{h}, \epsilon)$, we have:

$$\int_0^{L_y} |\rho_h^* - \rho^*|^2 dy \le \epsilon(\beta - \alpha)^2.$$

Together with

$$\int_0^{L_y} |\rho_h^* - \rho^*|^2 dy \ge |y_{j_2} - z_2^\rho|(\beta - \alpha)^2,$$

it implies that $|y_{j_2} - z_2^\rho| \le \epsilon$.

On the other hand, if $y_{j_1} \ge z_1^\rho$, then

$$\int_0^{L_y} |\rho_h^* - \rho^*|^2 dy \le \epsilon(\beta - \alpha)^2, \quad \int_0^{L_y} |\rho_h^* - \rho^*|^2 dy \ge |y_{j_1} - z_1^\rho|(\beta - \alpha)^2,$$

leads to $|y_{j_1} - z_1^\rho| \le \epsilon$.

If $y_{j_1} < z_1^\rho \le y_{j_1+1}$, $|y_{j_1} - z_1^\rho| < h \le \epsilon$. If $y_{j_1+1} < z_1^\rho$,

$$\int_0^{L_y} |\rho_h^* - \rho^*|^2 dy \le \epsilon(\beta - \alpha)^2, \quad \int_0^{L_y} |\rho_h^* - \rho^*|^2 dy \ge |y_{j_1+1} - z_1^\rho|(\beta - \alpha)^2,$$

which gives $|y_{j_1} - z_1^\rho| \le h + \epsilon = 2\epsilon$. Overall, let $h \to 0$, we have $y_{j_1} \to z_1^\rho$, $y_{j_2} \to z_2^\rho$. $\square$

Lemma 3.7. *Recall the formula* (3.14) *and Lemma* 2.1 *for the discrete optimal conductivity* σ_h^* *and the continuous optimal conductivity* σ^* *respectively. Let* $h \to 0$*, then we have* $x_{j_1} \to z_1^\sigma$ *and* $x_{j_2} \to z_2^\sigma$.

Proof. From the discrete and continuous inverse Liouville transformation, we can derive that:

$$\begin{aligned} z_1^\sigma &= \alpha z_1^\rho, & x_{j_1} &= \alpha y_{j_1}, \\ L_x - z_2^\sigma &= \alpha(L_y - z_2^\rho), & L_x - x_{j_2} &= \alpha(L_y - y_{j_2}). \end{aligned}$$

Combined with Lemma 3.6, we can obtain $x_{j_1} \to z_1^\sigma$, $z_{j_2} \to z_2^\sigma$. $\square$

Theorem 3.2. *Using the same notation as Lemma* 3.7, *we have $\sigma_h^* \to \sigma^*$ in $L^1(0, L_x)$ when $h \to 0$.*

Proof. By Lemma 3.7, $\forall \epsilon > 0$, $\exists \overline{h} > 0$, s.t. $\forall h < \min(\overline{h}, \epsilon)$, we have:

$$|x_{j_1} - z_1^\sigma| \le \frac{\epsilon}{8(\beta-\alpha)}, \quad |x_{j_2} - z_2^\sigma| \le \frac{\epsilon}{8(\beta-\alpha)}, \quad h \le \frac{\epsilon}{2\beta(\beta-\alpha)}.$$

Then by

$$\int_0^{L_x} |\sigma_h^* - \sigma^*| dx = \left(\int_0^{x_{j_1}} + \int_{x_{j_1}}^{x_{j_1+1}} + \int_{x_{j_1+1}}^{x_{j_2}} + \int_{x_{j_2}}^{L_x} \right) |\sigma_h^* - \sigma^*| dx,$$

we have

$$\begin{aligned}
&\int_0^{x_{j_1}} |\sigma_h^* - \sigma^*| dx \le (\beta-\alpha)|x_{j_1} - z_1^\sigma|, \\
&\int_{x_{j_1}}^{x_{j_1+1}} |\sigma_h^* - \sigma^*| dx \le h\beta(\beta-\alpha), \\
&\int_{x_{j_1+1}}^{x_{j_2}} |\sigma_h^* - \sigma^*| dx \le (\beta-\alpha)|x_{j_1} - z_1^\sigma| + (\beta-\alpha)|x_{j_2} - z_2^\sigma|, \\
&\int_{x_{j_2}}^{L_x} |\sigma_h^* - \sigma^*| dx \le (\beta-\alpha)|z_2^\sigma - x_{j_2}|.
\end{aligned}$$

Therefore

$$\int_0^{L_x} |\sigma_h^* - \sigma^*| dx \le \epsilon.$$

This completes the proof. ☐

3.4. Discrete monotonic decreasing algorithm

In this section, a discrete monotonic decreasing algorithm is used to solve Problem 3.1. See Algorithm 3.1. Given any initial guess $\rho_h^0 \in \mathscr{A}_{h,y}$, we solve the smallest eigenvalue and its corresponding eigenfunction v_h^0 of Eq. (2.3) for $\rho_h = \rho_h^0$, then we update ρ_h^1 from the eigenfunction v_h^0 such that it minimizes the Rayleigh's quotient $\mathscr{R}(\rho_h, v_h^0)$. Repeat this process until the stop rule is satisfied, where $\lambda_h^k = \mathscr{R}(\rho_h^k, v_h^k)$. Once we obtain the discrete optimal triple $(\lambda_{1,h}^*, \rho_h^*, v_h^*)$ from Algorithm 3.1, the discrete optimal σ_h^* can be recovered by the discrete inverse Liouville transformation (3.12).

In Algorithm 3.1, Step 1.2.1 can be obtained by MATLAB routine *eigs*; Step 1.2.2 follows the similar construction in Lemma 3.4. Without loss of generality let $k = 0$, we provide the details to compute the minimization of Rayleigh's quotient $\mathscr{R}(\rho_h, v_h^0)$. Firstly

Algorithm 3.1: Discrete version of monotonic decreasing algorithm.

1. Compute ρ_h, $\lambda_1(\rho_h)$.
   1.1 Initial guess for $\rho_h^0 \in \mathscr{A}_{h,y}$, calculate $(\lambda_{1,h}^0, v_h^0)$.
   1.2 Do while not optimal $(|\lambda_h^k - \lambda_h^{k-1}| \geq \epsilon)$.
      1.2.1 $v_h^k = v_h(\rho_h^k) \triangleq \underset{v_h \in \mathscr{V}_{h,y} \backslash \{0\}}{\operatorname{argmin}} \mathscr{R}(\rho_h^k, v_h)$, $\lambda^k = \mathscr{R}(\rho^k, v^k)$.
      1.2.2 $\rho_h^k = \rho_h(v_h^{k-1}) \triangleq \underset{\rho_h \in \mathscr{A}_{h,y}}{\operatorname{argmin}} \mathscr{R}(\rho_h, v_h^{k-1})$.
2. Recover σ_h^* by discrete inverse transformation.

the the mass integration $J(\Delta_y) = \fint_{\Delta_y} (v_h^0)^2$ are defined in each element Δ_y. To minimize the discrete version of Rayleigh's quotient (3.2), it is same as to maximize:

$$\int_{L_y} \rho_h (v_h^0)^2 = \sum_1^N \int_{\Delta_y} \rho_h (v_h^0)^2 = \sum_1^N \rho_{h,i} J(\Delta_{y,i}) |\Delta_{y,i}|.$$

Suppose $(\tau_1, \tau_2, \cdots, \tau_N)$ be a permutation of $(1, 2, \cdots, N)$ with

$$J(\Delta_{y,\tau_1}) \leq J(\Delta_{y,\tau_2}) \leq \cdots \leq J(\Delta_{y,\tau_N}).$$

Define ρ_h^1 as follows:

$$\rho_{h,\tau_1}^1 = \cdots = \rho_{h,\tau_{k-1}}^1 = \alpha, \quad \rho_{h,\tau_{k+1}}^1 = \cdots = \rho_{h,\tau_N}^1 = \beta,$$

where the subscript τ_k is satisfied with:

$$|\Delta_{y,\tau_1}| + \cdots + |\Delta_{y,\tau_{k-1}}| < \frac{\beta(\beta - c)}{(\beta - \alpha)(\alpha + \beta - c)} L_y \leq |\Delta_{y,\tau_1}| + \cdots + |\Delta_{y,\tau_{k-1}}| + |\Delta_{y,\tau_k}|.$$

To fulfill the constraint, we adopt

$$\rho_{h,\tau_k}^1 = \frac{1}{|\Delta_{y,\tau_k}|} \Big(L_x - \alpha \sum_{j=1}^{k-1} |\Delta_{y,\tau_j}| - \beta \sum_{j=k+1}^{N} |\Delta_{y,\tau_j}| \Big).$$

4. The extremum eigenvalue of Sturm-Liouville problem with potential

In this section, we will extend our method to the Sturm-Liouville problems with non-linear potential. It is not like Problem 2.1, the extremum eigenvalue for Sturm-Liouville problems with general potential does not have a closed form solution. And according to the authors' knowledge, there is no numerical result for this problem yet. The monotonic decreasing algorithm applied to σ-problem directly still have instability, and we will use Liouville transformation to reformulate the σ-problem to an equivalent ρ-problem and apply modified monotonic decreasing algorithm to solve it.

The eigenvalue problem for σ-problem with nonlinear potential reads:

$$\begin{cases} -(\sigma(x)u_x)_x + q(u) = \lambda(\sigma)u \quad \text{in } (0, L_x), \\ u(0) = u(L_x) = 0, \end{cases} \tag{4.1}$$

where $q(u)$ is a smooth function of the variable $u(x)$. The minimization problem is the same as Problem 2.1. Applying Liouville transformation, σ-problem can be reformulated into the following ρ-problem:

$$\begin{cases} -v_{yy} + \rho q(v) = \lambda \rho v \quad \text{in } (0, L_y), \\ v(0) = v(L_y) = 0. \end{cases} \tag{4.2}$$

Hence the corresponding Rayleigh's quotients of above systems be

$$\mathscr{R}(\rho, v) = \frac{\int_0^{L_y} |v_y|^2 dy + \int_0^{L_y} \rho v q(v) dy}{\int_0^{L_y} \rho v^2 dy}.$$

Unlike Algorithm 2.2, the subproblem

$$\rho^k = \operatorname*{argmin}_{\rho \in \mathscr{A}_y} \mathscr{R}(\rho, v^{k-1})$$

is not straightforward now. We notice that for any given v, the subproblem

$$\rho = \operatorname*{argmin}_{\rho \in \mathscr{A}_y} \frac{\int_0^{L_y} |v_y|^2 dy + \int_0^{L_y} \rho v q(v) dy}{\int_0^{L_y} \rho v^2 dy}$$

is a linear fractional program. To overcome the nonconvexity of the cost functional, we use the Dinkelbach's iterative algorithm [4, 21] to solve this linear fractional program. After standard finite element discretization as in Section 3.1, we need to solve subproblem

$$\min_{\rho_h \in \mathscr{A}_{h,y}} \mathscr{R}(\rho_h, v_h) = \frac{\int_0^{L_y} |(v_h)_y|^2 dy + \int_0^{L_y} \rho_h v_h q(v_h) dy}{\int_0^{L_y} \rho v_h^2 dy},$$

which is equivalent to solve

$$\max_{\rho_h \in \mathscr{A}_{h,y}} \frac{\int_0^{L_y} \rho v_h^2 dy}{\int_0^{L_y} |(v_h)_y|^2 dy + \int_0^{L_y} \rho_h v_h q(v_h) dy}.$$

From [4], the above optimization problem is equivalent to

$$F(t^*) = \operatorname*{argmax}_{\rho_h \in \mathscr{A}_{h,y}} \left\{ \int_0^{L_y} \rho_h v_h^2 dy - t^* \int_0^{L_y} \rho_h v_h q(v_h) dy \right\} = 0,$$

where

$$t^* = \frac{\int_0^{L_y} \rho_h^* v_h^2 dy}{\int_0^{L_y} \rho_h^* v_h q(v_h) dy}.$$

Then Dinkelbach's algorithm is used to find t^* iteratively, see Algorithm 4.1. At each iteration of Dinkelbach's algorithm, we may use the same technique from Algorithm 3.1.

Algorithm 4.1: Modified monotonic decreasing algorithm.

---

1. Given a tolerance $\epsilon > 0$ and initial guess $\rho_h^0 \in \mathscr{A}_{h,y}$, compute $(\lambda_{1,h}^0, v_h^0)$. Set $k = 0$.
2. Do while $|\lambda_h^k - \lambda_h^{k-1}| \geq \epsilon$. Let $\widetilde{\rho}_h^0 = \rho_h^k$, $\widetilde{v}_h = v_h^k$.
   - 2.1 Calculate $t^{(1)} = \frac{\Sigma \widetilde{\rho}_h^0 (\widetilde{v}_h)^2}{\Sigma \widetilde{\rho}_h^0 \widetilde{v}_h q(\widetilde{v}_h)}$, set $m = 1$.
   - 2.2 (Dinkelbach's iteration) Do while $F(t^{(m)}) \geq \epsilon$.
     - 2.2.1 update $\widetilde{\rho}_h^m = \underset{\widetilde{\rho}_h \in \mathscr{A}_{h,y}}{\operatorname{argmax}} \{\Sigma \widetilde{\rho}_h^{m-1} (\widetilde{v}_h)^2 - t^{(m)} \Sigma \widetilde{\rho}_h^{m-1} \widetilde{v}_h q(\widetilde{v}_h)\ \}$,
     - 2.2.2 $t^{(m+1)} = \frac{\Sigma \widetilde{\rho}_h^m (\widetilde{v}_h)^2}{\widetilde{\rho}_h^m \widetilde{v}_h q(\widetilde{v}_h)}$, set $m = m + 1$.
   - 2.3 update $\rho_h^k = \widetilde{\rho}_h^m$, compute $(\lambda_{1,h}^k, v_h^k)$, set $k = k + 1$.
3. Recover σ_h^* by discrete inverse transformation.

---

5. Numerical examples

In this section, we present some numerical experiments to verify our efficiency of our algorithm.

Example 5.1. Consider the one-dimensional interval is $L_x = 1.5$, the conductivities of two materials are $\alpha = 1.0$, $\beta = 2.0$, the volume constraint c is 1.5, namely two materials have equal areas. After Liouville transformation, it can be deduced that: $L_y = 1.125$.

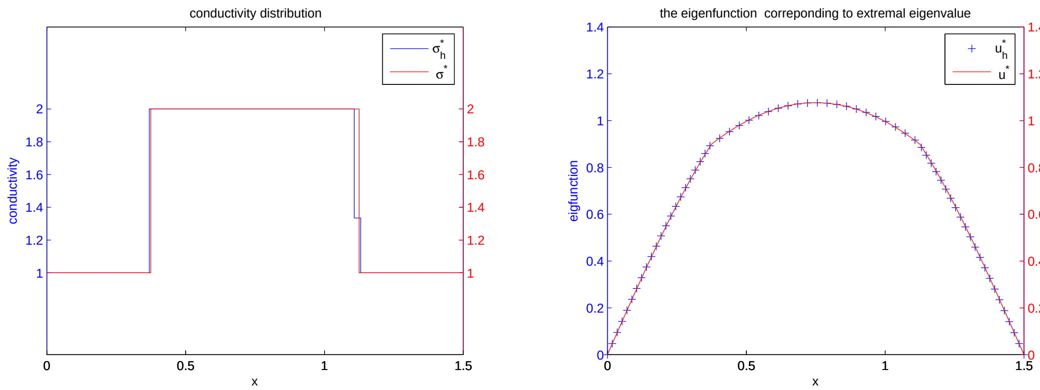
The exact extremal eigenvalue λ_1^* is same the smallest eigenvalue with exact density profile (solution of Problem 2.2) and it is approximately computed over a very fine mesh (3×2^{15} in our test). The mesh grids in our numerical example generate randomly. From the solution on the mesh with N grid points and $2N$ grid points, the convergence order can be calculated by

$$\log_2 \frac{|\lambda_N - \lambda_1^*|}{|\lambda_{2N} - \lambda_1^*|}.$$

From Table 1, the minimum of λ_1 decreases as the mesh grids increasing. The convergence order of the eigenvalue is $\mathscr{O}(h^2)$. The Fig. 1 shows the conductivity σ_h^* and the eigenfunction u_h^* corresponding with the extremal eigenvalue $\lambda_{1,h}^*$ in 65 mesh grids. We observe that

Table 1: A grid refinement analysis for the extremal eigenvalue problem.

N	$Iterations$	$Min\lambda_1$	Order
2^4	4	4.83667467357407	
2^5	4	4.81371122780496	1.97742704964463
2^6	5	4.80784148622367	2.00557426484817
2^7	6	4.80638460008228	1.99109294762469
2^8	6	4.80601686034790	2.00595789305809
2^9	7	4.80592547342413	1.99794092714717
2^{10}	8	4.80590260241290	1.99636507777688



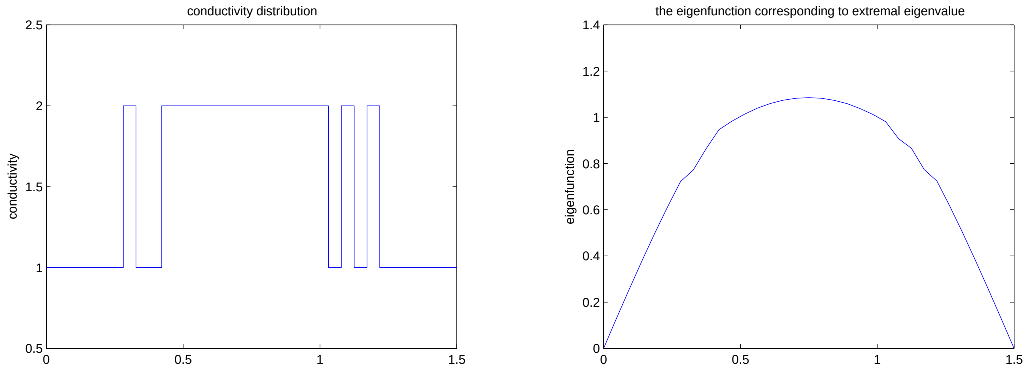
(a) Conductivity distribution (b) Eigenfunction

Figure 1: The conductivity σ_h^* and eigenfunction u_h^* with the extremal eigenvalue $\lambda_{1,h}^*$.

the conductivity σ_h^* in only one element is neither 1.0 nor 2.0, which matches our analysis result. It can be found that the eigenfunction u_h^* is non-smooth, which means that there is jump for the conductivity where material property changed.

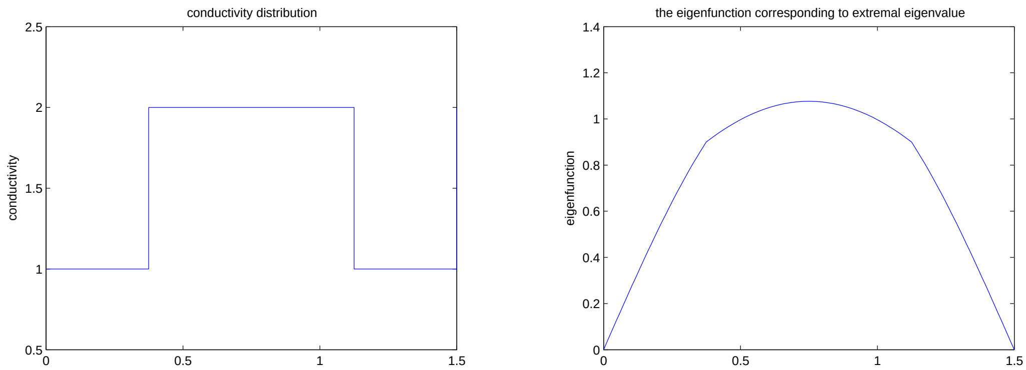
Example 5.2. The optimization problem is the same as Example 5.1. But we will solve the extremum eigenvalue problem in two ways. One way is to solve the conductivity problem directly, we can find the solution to the monotone decreasing algorithm depends on the choice of the initial data. But if we use the Liouville transformation, to reformulate the optimal conductivity distribution problem into the density configuration problem, the numerical experiment shows the algorithm is stable and does not depend on the initial data.

In this example, we use uniform mesh with $3*2^5$ elements and choose the initial conductivity(density) distribution randomly. Without the Liouville transformation, we get the approximate minimum λ_1 is 4.89377248044174. The corresponding eigenfunction u_h^* and



(a) Conductivity distribution (b) Eigenfunction

Figure 2: The conductivity σ_h^* and eigenfunction u_h^* without transformation.



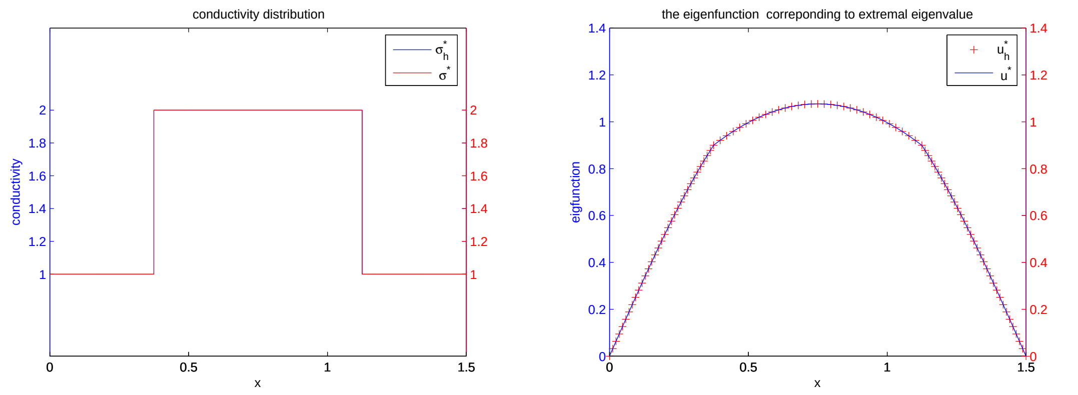
(a) Conductivity distribution (b) Eigenfunction

Figure 3: The conductivity σ_h^* and eigenfunction u_h^* with Liouville transformation.

the conductivity distribution σ_h^* have been described in Fig. 2. Applying Liouville transformation and discrete inverse transformation, we can obtain the optimal minimum λ_1 is 4.80636482143785. We also plot the corresponding eigenfunction u_h^* and the conductivity distribution σ_h^* have been described in Fig. 3.

Example 5.3. For above numerical experiments, the finite element space is chosen as continuous piecewise linear function. Next we will employ the continuous piecewise quadratic finite element space. The optimization problem is the same as Example 5.2.

We use a uniform mesh with $3 * 2^5$ intervals and choose the initial conductivity (density) distribution randomly. By applying Liouville transformation and the discrete inverse transformation, we can obtain the optimal minimum λ_1 is 4.80589495485539 and the



(a) Conductivity distribution (b) Eigenfunction

Figure 4: The conductivity σ_h^* and eigenfunction u_h^* with Liouville transformation by P_2 finite element.

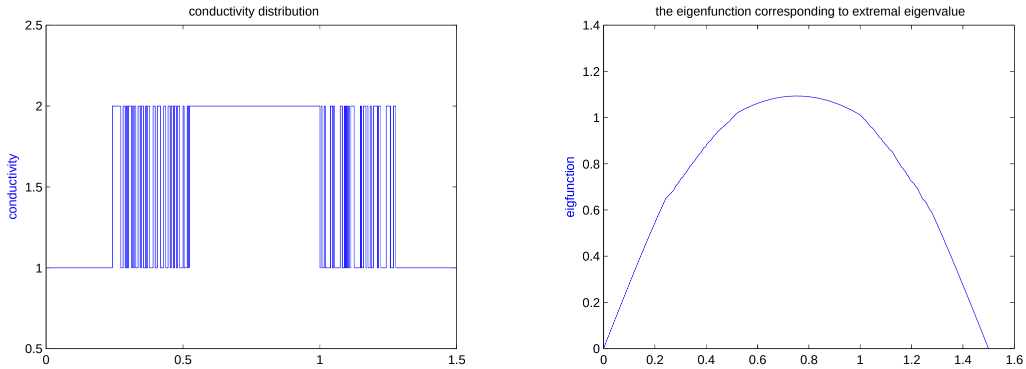
corresponding eigenfunction u_h^* and the conductivity distribution σ_h^* have been plotted in Fig. 4.

Example 5.4. We will consider the Sturm-Liouville problems with potential $q(u) = u^3$. We will also solve this problem by direct solver and by Liouville transformation.

In numerical experiments, let the interval $L_x = 1.5$, the conductivities of two materials are $\alpha = 1.0$, $\beta = 2.0$, the volume constraint c is 1.5. The mesh grids with $3 * 2^7$ and initial conductivity(density) distribution generate randomly. By the direct solver, we could get the approximate minimum eigenvalue is 5.48770808403623. The corresponding eigenfunction u_h^* and the conductivity distribution σ_h^* have been described in Fig. 5. By the Liouville transformation and inverse transformation solver, we can obtain the optimal minimum λ_1 is 5.30592595833589. We also plot the corresponding eigenfunction u_h^* and the conductivity distribution σ_h^* in Fig. 6.

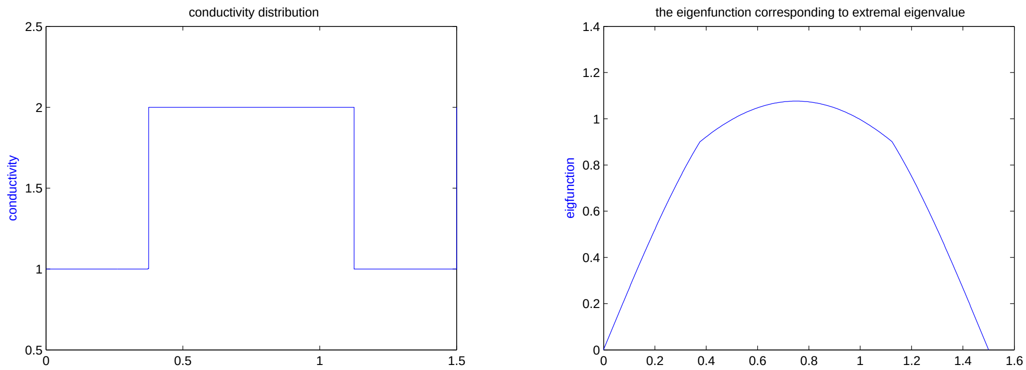
6. Conclusions

In this paper, finite element method with a monotonic decreasing algorithm are applied to solve the extremal eigenvalue of the Sturm-Liouville problem on an interval with variable conductivities. Convergence analysis and numerical experiments are obtained in the paper. We also determine the extremal eigenvalue of the Sturm-Liouville problem with nonlinear potential. Our work can be generalized in several directions. Firstly Dirichlet boundary condition can be replaced by Neumann boundary condition or Robin boundary condition. Secondly, a singular Sturm-Liouville system may be under consideration. Lastly, our method may be extend to find a radially symmetric minimizer when the domain is a ball.



(a) Conductivity distribution (b) Eigenfunction

Figure 5: The conductivity σ_h^* and eigenfunction u_h^* without transformation.



(a) Conductivity distribution (b) Eigenfunction

Figure 6: The conductivity σ_h^* and eigenfunction u_h^* with Liouville transformation.

Acknowledgments The author would like to thank the anonymous referees for the helpful suggestions. This work was supported by the National Natural Science Foundation of China (10971159, 91130022, 11101316).

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