Commun. Comput. Phys.
doi: 10.4208/cicp.221209.030111s

Vol. **11**, No. 2, pp. 555-572
February 2012

Time-Harmonic Acoustic Scattering in a Complex Flow: A Full Coupling Between Acoustics and Hydrodynamics

A. S. Bonnet-Ben Dhia[1,2,*], J. F. Mercier[1], F. Millot[2], S. Pernet[2] and E. Peynaud[2]

[1] *POEMS, CNRS-INRIA-ENSTA UMR 7231, 32 Boulevard Victor, 75015 Paris, France.*
[2] *CERFACS, 42, Avenue Gaspard Coriolis, 31057 Toulouse Cedex 01, France.*

Received 22 December 2009; Accepted (in revised version) 3 January 2011

Available online 24 October 2011

Abstract. For the numerical simulation of time harmonic acoustic scattering in a complex geometry, in presence of an arbitrary mean flow, the main difficulty is the coexistence and the coupling of two very different phenomena: acoustic propagation and convection of vortices. We consider a linearized formulation coupling an augmented Galbrun equation (for the perturbation of displacement) with a time harmonic convection equation (for the vortices). We first establish the well-posedness of this time harmonic convection equation in the appropriate mathematical framework. Then the complete problem, with Perfectly Matched Layers at the artificial boundaries, is proved to be coercive + compact, and a hybrid numerical method for the solution is proposed, coupling finite elements for the Galbrun equation and a Discontinuous Galerkin scheme for the convection equation. Finally a 2D numerical result shows the efficiency of the method.

AMS subject classifications: 35Q35, 65N30

Key words: Aeroacoustics, scattering of sound in flows, Galbrun equation, advection equation, finite elements, discontinuous Galerkin method.

1 Introduction

The reduction of noise is becoming today a main objective whose progress is, in particular, related to a better understanding of the complex phenomena occurring when acoustic waves propagate in presence of a mean flow. For instance, the radiation of the sound produced by aircraft engines is strongly influenced by the presence of the flow around the

*Corresponding author. *Email addresses:* `anne-sophie.bonnet-bendhia@ensta.fr` (A. S. Bonnet-Ben Dhia), `jmercier@ensta.fr` (J. Mercier), `millot@cerfacs.fr` (F. Millot), `pernet@cerfacs.fr` (S. Pernet), `peynaud@cerfacs.fr` (E. Peynaud)

airplane. Several methods have been developed to solve the time-domain Linearized Euler Equations, but the treatment of the artificial boundaries still raises open questions. On the other hand, the time-harmonic problem has been considered only in the simplest case of a potential mean flow, apart for some attempts to solve the model of Galbrun in a general flow [9]. Galbrun's system corresponds to a linearized model whose unknown $\mathbf{u}$ is the perturbation of the Lagrangian displacement. It results in second order equations in time and in space, at first sight similar to more classical wave models. Contrary to the Linearized Euler Equations, Galbrun's system does not involve any derivatives of the mean flow quantities.

Our objective is to develop a numerical method to solve time-harmonic Galbrun's system, in a quite general case in the sense that the geometry, and therefore the mean flow, can be complex. As a consequence, discretization methods written on an unstructured mesh will be privileged. It is now well-known that a direct resolution using finite elements combined with Perfectly Matched Layers does not work. Extending an approach originally applied to time-harmonic Maxwell equations, we have shown that the difficulties can be overcome by writing a so-called augmented equation. This augmented equation requires the evaluation of

$$\psi = \operatorname{curl} \mathbf{u},$$

which becomes the main difficulty.

This approach has been developed in 2D and applied successively to the case of a uniform flow and to the case of a non-vanishing parallel shear flow. In the first case, ψ can be computed a priori [2,8] and in the second case, it is explicitly related to $\mathbf{u}$ by a non-local convolution formula [4]. A simplified approach has been proposed in the case of a low Mach flow [3]: we can then replace the exact non-local expression of ψ by a simple local formula. This low Mach approach has been validated in the case of both a potential and a parallel flow, for which reference solutions are available.

The objective of the present paper is to get rid of the low Mach hypothesis. The main part of the paper is devoted to the theoretical study of the time-harmonic advection equation satisfied by ψ. The well-posedness results that we establish cannot be directly deduced from known results on the classical advection equation [7], but the techniques we use are inspired from [1].

The outline of the paper is the following. The model is briefly described in Section 2, including the augmented equation for $\mathbf{u}$, the hydrodynamic equation for ψ and the Perfectly Matched Layers. Details can be found in [3]. Section 3 is devoted to the theoretical study of the time-harmonic advection equation. Well-posedness is deduced from an inf-sup condition, which is proved for a flow which "fills" the domain, in the sense of [1]. These results are used in Section 4 to prove that the complete problem in $(\mathbf{u},\psi)$ with Perfectly Matched Layers is of Fredholm type if the flow varies slowly. A numerical method, coupling classical finite elements for $\mathbf{u}$ with a Discontinuous Galerkin scheme for ψ is finally described in Section 5 and some numerical results are presented.

2 A model for acoustic scattering in a complex flow

2.1 Geometry and flow

Let $\Omega_\infty = \{(x_1,x_2)\in\mathbb{R}^2;\ x_2>h(x_1)\}$ where h is a continuous positive function such that, for some positive r, $h(x_1)=0$ for $|x_1|>r$. We suppose that Ω_∞ is filled with a compressible inviscid fluid and that the boundary $\Gamma_\infty=\{(x_1,x_2)\in\mathbb{R}^2;\ x_2=h(x_1)\}$ is rigid. The fluid is moving and the flow, which is stationary, is characterized by its non uniform fields of velocity $\mathbf{v}_0$, density ρ_0, pressure p_0 and sound velocity c_0, which solve in Ω_∞ the stationary Euler equations:

$$\begin{cases} \operatorname{div}(\rho_0\mathbf{v}_0)=0, \\ \rho_0\mathbf{v}_0\cdot\nabla\mathbf{v}_0+\nabla p_0=0. \end{cases} \tag{2.1}$$

On the rigid boundary:

$$\mathbf{v}_0\cdot\mathbf{n}=0 \quad (\Gamma_\infty), \tag{2.2}$$

where $\mathbf{n}$ denotes the normal vector to Γ_∞ pointing to the exterior of Ω_∞. Finally, for a barotropic fluid, the state law reads:

$$\nabla p_0 = c_0^2\nabla\rho_0. \tag{2.3}$$

We suppose that the flow is subsonic and uniform far from the perturbation:

$$\exists R>0,\ \forall\ |x|>R,\quad \mathbf{v}_0(x)=v_\infty\mathbf{e}_1 \quad \text{and} \quad (\rho_0(x),p_0(x),c_0(x))=(\rho_\infty,p_\infty,c_\infty).$$

This means that the half disk $D_R=\{(x_1,x_2);\ x_2>0 \text{ and } x_1^2+x_2^2<R^2\}$ of radius R contains the perturbed area of the propagation domain. We suppose for instance that $v_\infty>0$. Finally, we assume for simplicity that all quantities related to the mean flow are regular enough in the sense that ρ_0, p_0, c_0 and $\mathbf{v}_0$ are in $\mathcal{C}^2(\overline{\Omega}_\infty)$. This regularity will be used to define the operators $\mathcal{B}$ and $\mathcal{C}$ below.

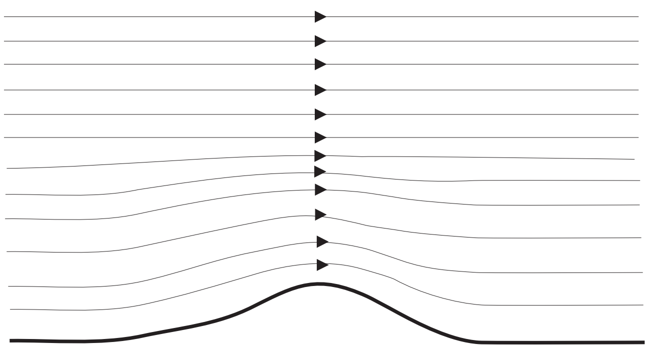

Figure 1: Mean flow.

2.2 The augmented Galbrun equation

The Galbrun equation is a linear equation which models the propagation of small perturbations of the previous mean flow, which can be produced by an acoustic source. In time harmonic regime (with a $e^{-i\omega t}$ time dependence, $\omega>0$), this equation takes the following form:

$$\rho_0\frac{D^2\mathbf{u}}{Dt^2}-\nabla\big(\rho_0 c_0^2\mathrm{div}\,\mathbf{u}\big)+\mathrm{div}\,\mathbf{u}\nabla p_0-{}^t\nabla\mathbf{u}\cdot\nabla p_0=\mathbf{f}\quad(\Omega_\infty),\tag{2.4}$$

where the convective derivative $D\mathbf{u}/Dt$ is defined by:

$$\frac{D\mathbf{u}}{Dt}=-i\omega\mathbf{u}+\nabla\mathbf{u}\cdot\mathbf{v}_0$$

and $\mathbf{f}$ is a source term, compactly supported in $D_R\cap\Omega_\infty$, such that $\mathrm{curl}\,\mathbf{f}\in L^2(\Omega_\infty)$. The unknown $\mathbf{u}$ is the perturbation of displacement. Its normal component vanishes on the rigid boundary:

$$\mathbf{u}\cdot\mathbf{n}=0\quad(\Gamma_\infty).\tag{2.5}$$

Let us notice that outside the perturbed area D_R, Eq. (2.4) takes the following simplified form:

$$\frac{D^2\mathbf{u}}{Dt^2}-c_\infty^2\nabla(\mathrm{div}\,\mathbf{u})=\frac{1}{\rho_\infty}\mathbf{f}.\tag{2.6}$$

It is well-known that a direct finite element discretization of (2.4) (using Lagrange elements) leads to a polluted result, due to a lack of H^1 coerciveness. A way to restore coerciveness is to consider the following "augmented" formulation:

$$\rho_0\frac{D^2\mathbf{u}}{Dt^2}-\nabla(\rho_0c_0^2\mathrm{div}\,\mathbf{u})+\mathbf{curl}\big(\rho_0c_0^2(\mathrm{curl}\,\mathbf{u}-\psi)\big)+\mathrm{div}\,\mathbf{u}\nabla p_0-{}^t\nabla\mathbf{u}\cdot\nabla p_0=\mathbf{f},\tag{2.7}$$

where we have introduced a new unknown

$$\psi=\mathrm{curl}\,\mathbf{u}$$

called here the "vorticity" (in the literature, the vorticity is usually defined as the curl of the Eulerian velocity). It has been proved in [3] that ψ satisfies the following equation

$$\frac{D^2\psi}{Dt^2}=-2\frac{D}{Dt}(\mathcal{B}\mathbf{u})-\mathcal{C}\mathbf{u}+\frac{1}{\rho_0}\mathrm{curl}\,\mathbf{f}\tag{2.8}$$

with

$$\mathcal{B}\mathbf{u}=\sum_{j=1}^{2}\nabla v_{0,j}\wedge\frac{\partial\mathbf{u}}{\partial x_j}\tag{2.9}$$

and

$$\mathcal{C}\mathbf{u}=\sum_{j,k=1}^{2}\Big(\frac{\partial v_{0,k}}{\partial x_j}\nabla v_{0,j}\wedge\frac{\partial\mathbf{u}}{\partial x_k}-v_{0,j}\nabla\frac{\partial v_{0,k}}{\partial x_j}\wedge\frac{\partial\mathbf{u}}{\partial x_k}\Big)+\frac{1}{\rho_0}\sum_{j=1}^{2}\Big(\frac{1}{\rho_0c_0^2}\frac{\partial p_0}{\partial x_j}\nabla p_0-\nabla\Big(\frac{\partial p_0}{\partial x_j}\Big)\Big)\wedge\nabla u_j.\tag{2.10}$$

Notice that $\mathcal{C}\mathbf{u}=\mathcal{B}\mathbf{u}=0$ outside the perturbed area D_R and that $\mathcal{C}\mathbf{u}$ vanishes everywhere for a parallel shear flow.

For the coupled problems (2.5), (2.7), (2.8) to be equivalent with the initial problems (2.4, 2.5), the following additional boundary condition must be imposed:

$$\operatorname{curl}\mathbf{u}-\psi=0 \quad (\Gamma_\infty). \tag{2.11}$$

Moreover let us point out that the equivalence (with $\mathbf{u}\in H^1(\Omega_\infty)^2$) requires the regularity of $\partial\Omega_\infty$ (see for instance the Remark 3.5 in [5]), and the treatment of reentrant corners still raises open questions of modelling.

2.3 The perfectly matched layers

Our objective is the computation of the "outgoing" solution $(\mathbf{u},\psi)$ of the coupled problem (2.7), (2.5), (2.8), (2.11). Following [2], for this outgoing solution, ψ must vanish upstream of the perturbation area: indeed, the vortices are produced by the source and by the coupling between acoustics and hydrodynamics in the case of a non uniform flow, and are then convected downstream by the flow. In practice, we use PMLs to select this outgoing solution. The computational domain is defined by $\Omega_L=B_L\cap\Omega_\infty$, where B_L is the following square

$$B_L=\Big\{(x_1,x_2);\ |x_1|<R+L \text{ and } 0<x_2<R+L\Big\}$$

and L denotes the width of the absorbing layers. The model in the PMLs involves a complex parameter α such that $\Re e(\alpha)>0$ and $\Im m(\alpha)<0$. In the following the index α means that the corresponding operator has been modified according to the substitution:

$$\frac{\partial}{\partial x_i}\to\alpha_i(x)\frac{\partial}{\partial x_i}$$

with α_i defined by $\alpha_i(x)=1$ if $|x_i|<R$ and $\alpha_i(x)=\alpha$ if $|x_i|>R$.

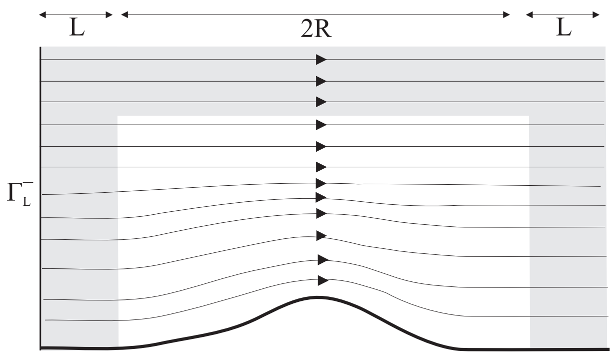

Figure 2: The computational domain with the perfectly matched layers.

For example,

$$\mathrm{div}_\alpha \mathbf{u} = \alpha_1(x)\frac{\partial u_1}{\partial x_1} + \alpha_2(x)\frac{\partial u_2}{\partial x_2}.$$

Finally, the problem that we solve is the following:

$$\begin{aligned}
&\rho_0 \frac{D_\alpha^2 \mathbf{u}}{Dt^2} - \nabla_\alpha(\rho_0 c_0^2 \mathrm{div}_\alpha \mathbf{u}) + \mathbf{curl}_\alpha(\rho_0 c_0^2(\mathrm{curl}_\alpha \mathbf{u} - \psi)) \\
&\qquad + \mathrm{div}_\alpha \mathbf{u} \nabla p_0 - {}^t\nabla_\alpha \mathbf{u} \cdot \nabla p_0 = \mathbf{f}, && \text{in } \Omega_L, && (2.12a) \\
&\frac{D_\alpha^2 \psi}{Dt^2} = -2\frac{D_\alpha}{Dt}(\mathcal{B}\mathbf{u}) - \mathcal{C}\mathbf{u} + \frac{1}{\rho_0}\mathrm{curl}\mathbf{f}, && \text{in } \Omega_L, && (2.12b) \\
&\mathbf{u}\cdot\mathbf{n} = \mathrm{curl}_\alpha \mathbf{u} - \psi = 0, && \text{on } \partial\Omega_L, && (2.12c) \\
&\psi = \frac{D_\alpha \psi}{Dt} = 0, && \text{on } \Gamma_L^-, && (2.12d)
\end{aligned}$$

where Γ_L^- is the inflow boundary of the computational domain:

$$\Gamma_L^- = \Big\{(x_1,x_2);\ x_1 = -R-L \text{ and } 0 < x_2 < R+L\Big\}.$$

The boundary condition on Γ_L^- will allow to ensure the causality of ψ, as described below.

Notice that it is useless to introduce the notations $\mathcal{B}_\alpha$ or $\mathcal{C}_\alpha$ since $\mathcal{B}$ and $\mathcal{C}$ vanish in the absorbing layers. For the same reason, we wrote ∇p_0 instead of $\nabla_\alpha p_0$ in the first equation of (2.12).

It has been already proved (see for instance [3]) that for a given ψ, the problem in $\mathbf{u}$ is of Fredholm type in $H^1(\Omega_L)^2$. In the next section, we consider the problem in ψ for a given $\mathbf{u}$. Results for the coupled problem (2.12) are finally discussed in Section 4.

3 The time-harmonic convective equation

3.1 A model problem

Let us first consider the following model problem :

$$\begin{aligned}
&-i\omega\psi + \mathbf{v}\cdot\nabla\psi = g, && \text{in } \Omega, && (3.1a) \\
&\psi = 0, && \text{on } \Gamma^-, && (3.1b)
\end{aligned}$$

where Ω is a bounded domain of $\mathbb{R}^2$, $\mathbf{v}$ is a vector field defined on Ω such that

$$\mathbf{v} \in \mathcal{C}^1(\overline{\Omega})^2 \quad \text{and} \quad \mathrm{div}\mathbf{v} = 0$$

and Γ^- (resp. Γ^+) is the inflow (resp. outflow) boundary ($\mathbf{n}$ is the exterior normal vector to Ω):

$$\Gamma^\pm = \{x \in \partial\Omega;\ \pm\mathbf{v}\cdot\mathbf{n} > 0\}.$$

Contrary to the standard advection equation with $\omega = ia$ with $a \geq 0$ (see [7]), this time-harmonic equation does not seem to have been studied before. Following [7], we notice that $\mathrm{div}(\psi \mathbf{v}) = \mathbf{v}\cdot\nabla\psi$ and we consider the Hilbert space

$$H(\Omega,\mathbf{v}) = \left\{\psi \in L^2(\Omega);\ \mathrm{div}(\psi\mathbf{v}) \in L^2(\Omega)\right\} \tag{3.2}$$

equipped with the following norm:

$$\|\psi\|_{H(\Omega,\mathbf{v})} = \sqrt{\int_\Omega \omega^2|\psi|^2 + |\mathbf{v}\cdot\nabla\psi|^2}, \tag{3.3}$$

where ω is introduced for homogeneity reasons. Then, we deduce from classical properties of the space $H(\Omega,\mathrm{div})$ the existence of a continuous trace application:

$$\psi \in H(\Omega,\mathbf{v}) \mapsto \psi\mathbf{v}\cdot\mathbf{n} \in H^{-\frac{1}{2}}(\partial\Omega).$$

As a consequence, the space

$$H(\Omega,\mathbf{v},\Gamma^-) = \left\{\psi \in L^2(\Omega);\ \mathbf{v}\cdot\nabla\psi \in L^2(\Omega) \text{ and } \psi = 0 \text{ on } \Gamma^-\right\} \tag{3.4}$$

is a closed subspace of $H(\Omega,\mathbf{v})$. Naturally, the space $H(\Omega,\mathbf{v},\Gamma^+)$ can be defined in the same manner. Finally, we have the:

Lemma 3.1. *For all $\psi \in H(\Omega,\mathbf{v},\Gamma^-)$, $|\psi|^2\mathbf{v}\cdot\mathbf{n} \in L^1(\Gamma^+)$ and the following identities hold:*

$$\forall\psi \in H(\Omega,\mathbf{v},\Gamma^-),\ \forall\varphi \in H(\Omega,\mathbf{v}),\ \int_\Omega (\mathbf{v}\cdot\nabla\psi)\varphi = -\int_\Omega \psi(\mathbf{v}\cdot\nabla\varphi) + \int_{\Gamma^+} \mathbf{v}\cdot\mathbf{n}\psi\varphi d\gamma, \tag{3.5a}$$

$$\forall\psi \in H(\Omega,\mathbf{v},\Gamma^-),\qquad \int_{\Gamma^+} \mathbf{v}\cdot\mathbf{n}|\psi|^2 d\gamma \leq \frac{1}{\omega}\|\psi\|^2_{H(\Omega,\mathbf{v})}. \tag{3.5b}$$

We can now specify the functional framework well-suited for problem (3.1): for a data g in $L^2(\Omega)$, the solution ψ is sought in $H(\Omega,\mathbf{v},\Gamma^-)$.

3.2 Explicit solution in the case of a uniform flow

Let us first consider the case of a uniform vector field $\mathbf{v} = v\mathbf{e}_1$ $(v > 0)$ in a rectangular domain $\Omega = [0,d]\times[0,\ell]$, so that $\Gamma^- = \left\{(0,x_2);\ 0 < x_2 < \ell\right\}$. Then problem (3.1) consists in finding $\psi \in H(\Omega,\mathbf{v})$ such that

$$v\frac{\partial\psi}{\partial x_1}(x_1,x_2) - i\omega\psi(x_1,x_2) = g(x_1,x_2), \qquad \text{in } \Omega, \tag{3.6a}$$

$$\psi(0,x_2) = 0, \qquad \text{for } 0 < x_2 < \ell, \tag{3.6b}$$

where $g \in L^2(\Omega)$. This is a family of first order differential equations in x_1 (with constant coefficients) parametrized by x_2, whose solution $\psi = \psi_g$ is given by the following convolution formula

$$\psi_g(x_1,x_2) = \frac{1}{v}\int_0^{x_1} g(s,x_2)e^{i\frac{\omega}{v}(x_1-s)}ds. \tag{3.7}$$

Notice the oscillating behavior (the smaller v, the smaller the wavelength). From formula (3.7) results the following stability estimate:

$$\|\psi_g\|_{H(\Omega,\mathbf{v})} \leq \sqrt{1+\sqrt{2}\frac{\omega d}{v}+\frac{\omega^2 d^2}{v^2}}\|g\|_{L^2(\Omega)}. \tag{3.8}$$

Indeed, we have

$$\begin{aligned}\|\psi_g\|^2_{H(\Omega,\mathbf{v})} &= \|\omega\psi_g\|^2_{L^2(\Omega)} + \left\|v\frac{\partial\psi_g}{\partial x_1}\right\|^2_{L^2(\Omega)} \\ &= \|\omega\psi_g\|^2_{L^2(\Omega)} + \|g+i\omega\psi_g\|^2_{L^2(\Omega)} \\ &\leq 2\|\omega\psi_g\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} + 2\|g\|_{L^2(\Omega)}\|\omega\psi_g\|_{L^2(\Omega)}\end{aligned} \tag{3.9}$$

and we directly obtain (3.8) by using the estimate

$$\begin{aligned}\|\omega\psi_g\|^2_{L^2(\Omega)} &\leq \int_0^d\int_0^\ell \frac{\omega^2}{v^2}\Big(\int_0^{x_1}|g(s,x_2)|ds\Big)^2 dx_2dx_1 \\ &\leq \frac{\omega^2}{v^2}\int_0^d\int_0^\ell\Big(\int_0^{x_1}|g(s,x_2)|^2ds\Big)\Big(\int_0^{x_1}1^2ds\Big)dx_2dx_1 \\ &\leq \frac{\omega^2 d^2}{2v^2}\|g\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.10}$$

Let us point out that the estimate (3.8) deteriorates when the length d of the domain increases (with a linear dependence in d) or when the velocity v decreases.

3.3 Well-posedness in the general case

In the case of an arbitrary vector field $\mathbf{v}$, the previous approach is not generalizable and an explicit solution of problem (3.1) is not available. We will use instead a variational approach for problem (3.1) written in the following weak form:

$$\psi \in H(\Omega,\mathbf{v},\Gamma^-), \quad a(\psi,\phi) = \int_\Omega g\bar{\phi}, \quad \forall\phi \in L^2(\Omega), \tag{3.11}$$

where $a(\psi,\phi) = \int_\Omega(-i\omega\psi+\mathbf{v}\cdot\nabla\psi)\bar{\phi}$. Following [1], we suppose that the vector field $\mathbf{v}$ satisfies the following additional hypothesis:

$$v^- = \inf_{x\in\Omega}\mathbf{v}(x)\cdot\mathbf{e}_1 > 0. \tag{3.12}$$

This condition implies that $\mathbf{v}$ is Ω-filling (see [1]) in the sense that every point in Ω can be reached by the flow associated to $\mathbf{v}$ in a finite time. In particular, recirculation zones are forbidden.

It is proved in [1] that, under condition (3.12), problem (3.1) with $\omega=0$ is well-posed. The same result holds if Γ^- is replaced by Γ^+. In particular, there exists a unique real-valued function τ such that

$$\tau\in H(\Omega,\mathbf{v},\Gamma^+) \quad \text{and} \quad \mathbf{v}\cdot\nabla\tau=-2, \quad \text{in } \Omega. \tag{3.13}$$

Moreover $\tau\in L^\infty(\Omega)$ and $\|\tau\|_{L^\infty(\Omega)}$ is twice the maximum time necessary for a particle convected by the flow $\mathbf{v}$ to go across the domain Ω, so that:

$$\|\tau\|_{L^\infty(\Omega)}\leq 2\frac{d(\Omega)}{v^-}, \tag{3.14}$$

where $d(\Omega)=\max_{x\in\Omega}\{x_1\}-\min_{x\in\Omega}\{x_1\}$. The function τ will be used to establish the following result:

Proposition 3.1. Under condition (3.12), the following inf-sup condition holds:

$$\inf_{\psi\in H(\Omega,\mathbf{v},\Gamma^-)}\sup_{\phi\in L^2(\Omega)} \Re e\big(a(\psi,\phi)\big)\geq\frac{1}{\beta}\|\psi\|_{H(\Omega,\mathbf{v})}\|\phi\|_{L^2(\Omega)},$$

where

$$\beta=2\sqrt{2+4\Big(\frac{\omega d(\Omega)}{v^-}\Big)^2}.$$

Proof. Let $\psi\in H(\Omega,\mathbf{v},\Gamma^-)$. Taking $\phi=\omega^2\tau\psi+\mathbf{v}\cdot\nabla\psi$, we get

$$\begin{aligned}\Re e(a(\psi,\phi))&=\Re e\Big(\int_\Omega -i\omega^3\tau|\psi|^2+|\mathbf{v}\cdot\nabla\psi|^2+\omega^2(\mathbf{v}\cdot\nabla\psi)\tau\bar{\psi}-i\omega\psi(\mathbf{v}\cdot\nabla\bar{\psi})\Big)\\&\geq\int_\Omega|\mathbf{v}\cdot\nabla\psi|^2+\omega^2\Re e\Big(\int_\Omega(\mathbf{v}\cdot\nabla\psi)\tau\bar{\psi}\Big)-\omega\|\psi\|_{L^2(\Omega)}\|\mathbf{v}\cdot\nabla\psi\|_{L^2(\Omega)}.\end{aligned} \tag{3.15}$$

Then applying (3.5), we get:

$$\int_\Omega(\mathbf{v}\cdot\nabla\psi)\tau\bar{\psi}=-\int_\Omega\tau\psi(\mathbf{v}\cdot\nabla\bar{\psi})-\int_\Omega|\psi|^2\mathbf{v}\cdot\nabla\tau,$$

which gives, using the properties of τ:

$$\Re e\Big(\int_\Omega(\mathbf{v}\cdot\nabla\psi)\tau\bar{\psi}\Big)=\|\psi\|^2_{L^2(\Omega)}. \tag{3.16}$$

Combining (3.15) and (3.16), we obtain finally the following inequality:

$$\Re e\big(a(\psi,\phi)\big)\geq\frac{1}{2}\|\psi\|^2_{H(\Omega,\mathbf{v})}. \tag{3.17}$$

On the other hand:

$$\|\phi\|^2_{L^2(\Omega)}=\|\mathbf{v}\cdot\nabla\psi\|^2_{L^2(\Omega)}+\omega^4\|\tau\psi\|^2_{L^2(\Omega)}+2\omega^2\Re e\Big(\int_\Omega(\mathbf{v}\cdot\nabla\psi)\tau\bar{\psi}\Big),$$

which leads, using (3.16) to the following estimate:

$$\|\varphi\|^2_{L^2(\Omega)} \leq \big(2+\omega^2\|\tau\|^2_{L^\infty(\Omega)}\big)\|\psi\|^2_{H(\Omega,\mathbf{v})}. \tag{3.18}$$

The theorem results from (3.17) and (3.18). □

Well-posedness of problem (3.1) is then a simple consequence of the previous proposition:

Theorem 3.1. *Under condition* (3.12), (3.1) *is well-posed and its solution* ψ *satisfies the following estimate:*

$$\|\psi\|_{H(\Omega,\mathbf{v})} \leq \beta\|g\|_{L^2(\Omega)}, \tag{3.19}$$

where β *has been defined in Proposition* 3.1.

Proof. Let us consider the operator A from $H(\Omega,\mathbf{v},\Gamma^-)$ to $L^2(\Omega)$ defined by $A\psi=-i\omega\psi+\mathbf{v}\cdot\nabla\psi$. From Proposition 3.1, it results that A is injective and has a closed range. To prove the surjectivity, notice that the adjoint A^* of A which is defined from $H(\Omega,\mathbf{v},\Gamma^+)$ into $L^2(\Omega)$ by $A^*\psi=i\omega\psi+\mathbf{v}\cdot\nabla\psi$ is also injective (by a similar argument), so that the range of A is dense. □

Remark 3.1. In the particular case studied in Subsection 3.2, estimate (3.19) becomes:

$$\|\psi\|_{H(\Omega,\mathbf{v})} \leq 2\sqrt{2+4\frac{\omega^2 d^2}{v^2}}\|g\|_{L^2(\Omega)},$$

which is in accordance with (3.8).

3.4 Some straightforward generalizations

Some simple extensions are required in order to apply the previous results to the acoustic problem (2.12).

3.4.1 The case of a compressible flow

The flow which is considered in the acoustic problem is a solution of Euler's equations (2.1). In particular, the velocity field $\mathbf{v}_0$ satisfies $\operatorname{div}(\rho_0\mathbf{v}_0)=0$ but not $\operatorname{div}\mathbf{v}_0=0$. As a consequence, the above results cannot be directly applied to the following problem

$$-i\omega\psi+\mathbf{v}_0\cdot\nabla\psi=g, \qquad \text{in } \Omega, \tag{3.20a}$$

$$\psi=0, \qquad \text{on } \Gamma^-. \tag{3.20b}$$

The idea is to write the first equation of (3.20) in the following equivalent form:

$$-i\omega\rho_0\psi+\rho_0\mathbf{v}_0\cdot\nabla\psi=\rho_0 g, \quad \text{in } \Omega, \tag{3.21}$$

in order to use the equation $\operatorname{div}(\rho_0\mathbf{v}_0)=0$, and then to introduce a modified definition of function τ:

$$\tau\in H(\Omega,\rho_0\mathbf{v}_0,\Gamma^+) \quad \text{and} \quad \rho_0\mathbf{v}_0\cdot\nabla(\rho_0\tau)=-2\rho_0^2, \quad \text{in } \Omega \tag{3.22}$$

and a modified definition of the norm in the space $H(\Omega,\rho_0\mathbf{v}_0)$:

$$\|\psi\|_{H(\Omega,\rho_0\mathbf{v}_0)}=\sqrt{\int_\Omega \omega^2\rho_0^2|\psi|^2+|\rho_0\mathbf{v}_0\cdot\nabla\psi|^2}.$$

If we assume that the density ρ_0 belongs to $L^\infty(\Omega)$ and is bounded from below by a strictly positive constant $\rho_0^{\inf}>0$, then we obtain with a very similar approach as above the following result:

Theorem 3.2. *If* $\mathbf{v}_0$ *satisfies condition* (3.12), (3.20) *is well-posed and its solution* ψ *satisfies the following estimate:*

$$\|\psi\|_{H(\Omega,\rho_0\mathbf{v}_0)}\le\beta_0\|\rho_0 g\|_{L^2(\Omega)}, \tag{3.23}$$

where

$$\beta_0=2\sqrt{2+4\Big(\frac{\omega d(\Omega)}{v_0^-}\frac{\rho_0^{\sup}}{\rho_0^{\inf}}\Big)^2}$$

and $\rho_0^{\sup}=\|\rho_0\|_\infty$.

3.4.2 The case of a second order time-harmonic convective equation

As the hydrodynamic equation (2.8) is a second order one, it will be useful to notice that the following problem

$$\frac{D^2\psi}{Dt^2}=(-i\omega+\mathbf{v}_0\cdot\nabla)^2\psi=g, \qquad \text{in } \Omega, \tag{3.24a}$$

$$\psi=\frac{D\psi}{Dt}=0, \qquad \text{on } \Gamma^-, \tag{3.24b}$$

can be very simply solved by introducing the intermediary unknown $\tilde{\psi}=D\psi/Dt$. By Theorem 3.2, there exists a unique $\tilde{\psi}\in H(\Omega,\rho_0\mathbf{v}_0,\Gamma^-)$ solution of $D\tilde{\psi}/Dt=g$ and a unique $\psi\in H(\Omega,\rho_0\mathbf{v}_0,\Gamma^-)$ solution of $D\psi/Dt=\tilde{\psi}$ (therefore solution of (3.24)) which satisfy the following estimates:

$$\|\tilde{\psi}\|_{H(\Omega,\rho_0\mathbf{v}_0)}\le\beta_0\|\rho_0 g\|_{L^2(\Omega)} \quad \text{and} \quad \|\psi\|_{H(\Omega,\rho_0\mathbf{v}_0)}\le\beta_0\|\rho_0\tilde{\psi}\|_{L^2(\Omega)}.$$

Summing up, we obtain the following estimate:

$$\|\psi\|_{H(\Omega,\rho_0\mathbf{v}_0)}\le\frac{{\beta_0}^2}{\omega}\|\rho_0 g\|_{L^2(\Omega)}. \tag{3.25}$$

Indeed, by using the definition of the norm $\|\cdot\|_{H(\Omega,\rho_0\mathbf{v}_0)}$, we immediately obtain:

$$\|\rho_0\tilde{\psi}\|_{L^2(\Omega)}\le\frac{1}{\omega}\|\tilde{\psi}\|_{H(\Omega,\rho_0\mathbf{v}_0)}. \tag{3.26}$$

4 Well-posedness of the coupled problem

We will now use the previous results on the time harmonic convective equation to prove, under some conditions on the mean flow, the well-posedness of the coupled problem (2.12). We suppose that $\mathbf{v}_0$ satisfies condition (3.12).

4.1 Estimates on ψ

Suppose first that $\mathbf{u}\in H_0^1(\Omega_L)^2$ and let us consider the following problem for ψ, which is part of problem (2.12):

$$\frac{D_\alpha^2\psi}{Dt^2}=-2\frac{D_\alpha}{Dt}(\mathcal{B}\mathbf{u})-\mathcal{C}\mathbf{u}+\frac{1}{\rho_0}\mathrm{curl}\,\mathbf{f},\qquad \text{in } \Omega_L, \tag{4.1a}$$

$$\psi=\frac{D_\alpha\psi}{Dt}=0,\qquad \text{on } \Gamma_L^-. \tag{4.1b}$$

Simple considerations show that ψ vanishes in $\Omega_L\backslash(\Omega_R\cup Q_L^+)$ where we have set

$$\Omega_R=\Big\{(x_1,x_2);\ |x_1|<R \text{ and } 0<x_2<R\Big\}\cap\Omega_\infty,$$

$$Q_L^+=\Big\{(x_1,x_2);\ R<x_1<R+L \text{ and } 0<x_2<R\Big\}.$$

Problem (4.1) can be solved by solving first the problem in Ω_R:

$$\frac{D^2\psi}{Dt^2}=-2\frac{D}{Dt}(\mathcal{B}\mathbf{u})-\mathcal{C}\mathbf{u}+\frac{1}{\rho_0}\mathrm{curl}\,\mathbf{f},\qquad \text{in } \Omega_R, \tag{4.2a}$$

$$\psi=\frac{D\psi}{Dt}=0,\qquad \text{on } \Gamma_R^-, \tag{4.2b}$$

where $\Gamma_R^\pm=\{(x_1,x_2)/\pm x_1=R \text{ and } 0<x_2<R\}$. The solution of (4.2) then provides initial conditions on Γ_R^+ (which is the inflow boundary of Q_L^+) to compute ψ in Q_L^+, which is a solution of the following homogeneous equation (since $\mathcal{B}$, $\mathcal{C}$, and $\mathbf{f}$ are supported in Ω_R) with constant coefficients:

$$\frac{D_\alpha^2\psi}{Dt^2}=\Big(-i\omega+\alpha v_\infty\frac{\partial}{\partial x_1}\Big)^2\psi=0,\quad \text{in } Q_L^+. \tag{4.3}$$

By linearity, the solution ψ of (4.2) is given by $\psi=\psi_{\mathcal{B}}+\psi_{\mathcal{C}}+\psi_{\mathbf{f}}$, where $\psi_{\mathcal{B}}$ is a solution of

$$\frac{D\psi_{\mathcal{B}}}{Dt}=-2\mathcal{B}\mathbf{u},\qquad \text{in } \Omega_R, \tag{4.4a}$$

$$\psi_{\mathcal{B}}=0,\qquad \text{on } \Gamma_R^-, \tag{4.4b}$$

and $\psi_{\mathcal{C}}$ and $\psi_{\mathbf{f}}$ satisfy the same homogeneous initial conditions on Γ_R^- as ψ and the following equations in Ω_R:

$$\frac{D^2\psi_{\mathcal{C}}}{Dt^2}=-\mathcal{C}\mathbf{u} \quad \text{and} \quad \frac{D^2\psi_{\mathbf{f}}}{Dt^2}=\frac{1}{\rho_0}\mathrm{curl}\mathbf{f}.$$

Results of subsection 3.4 prove the existence of $\psi_{\mathcal{B}}$ and $\psi_{\mathcal{C}}$ ($\psi_{\mathbf{f}}$ can be treated like $\psi_{\mathcal{C}}$) and the following estimates:

$$\|\psi_{\mathcal{B}}\|_{H(\Omega_R,\rho_0\mathbf{v}_0)}\leq 2\beta_0\|\rho_0\mathcal{B}\mathbf{u}\|_{L^2(\Omega_R)}, \tag{4.5a}$$

$$\|\psi_{\mathcal{C}}\|_{H(\Omega_R,\rho_0\mathbf{v}_0)}\leq \frac{{\beta_0}^2}{\omega}\|\rho_0\mathcal{C}\mathbf{u}\|_{L^2(\Omega_R)} \;\text{ and }\; \left\|\frac{D\psi_{\mathcal{C}}}{Dt}\right\|_{H(\Omega_R,\rho_0\mathbf{v}_0)}\leq \beta_0\|\rho_0\mathcal{C}\mathbf{u}\|_{L^2(\Omega_R)}. \tag{4.5b}$$

Then $\psi_{\mathcal{B}}$ and $\psi_{\mathcal{C}}$ can be extended in Q_L^+ by solving (4.3):

$$\psi_{\mathcal{B}}(x_1,x_2)=\psi_{\mathcal{B}}(R,x_2)e^{i\frac{\omega}{\alpha v_\infty}(x_1-R)}, \qquad R<x_1<R+L, \tag{4.6a}$$

$$\psi_{\mathcal{C}}(x_1,x_2)=\Big(\psi_{\mathcal{C}}(R,x_2)+\frac{x_1-R}{\alpha v_\infty}\frac{D\psi_{\mathcal{C}}}{Dt}(R,x_2)\Big)e^{i\frac{\omega}{\alpha v_\infty}(x_1-R)}, \qquad R<x_1<R+L. \tag{4.6b}$$

Combining (3.5b), (4.5) and (4.6) and setting $\gamma_L=\omega L/v_\infty$, we finally get (after some calculations using the rough estimate $|e^{i\frac{\omega}{\alpha v_\infty}(x_1-R)}|\leq 1$):

$$\|\rho_0\psi_{\mathcal{B}}\|_{L^2(\Omega_L)}\leq\frac{2\beta_0}{\omega}\sqrt{1+\frac{\rho_\infty}{\rho_0^{\inf}}\gamma_L}\|\rho_0\mathcal{B}\mathbf{u}\|_{L^2(\Omega_R)}, \tag{4.7a}$$

$$\|\rho_0\psi_{\mathcal{C}}\|_{L^2(\Omega_L)}\leq\frac{\beta_0}{\omega^2}\sqrt{\beta_0^2+2\frac{\rho_\infty}{\rho_0^{\inf}}\gamma_L\Big(\beta_0^2+\frac{\gamma_L^2}{3|\alpha|^2}\Big)}\|\rho_0\mathcal{C}\mathbf{u}\|_{L^2(\Omega_R)}. \tag{4.7b}$$

4.2 Coercivity condition

The main result that will be proved now is the well-posedness of problem (2.12). We denote by V the functional space for the fields $\mathbf{u}$:

$$V=\big\{\mathbf{u}\in H^1(\Omega_L)^2;\,\mathbf{u}\cdot\mathbf{n}=0 \text{ on } \partial\Omega_L\big\}.$$

Let us introduce now the operator $\mathcal{T}$, defined from V into $L^2(\Omega_L)$, such that $\mathcal{T}\mathbf{u}=\psi_{\mathcal{B}}+\psi_{\mathcal{C}}$. Then the solution ψ of (4.2) is given by

$$\psi=\mathcal{T}\mathbf{u}+\psi_{\mathbf{f}}. \tag{4.8}$$

By (4.7), $\mathcal{T}$ is a bounded operator satisfying:

$$\big\|\rho_0^{\frac{1}{2}}c_0\mathcal{T}\mathbf{u}\big\|_{L^2(\Omega_L)}\leq\frac{\beta_0}{\omega}\Big(K_1\|\mathcal{B}\|+\frac{K_2\beta_0+K_3}{\omega}\|\mathcal{C}\|\Big)\big\|\rho_0^{\frac{1}{2}}c_0\nabla\mathbf{u}\big\|_{L^2(\Omega_R)}, \tag{4.9}$$

where K_i are dimensionless constants depending only on γ_L and ρ_0, and where the norms of operators $\mathcal{B}$ and $\mathcal{C}$ are defined by (the weight $\rho_0^{1/2}c_0$ will be well-suited in what follows):

$$\|\mathcal{B}\| = \sup_{\mathbf{u}\in V,\mathbf{u}\neq 0} \frac{\|\rho_0^{\frac{1}{2}}c_0\mathcal{B}\mathbf{u}\|_{L^2(\Omega_R)}}{\|\rho_0^{\frac{1}{2}}c_0\nabla\mathbf{u}\|_{L^2(\Omega_R)}} \quad \text{and} \quad \|\mathcal{C}\| = \sup_{\mathbf{u}\in V,\mathbf{u}\neq 0} \frac{\|\rho_0^{\frac{1}{2}}c_0\mathcal{C}\mathbf{u}\|_{L^2(\Omega_R)}}{\|\rho_0^{\frac{1}{2}}c_0\nabla\mathbf{u}\|_{L^2(\Omega_R)}}.$$

Now using (4.8), we can eliminate ψ in problem (2.12) which is then rewritten as follows: Find $\mathbf{u}\in V$ such that:

$$\begin{aligned}
&\rho_0\frac{D_\alpha^2\mathbf{u}}{Dt^2} - \nabla_\alpha(\rho_0c_0^2\mathrm{div}_\alpha\mathbf{u}) + \mathbf{curl}_\alpha\big(\rho_0c_0^2(\mathrm{curl}_\alpha\mathbf{u} - \mathcal{T}\mathbf{u})\big)\\
&\qquad +\mathrm{div}_\alpha\mathbf{u}\nabla p_0 - {}^t\nabla_\alpha\mathbf{u}\cdot\nabla p_0 = \mathbf{f} + \mathbf{curl}_\alpha(\rho_0c_0^2\psi_\mathbf{f}), \qquad \text{in } \Omega_L, &&(4.10a)\\
&\mathrm{curl}_\alpha\mathbf{u} - \mathcal{T}\mathbf{u} = \psi_\mathbf{f}, \qquad \text{on } \partial\Omega_L. &&(4.10b)
\end{aligned}$$

This leads to the following variational form: find $\mathbf{u}\in V$, such that

$$\forall\mathbf{v}\in V, \quad a(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{v}) = \ell(v), \tag{4.11}$$

where

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) &= \int_{\Omega_L}\frac{\rho_0}{\alpha_1\alpha_2}\big(c_0^2\mathrm{div}_\alpha\mathbf{u}\,\mathrm{div}_\alpha\bar{\mathbf{v}} + c_0^2\mathrm{curl}_\alpha\mathbf{u}\,\mathrm{curl}_\alpha\bar{\mathbf{v}} - (\mathbf{v}_0\cdot\nabla_\alpha)\mathbf{u}(\mathbf{v}_0\cdot\nabla_\alpha)\bar{\mathbf{v}}\big) - \int_{\Omega_L}\frac{\rho_0c_0^2}{\alpha_1}\mathcal{T}\mathbf{u}\,\mathrm{curl}_\alpha\bar{\mathbf{v}},\\
b(\mathbf{u},\mathbf{v}) &= \int_{\Omega_L}\frac{-\rho_0\omega}{\alpha_1\alpha_2}(2i(\mathbf{v}_0\cdot\nabla_\alpha)\mathbf{u} + \omega\mathbf{u})\cdot\bar{\mathbf{v}} + \int_{\Omega_R}\big(\mathrm{div}\mathbf{u}\nabla p_0 - {}^t\nabla\mathbf{u}\cdot\nabla p_0\big)\bar{\mathbf{v}},\\
\ell(v) &= \int_{\Omega_R}\mathbf{f}\cdot\bar{\mathbf{v}} + \int_{\Omega_L}\frac{\rho_0c_0^2}{\alpha_1}\psi_\mathbf{f}\mathrm{curl}_\alpha\bar{\mathbf{v}}.
\end{aligned}$$

Here for the sake of simplicity we have omitted negligible terms (on the boundary at $x_2 = R+L$) in the definitions of $b(u,v)$ and $l(v)$ using the decreasing behavior of ψ in the PMLs (see (4.6)).

Theorem 4.1. *Suppose* $\mathbf{v}_0$ *satisfies condition* (3.12). *Problem* (4.11) *is of Fredholm type if*

$$\inf_{x\in\Omega_R}\Big(1 - \frac{|\mathbf{v}_0|^2}{c_0^2}\Big) > \frac{\sqrt{2}}{K_\alpha}\frac{\beta_0}{\omega}\Big(K_1\|\mathcal{B}\| + \frac{K_2\beta_0 + K_3}{\omega}\|\mathcal{C}\|\Big), \tag{4.12}$$

where the constant K_i *are given in* (4.9) *and* $K_\alpha = \min(1,|\alpha|)\min\big(\Re e\alpha, \Re e\frac{1}{\alpha}\big)$.

Proof. Following the proof of Theorem 1 of [3], we will prove that, under hypothesis (4.12), the bilinear form $a(\mathbf{u},\mathbf{v})$ has a coercive+compact decomposition on V. This proves the theorem since $b(\mathbf{u},\mathbf{v})$ is clearly compact (i.e., associated to a compact operator on V).

It is established in [3] that $\forall\mathbf{u},\mathbf{v}\in V$:

$$\int_{\Omega_L}\frac{\rho_0c_0^2}{\alpha_1\alpha_2}\big(\mathrm{div}_\alpha\mathbf{u}\,\mathrm{div}_\alpha\bar{\mathbf{v}} + \mathrm{curl}_\alpha\mathbf{u}\,\mathrm{curl}_\alpha\bar{\mathbf{v}}\big) = \int_{\Omega_L}\frac{\rho_0c_0^2}{\alpha_1\alpha_2}\nabla_\alpha\mathbf{u}\cdot\nabla_\alpha\bar{\mathbf{v}} + d(\mathbf{u},\mathbf{v})$$

with

$$d(\mathbf{u},\mathbf{v}) = \int_{\Omega_L} \Big(\frac{\partial(\rho_0 c_0^2)}{\partial x_1}\frac{\partial \mathbf{u}}{\partial x_2} - \frac{\partial(\rho_0 c_0^2)}{\partial x_2}\frac{\partial \mathbf{u}}{\partial x_1} \Big) \times \bar{\mathbf{v}} - \int_{\partial \mathcal{O}} \rho_0 c_0^2 \big[(\mathbf{u}\cdot\nabla)\mathbf{n}\times\mathbf{n} \big] (\mathbf{n}\times\bar{\mathbf{v}}),$$

so that $d(\mathbf{u},\mathbf{v})$ is compact. The theorem then requires the existence of a constant $\delta > 0$ such that $\forall \mathbf{u} \in V$:

$$\Big| \int_{\Omega_L} \frac{\rho_0 c_0^2}{\alpha_1\alpha_2} \nabla_\alpha \mathbf{u}\cdot\nabla_\alpha \bar{\mathbf{u}} - \frac{\rho_0}{\alpha_1\alpha_2} (\mathbf{v}_0\cdot\nabla_\alpha)\mathbf{u}\cdot(\mathbf{v}_0\cdot\nabla_\alpha)\bar{\mathbf{u}} - \frac{\rho_0 c_0^2}{\alpha_1} \mathcal{T}\mathbf{u}\,\mathrm{curl}_\alpha \bar{\mathbf{u}} \Big| \geq \delta \int_{\Omega_L} \rho_0 c_0^2 |\nabla \mathbf{u}|^2.$$

The existence of $\delta > 0$ is obtained under hypothesis (4.12) by using (4.9) and the following inequality:

$$\Big| \int_{\Omega_L} \frac{\rho_0}{\alpha_1\alpha_2} \big(c_0^2 \nabla_\alpha \mathbf{u}\cdot\nabla_\alpha\bar{\mathbf{u}} - (\mathbf{v}_0\cdot\nabla_\alpha)\mathbf{u}\cdot(\mathbf{v}_0\cdot\nabla_\alpha)\bar{\mathbf{u}} \big) \Big| \geq \min\Big(\Re e\alpha, \Re e\frac{1}{\alpha} \Big) \int_{\Omega_L} \rho_0 (c_0^2 - |\mathbf{v}_0|^2) |\nabla \mathbf{u}|^2.$$

Thus, the theorem is proved. □

Remark 4.1. 1. Coerciveness is obtained for small values of $\|\mathcal{B}\|$ and $\|\mathcal{C}\|$, that is for a slowly varying flow.

2. The right member of (4.12) diverges as $\omega \to 0$. This is due to the dependence versus ω of the norm (3.3) which is not appropriate at low frequency. The divergence can be easily removed by replacing ω in the definition (3.3) by some arbitrary value $\omega_0 > \omega$.

3. Estimates (4.7) can be improved by taking into account the decreasing behavior in the PMLs, leading to constants K_i, $i = 1,2,3$, depending only on ρ_0, α and v_∞, and therefore independent of ω. As β_0/ω is a decreasing function of ω, we see then that condition (4.12) is easier to satisfy when ω increases.

5 Numerical solution of the coupled problem

5.1 The numerical scheme

Numerical results for the coupled problem can be obtained by combining the Finite Element solution of (2.7) and the Discontinuous Galerkin solution of (2.8).

Let $\mathcal{M}_h$ be a triangulation [6] of the computational domain Ω_L. The construction of the approximation is based on the following approximate spaces:

$$V_h^k := \{ \mathbf{v}_h \in V;\ \forall T \in \mathcal{M}_h,\ \mathbf{v}_{h|T} \in (\mathcal{P}^k(T))^2 \}, \tag{5.1a}$$

$$W_h^k := \{ \varphi_h \in L^2(\Omega_L);\ \forall T \in \mathcal{M}_h,\ \varphi_{h|T} \in \mathcal{P}^k(T) \}, \tag{5.1b}$$

where $k \in \mathbb{N}^*$ and $\mathcal{P}^k(T)$ is the space of polynomial functions of total degree at most k. Moreover, for each $T \in \mathcal{M}_h$, we denote by $\mathbf{n}_T$ the outward unit normal to ∂T and $\mathcal{F}_T^-$ is the subset of ∂T where $\mathbf{v}_0\cdot\mathbf{n}_T < 0$. Finally, $\mathcal{F}_T^{-,i}$ is the subset of $\mathcal{F}_T^-$ corresponding to the interior faces i.e., $\forall F \in \mathcal{F}_T^{-,i}$, $F \cap \Gamma_L^- = \emptyset$.

The approximate formulation is then written as follows: find $\mathbf{u}_h \in V_h^k$ such that $\forall \mathbf{v}_h \in V_h^k$,

$$a_h(\mathbf{u}_h,\mathbf{v}_h)+b(\mathbf{u}_h,\mathbf{v}_h)=\ell_h(\mathbf{v}_h), \tag{5.2}$$

where a_h and ℓ_h are defined as the bilinear and linear forms a and ℓ (see subsection 4.2) by replacing $\mathcal{T}$ by an approximate operator $\mathcal{T}_h$ and $\psi_{\mathbf{f}}$ by an approximation $\psi_{\mathbf{f},h}$.

The discrete operator $\mathcal{T}_h$ and function $\psi_{\mathbf{f},h}$ are then defined by the successive solution of the following first order problem:

$$\begin{cases} \dfrac{D_\alpha \psi}{Dt}=g, & \text{in } \Omega_L, \\ \psi=0, & \text{on } \Gamma_L^-, \end{cases} \tag{5.3}$$

by using a classical discontinuous Galerkin method [7] well-adapted to take into account transport phenomena: find $\psi_h \in W_h^k$ such that $\forall \phi_h \in W_h^k$ and $\forall T \in \mathcal{M}_h$,

$$a_{DG,T}(\psi_h,\phi_h)=\ell_{T,g}(\phi_h), \tag{5.4}$$

where

$$a_{DG,T}(\psi_h,\phi_h)=-i\omega\int_T \frac{1}{\alpha_1\alpha_2}\psi_h\bar{\phi}_h dx+\int_T \frac{1}{\alpha_1\alpha_2}\mathbf{v}_0\cdot\nabla_\alpha\psi_h\bar{\phi}_h dx+\sum_{E=T\cap\Gamma_L^-}\int_E |\mathbf{v}_0\cdot\mathbf{n}_{T,\alpha}|\psi_{h|T}\bar{\phi}_h d\sigma$$
$$-\sum_{E=T\cap T'}\frac{1}{2}\int_{E\cap\mathcal{F}_T^{-,i}}\big((\mathbf{v}_{0|T'}\cdot\mathbf{n}_{T',\alpha})\psi_{h|T'}+(\mathbf{v}_{0|T}\cdot\mathbf{n}_{T,\alpha})\psi_{h|T}\big)\bar{\phi}_h d\sigma,$$

with

$$\mathbf{n}_{T,\alpha}=\frac{1}{\alpha_{1|T}\alpha_{2|T}}\left(\alpha_{1|T}n_T^1,\alpha_{2|T}n_T^2\right)^T \quad \text{and} \quad \ell_{T,g}(\phi_h)=\int_T \frac{1}{\alpha_1\alpha_2}g\bar{\phi}_h dx.$$

Now, we define the operator $\mathcal{D}_h$ from $L^2(\Omega_L)$ into W_h^k by $\mathcal{D}_h g=\psi_h$ where ψ_h is the solution of (5.4). The operator $\mathcal{T}_h$ is then constructed in the following way:

$$\mathcal{T}_h=\mathcal{D}_h(-2\mathcal{B}\mathbf{u}_h)+\mathcal{D}_h\circ\mathcal{D}_h(-\mathcal{C}\mathbf{u}_h). \tag{5.5}$$

Finally, $\psi_{\mathbf{f},h}$ is defined by $\mathcal{D}_h\circ\mathcal{D}_h(\frac{\mathrm{curl}\mathbf{f}}{\rho_0})$.

5.2 Numerical results

We present here a numerical result where the depth function h introduced in subsection 2.1 is defined by:

$$h(x_1)\left(x_1^2+(h(x_1)+b)^2-a^2\right)=a^2b$$

and the mean flow is a potential incompressible flow (with a constant density $\rho_0=\rho_\infty$ and a constant velocity $c_0=c_\infty$) given by:

$$\mathbf{v}_0(x_1,x_2)=v_\infty\begin{pmatrix} 1+\dfrac{a^2}{x_1^2+(x_2+b)^2}-\dfrac{2a^2x_1^2}{(x_1^2+(x_2+b)^2)^2} \\ \dfrac{-2a^2x_1(x_2+b)}{(x_1^2+(x_2+b)^2)^2} \end{pmatrix} \quad \text{and} \quad \nabla p_0=-\frac{1}{2}\rho_\infty\nabla(|\mathbf{v}_0|^2),$$

where a and b are strictly positive constants such that $a<R$. This is the potential flow around a cylinder of center $(0,-b)$ and radius a, and the definition of h is such that Γ_∞ is a stream line of the flow located strictly above the cylinder. As a consequence, $\mathbf{v}\cdot\mathbf{e}_1$ does not vanish and condition (3.12) is fulfilled.

Notice that the hypotheses of uniformity of the geometry ($h(x_1)=0$ for $|x_1|>r$) and of the flow ($\mathbf{v}_0(x)=v_\infty$ for $x_1^2+x_2^2>R^2$) are not satisfied for finite values of r and R, but the variations of h and $\mathbf{v}_0$ will be supposed negligible far enough. Also this incompressible flow does not satisfy the state law (2.3), except asymptotically for $c_0\to+\infty$, but we take advantage of its analytical expression (similar expressions for compressible potential flows do not exist). Here we take:

$$a=0.5,\quad b=0.1,\quad v_\infty=0.4c_0,\quad R=3,\quad L=1,\quad \alpha=0.65(1-i).$$

The frequency is such that $\omega/c_0=4\pi/3$ and we consider a source term $\mathbf{f}$ of the following form:

$$\mathbf{f}=\mu\nabla\varphi+\beta\,\mathbf{curl}\,\varphi,\quad \text{where}\;\; \varphi(x_1,x_2)=\exp\Big\{-\frac{(x_1-x_1^s)^2+(x_2-x_2^s)^2}{r_s^2}\Big\}.$$

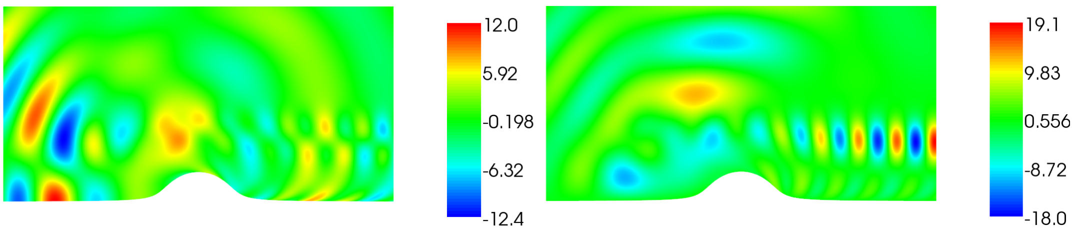

Figure 3: Real part of u_1 and u_2.

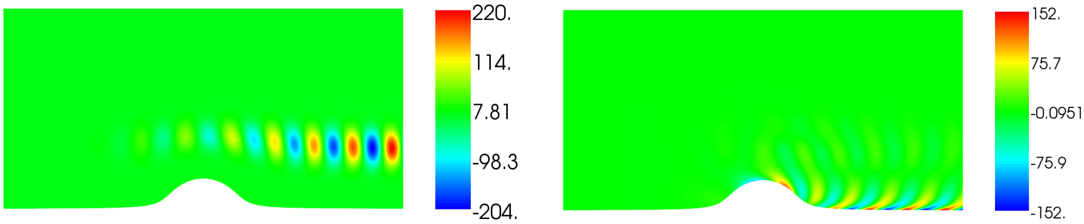

Figure 4: Real part of ψ_f and $\mathcal{T}u$.

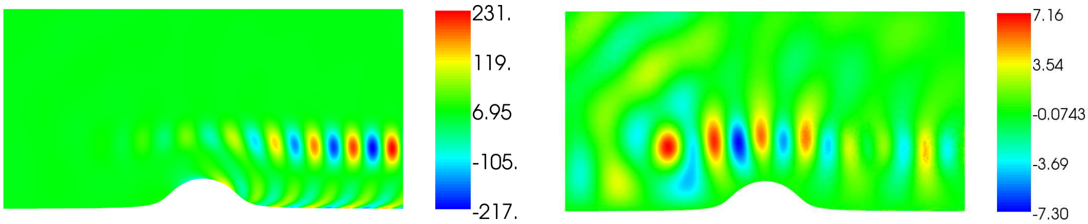

Figure 5: Real part of $\mathrm{curl}\,\mathbf{u}$ and of $\mathrm{curl}\,\mathbf{u}-\psi$.

Here we take:

$$\mu=100, \quad \beta=3, \quad x_1^s=-1.5, \quad x_2^s=1.05, \quad r^s=0.3.$$

Moreover, we have neglected the term ψ_C in first approximation to evaluate the proposed method.

Fig. 3 shows the isovalues of the real part of both components of $\mathbf{u}$ in the domain Ω_R. One can observe two kinds of structures, corresponding to acoustic and hydrodynamic phenomena. The acoustic wave is particularly visible upstream of the source and partially hidden downstream by the vortices. These ones are mainly produced by the source and convected along the stream lines of the flow, but we can also notice some vortices generated by the perturbed part of the mean flow (where $\mathcal{B}\mathbf{u}$ takes significant values) and convected along the rigid boundary. This interpretation is confirmed by the representation of ψ_f and $\mathcal{T}u$ in Fig. 4.

Let us recall that we have solved the augmented equation (2.7) instead of (2.4). Equivalence is achieved if $\mathrm{curl}\,\mathbf{u}=\psi$, which can be checked a posteriori as illustrated by Fig. 5 (note the different scales).

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