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Composite Coherent States Approximation for One-Dimensional Multi-Phased Wave Functions

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Abstract. The coherent states approximation for one-dimensional multi-phased wave functions is considered in this paper. The wave functions are assumed to oscillate on a characteristic wave length $\mathcal{O}(\epsilon)$ with $\epsilon \ll 1$. A parameter recovery algorithm is first developed for single-phased wave function based on a moment asymptotic analysis. This algorithm is then extended to multi-phased wave functions. If cross points or caustics exist, the coherent states approximation algorithm based on the parameter recovery will fail in some local regions. In this case, we resort to the windowed Fourier transform technique, and propose a composite coherent states approximation method. Numerical experiments show that the number of coherent states derived by the proposed method is much less than that by the direct windowed Fourier transform technique.

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1 Introduction

This paper aims at an efficient coherent states approximation method for wave functions oscillating at most on the $\mathcal{O}(\epsilon)$ scale with $\epsilon \ll 1$. By *coherent state* we mean a function of the following form

$$\tilde{A}\exp\left(\frac{i}{\epsilon}\left(p(x-q)+\frac{\gamma}{2}(x-q)^2\right)\right), \tag{1.1}$$

where $q\in\mathbb{R}$ is termed *spatial center*, $p\in\mathbb{R}$ *momentum*, $\gamma\in\mathbb{C}$ *spread*, and $\tilde{A}\in\mathbb{C}$ *amplitude*. The imaginary part of γ should be positive, which renders to the coherent state function a Gaussian profile centered at point q. A *coherent states approximation* (CSA) is a set of

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coherent states parameterized by $\{q_j,p_j,\gamma_j,\tilde{A}_j\}$, whose summation approximates a given family of wave functions $u_\epsilon(x)$ by an asymptotic error $\mathcal{O}(\mathscr{E})$ with $\lim_{\epsilon\to 0}\mathscr{E}=0$, i.e.,

$$u_\epsilon(x)=\sum_j \tilde{A}_j \exp\left(\frac{i}{\epsilon}\left(p_j(x-q_j)+\frac{\gamma_j}{2}(x-q_j)^2\right)\right)+\mathcal{O}(\mathscr{E}). \tag{1.2}$$

The CSA problem exists in various disciplines, such as quantum chemistry [6–8, 10], geophysics [4, 5, 9, 14], and signal processing [2, 11, 13]. For example, in geophysics, if one wants to perform seismic migration with the Gaussian beam approach, the first issue faced by the practitioner is to decompose the acquired seismic signal into a set of coherent state functions. The approximating accuracy is very important for an accurate and reliable exploration. On the other hand, however, the number of derived coherent states should not be too large so that the migration algorithm can be implemented within the limited computing power. This dilemma situation also appears in quantum mechanics whenever a semi-classical approximation for the propagator based on Gaussian coherent states is employed to evolve the quantum wave field.

It is well known that the following set of coherent states

$$\varphi_{pq}(x)=\frac{1}{(\pi\epsilon a)^{1/4}(2\pi\epsilon)^{\frac{1}{2}}}\exp\left(\frac{i}{\epsilon}\left(p(x-q)+\frac{i}{2a}(x-q)^2\right)\right),$$

parameterized in the phase space $\mathbb{R}_p\times\mathbb{R}_q$, form a tight frame in $L^2(\mathbb{R})$ for any $a>0$ (see Appendix C). This means that for any $f(x)\in L^2(\mathbb{R})$ the following holds

$$f(x)=\iint_{\mathbb{R}_p\times\mathbb{R}_q}(f,\varphi_{pq})\varphi_{pq}(x)dpdq. \tag{1.3}$$

Here $(\cdot,\cdot)$ indicates the standard $L^2(\mathbb{R})$ inner product. Since

$$(f,f)=\iint_{\mathbb{R}_p\times\mathbb{R}_q}|(f,\varphi_{pq})|^2dpdq,$$

the "coordinates" (f,φ_{pq}) can be taken as the energy spectra of f in the phase space. A discrete CSA is thus derived by applying a suitable numerical quadrature on (1.3). For example, using the trapezoidal rule gives

$$f(x)\approx\Delta p\Delta q\sum_{j,k\in\mathbb{Z}}(f,\varphi_{p_jq_k})\varphi_{p_jq_k}(x), \tag{1.4}$$

where $p_j=j\Delta p$ and $q_k=k\Delta q$, with Δp and Δq being the momentum and spatial stepsizes respectively.

There exists an alternative way based on the dual frame technique to arrive at a CSA as (1.4) [2, 16]. Let a and b be two positive constants on the scale of $\mathcal{O}(1)$. Set $q_j=j\sqrt{\epsilon a}/b$ and

$$h_j(x)=h_j(a,b,x)=\frac{1}{\sqrt{2\pi b^2}}\exp\left(-\frac{(x-q_j)^2}{2\epsilon a}\right). \tag{1.5}$$

Under a mild condition on b (see Appendix A), $\{h_j(x)\}_{j\in\mathbb{Z}}$ forms an approximate partition of unity of $\mathbb{R}$. Thus for any $f(x)\in L^2(\mathbb{R})$ we have

$$f(x)\approx\sum_{j\in\mathbb{Z}}f(x)h_j(x)=\sum_{j\in\mathbb{Z}}f(x)h_j^{\frac{1}{2}}(x)h_j^{\frac{1}{2}}(x).$$

The *windowed* function $f(x)h_j^{\frac{1}{2}}(x)$ is largely supported in an interval I_j of size $L=\mathcal{O}(\sqrt{\epsilon})$ and centered at $x=q_j$. One can expand it into the Fourier series form as

$$f(x)h_j^{\frac{1}{2}}(x)\approx\frac{1}{L}\sum_{k\in\mathbb{Z}}C_{jk}\exp\left(\frac{2i\pi k(x-q_j)}{L}\right),$$

where

$$C_{jk}=\int_{I_j}f(x)h_j^{\frac{1}{2}}(x)\exp\left(-\frac{2i\pi k(x-q_j)}{L}\right)dx.$$

Thus then, it holds that

$$f(x)\approx\frac{1}{L}\sum_{j,k\in\mathbb{Z}}C_{jk}\exp\left(\frac{2i\pi k(x-q_j)}{L}\right)h_j^{\frac{1}{2}}(x). \tag{1.6}$$

Considering $\exp\big(2i\pi k(x-q_j)/L\big)h_j^{\frac{1}{2}}(x)$ expresses a coherent state with momentum $2\pi k\epsilon/L$ and centered at q_j, a CSA is then derived by removing those terms with negligible amplitudes in the double summation of (1.6). In [15], Qian and Ying developed a fast algorithm for computing the coefficients C_{jk}.

The approximating accuracy of (1.6) strongly relies on the characteristic wave length λ of the wave function $f(x)$. If $f(x)$ oscillates on the $\mathcal{O}(\epsilon)$ scale, the formula (1.6) presents a CSA with arbitrary accuracy provided a, b and L are set appropriately. In case the data is supported in an interval with an $\mathcal{O}(1)$ length, the number of coherent states is on the scale of $\mathcal{O}(\epsilon^{-1})$. In general this estimate cannot be improved. From the function approximation point of view, this treatment is not superior to the standard grid-resolving method.

As a matter of fact, the advantage of the CSA (1.6) only shows up when the energy spectra of the oscillating function has an appropriate *sparse structure*. To make clear this point, let us consider the wave function u_ϵ with an analytical expression of the WKB form, i.e.,

$$u_\epsilon(x)=A_\epsilon(x)\exp\left(\frac{iS_\epsilon(x)}{\epsilon}\right).$$

The energy bandwidth of the windowed function $u_\epsilon(x)h_j^{\frac{1}{2}}(x)$ is on the scale of $\mathcal{O}(\sqrt{\epsilon})$. Thus for each point q_j, there are at most $\mathcal{O}(\sqrt{\epsilon})/(\frac{2\pi\epsilon}{L})=\mathcal{O}(1)$ important coefficients C_{jk}. Remember that $L=\mathcal{O}(\sqrt{\epsilon})$. If $u_\epsilon(x)$ is supported in an interval of length $\mathcal{O}(1)$, the total number of coherent states is on the scale of $\mathcal{O}(\epsilon^{-\frac{1}{2}})$. The CSA thus presents a very economical approximation to the highly oscillating function u_ϵ.

Even though, the above CSA based on the windowed Fourier transform (WFT) is not the most efficient, especially when the approximating accuracy is not too demanded. Since each windowed function $u_\epsilon(x)h_j(x)$ is largely supported in an interval of length $\mathcal{O}(\sqrt{\epsilon})$, one could expand the amplitude and phase functions of $u_\epsilon(x)$ as

$$A_\epsilon(x)=A_\epsilon(q_j)+\mathcal{O}(\sqrt{\epsilon}),$$
$$S_\epsilon(x)=S_\epsilon(q_j)+S'_\epsilon(q_j)(x-q_j)+\frac{S''_\epsilon(q_j)}{2}(x-q_j)^2+\mathcal{O}(\epsilon^{\frac{3}{2}}).$$

By setting

$$\tilde{A}_j=\frac{1}{\sqrt{2\pi b^2}}A_\epsilon(q_j)\exp\left(\frac{iS_\epsilon(q_j)}{\epsilon}\right), \tag{1.7a}$$
$$p_j=S'_\epsilon(q_j), \qquad \gamma_j=S''_\epsilon(q_j)+\frac{i}{a}, \tag{1.7b}$$

the coherent state function

$$\tilde{A}_j\exp\left(\frac{i}{\epsilon}\left(p_j(x-q_j)+\frac{\gamma_j}{2}(x-q_j)^2\right)\right)$$

is a half-order approximation of the windowed function $u_\epsilon(x)h_j(x)$. Since this approximation is purely local, the beam summation remains at least the same accuracy, which implies $\mathscr{E}=\epsilon^{\frac{1}{2}}$ in (1.2).

The above analysis is essentially due to Tanushev [17]. However, as most numerical evidences suggested, the half-order accuracy is not optimal. In [12], Motamed and Runborg made a more sophisticated estimate. It was shown that under some mild conditions, the approximating error of (1.2) is on the $\mathcal{O}(\epsilon)$ scale due to the remarkable *error cancelation effect* between the neighboring coherent states. A brief analysis is also given in this paper.

Compared with the CSA based on WFT, the advantage of the above CSA for single-phased wave functions is such that *only one coherent state is associated with a specific spatial point q_j*. As a matter of fact, our numerical experiments demonstrate that the number of coherent states differs significantly for these two methods. This benefit motivates us greatly to find an efficient and reliable way to recover the phase information on the approximated wave function, provided an a priori information is known that the energy spectra of the given wave function has a multi-phased structure in the phase space.

The rest of this paper is organized as follows. In Section 2, a CSA version based on the WFT technique is proposed and the error analysis is performed. In Section 3, an error analysis on the CSA determined by (1.7) is made for a single-phased wave function. In Section 4, we propose a phase parameter recovery algorithm for the corrupted single-phased wave functions. Based on it, a composite coherent states approximation is proposed in Section 5. Numerical experiments are reported in Section 6, and this paper concludes in Section 7.

2 CSA based on the windowed Fourier transform

Throughout this paper, we set $I=[-L/2,L/2]$ and $I_j=[q_j-L/2,q_j+L/2]$ with $q_j=j\sqrt{\epsilon a}/b$. Here a, b and L are three prescribed positive constants. We indicate $\chi_j(x)$ the indicator function of I_j and set $\chi(x)=\chi_0(x)$. Besides, we put

$$\mathfrak{S}_0(a,b,x)=\sum_{j\in\mathbb{Z}}h_j(a,b,x).$$

See the definition of h_j in (1.5). For any $f(x)\in L^2(\mathbb{R})$, we define its *energy* as $\|f\|^2$, where $\|\cdot\|$ denotes the L^2 norm.

By Lemma A.2 it holds that

$$\begin{aligned}\sum_{j\in\mathbb{Z}}h_j^{\frac{1}{2}}(x)=\sum_{j\in\mathbb{Z}}h_j^{\frac{1}{2}}(a,b,x)&=(2\pi b^2)^{-\frac{1}{4}}\sum_{j\in\mathbb{Z}}\exp\left(-\frac{(x-j\sqrt{\epsilon a}/b)^2}{4\epsilon a}\right)\\&=\sqrt{2}(2\pi b^2)^{\frac{1}{4}}\mathfrak{S}_0(2a,\sqrt{2}b,x).\end{aligned}$$

If b is set appropriately, $\mathfrak{S}_0(2a,\sqrt{2}b,x)$ equals almost 1 by (A.2). The above expression implies that $\{h_j^{\frac{1}{2}}\}_{j\in\mathbb{Z}}$ forms an approximate partition of unity of $\mathbb{R}$ (up to a factor), just as $\{h_j\}_{j\in\mathbb{Z}}$ does. Furthermore, we have

$$\sum_{j\in\mathbb{Z}}h_j^{\frac{1}{2}}(x)\le\sqrt{2}(2\pi b^2)^{\frac{1}{4}}(1+4\exp(-4\pi^2b^2))\equiv C_4(b). \tag{2.1}$$

Lemma 2.1. *Let $f(x)\in L^2(\mathbb{R})$. Then*

$$[1-C_0(b)]||f||^2\le\sum_{j\in\mathbb{Z}}||fh_j^{\frac{1}{2}}||^2\le[1+C_0(b)]||f||^2.$$

Proof. A direct computation shows

$$\sum_{j\in\mathbb{Z}}||fh_j^{\frac{1}{2}}||^2=\sum_{j\in\mathbb{Z}}(fh_j^{\frac{1}{2}},fh_j^{\frac{1}{2}})=\sum_{j\in\mathbb{Z}}(fh_j,f)=(f\mathfrak{S}_0,f).$$

By Lemma A.2 we have $|\mathfrak{S}_0-1|\le C_0(b)$. This ends the proof. □

Remark 2.1. Lemma 2.1 implies that the *windowed functions* $\{fh_j^{\frac{1}{2}}\}_{j\in\mathbb{Z}}$ presents an approximate energy partition for the function f.

Lemma 2.2. *Suppose $g_j(x)\in L^2_{loc}(\mathbb{R})$. It holds that*

$$\Big\|\sum_{j\in\mathbb{Z}} g_j h_j^{\frac{1}{2}}\Big\|^2 \le [1+C_0(b)]\sum_{j\in\mathbb{Z}}||g_j||^2, \tag{2.2}$$

$$\Big\|\sum_{j\in\mathbb{Z}} g_j h_j^{\frac{1}{2}} - \sum_{j\in\mathbb{Z}} g_j\chi_j h_j^{\frac{1}{2}}\Big\|^2 \le C_5(a,b,L)\sum_{j\in\mathbb{Z}}||g_j||^2, \tag{2.3}$$

where

$$C_5(a,b,L)=h_0^{\frac{1}{2}}(L/2)C_4(b).$$

In case that $g_j(x)$ is L-periodic, then

$$\Big\|\sum_{j\in\mathbb{Z}} g_j h_j^{\frac{1}{2}} - \sum_{j\in\mathbb{Z}} g_j\chi_j h_j^{\frac{1}{2}}\Big\|^2 \le C_6(a,b,L)\sum_{j\in\mathbb{Z}}||g_j\chi_j||^2, \tag{2.4}$$

where

$$C_6(a,b,L)=2C_4(b)\sum_{k\ge 1} h_0^{\frac{1}{2}}(kL-L/2).$$

Proof. By Lemma A.2 we have

$$\Big|\sum_{j\in\mathbb{Z}} g_j(x)h_j^{\frac{1}{2}}(x)\Big|^2 \le \sum_{j\in\mathbb{Z}}|g_j(x)|^2\sum_{j\in\mathbb{Z}}h_j(x)\le[1+C_0(b)]\sum_{j\in\mathbb{Z}}|g_j(x)|^2.$$

(2.2) thus follows. Since

$$\begin{aligned}&\Big|\sum_{j\in\mathbb{Z}} g_j(x)h_j^{\frac{1}{2}}(x)-\sum_{j\in\mathbb{Z}} g_j(x)\chi_j(x)h_j^{\frac{1}{2}}(x)\Big|^2=\Big|\sum_{j\in\mathbb{Z}} g_j(x)(1-\chi_j(x))h_j^{\frac{1}{2}}(x)\Big|^2\\ \le&\sum_{j\in\mathbb{Z}}|g_j(x)|^2(1-\chi_j(x))h_j^{\frac{1}{2}}(x)\sum_{j\in\mathbb{Z}}h_j^{\frac{1}{2}}(x)\le\sum_{j\in\mathbb{Z}}|g_j(x)|^2(1-\chi_j(x))h_j^{\frac{1}{2}}(x)C_4(b),\end{aligned}$$

we have

$$\Big|\sum_{j\in\mathbb{Z}} g_j(x)h_j^{\frac{1}{2}}(x)-\sum_{j\in\mathbb{Z}} g_j(x)\chi_j(x)h_j^{\frac{1}{2}}(x)\Big|^2\le h_0^{\frac{1}{2}}(L/2)C_4(b)\sum_{j\in\mathbb{Z}}|g_j(x)|^2.$$

This proves (2.3). In case that g_j is L-periodic, a direct computation shows that

$$\begin{aligned}&\int_{\mathbb{R}}|g_j(x)|^2(1-\chi_j(x))h_j^{\frac{1}{2}}(x)dx\\ =&\int_{\mathbb{R}}|g_j(x+q_j)|^2(1-\chi_0(x))h_0^{\frac{1}{2}}(x)dx=\sum_{k\ne 0}\int_{kL-\frac{L}{2}}^{kL+\frac{L}{2}}|g_j(x+q_j)|^2h_0^{\frac{1}{2}}(x)dx\\ =&\sum_{k\ne 0}\int_{-\frac{L}{2}}^{\frac{L}{2}}|g_j(x+q_j)|^2h_0^{\frac{1}{2}}(x+kL)dx\le 2\sum_{k\ge 1}h_0^{\frac{1}{2}}(kL-L/2)||g_j\chi_j||^2.\end{aligned}$$

Thus then,

$$
\begin{aligned}
\Big\|\sum_{j\in\mathbb{Z}} g_j h_j^{\frac{1}{2}} - \sum_{j\in\mathbb{Z}} g_j\chi_j h_j^{\frac{1}{2}}\Big\|^2 &\le C_4(b)\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}} |g_j(x)|^2(1-\chi_j(x))h_j^{\frac{1}{2}}(x)dx \\
&\le 2C_4(b)\sum_{k\ge 1} h_0^{\frac{1}{2}}(kL-L/2)\sum_{j\in\mathbb{Z}} ||g_j\chi_j||^2.
\end{aligned}
$$

This ends the proof. ◻

Remark 2.2. If the parameters a, b and L are set appropriately, the constants C_5 and C_6 can be made very small. Lemma 2.2 thus implies that when summing up a sequence of windowed functions $g_j h_j^{\frac{1}{2}}$ together, the exponentially decaying tail of Gaussian functions $h_j^{\frac{1}{2}}$ has a negligible influence on the result.

Given a wave function $f(x)\in L^2(\mathbb{R})$. If we expand each cut-off windowed function $f\chi_j h_j^{\frac{1}{2}}$ into the Fourier series form, i.e.,

$$
f(x)\chi_j(x)h_j^{\frac{1}{2}}(x) = \frac{1}{L}\sum_{k\in\mathbb{Z}} C_{jk}\exp\left(\frac{2i\pi k(x-q_j)}{L}\right),
$$

then

$$
\|f\chi_j h_j^{\frac{1}{2}}\|^2 = \frac{1}{L}\sum_{k\in\mathbb{Z}}|C_{jk}|^2.
$$

Since

$$
\sum_{j\in\mathbb{Z}}\|f\chi_j h_j^{\frac{1}{2}}\|_2^2 = \sum_{j\in\mathbb{Z}}(f\chi_j h_j, f) = \|f\|_2^2 + \Big(\sum_{j\in\mathbb{Z}} f(\chi_j-1)h_j, f\Big) + (f(\mathfrak{S}_0-1), f),
$$

by Cauchy-Schwartz inequality and Lemma A.2, Lemma 2.2 and Lemma 2.1, we have

$$
\left|\frac{1}{L}\sum_{j,k}|C_{jk}|^2 - \|f\|^2\right| \le \left[C_0(b) + \sqrt{C_5(a,b,L)}\sqrt{1+C_0(b)}\right]\|f\|^2.
$$

This implies that if C_0 and C_5 are sufficiently small, then $|C_{jk}|^2/L$ depicts the energy distribution of f in the *discrete phase space* $\{(j\sqrt{\epsilon a}/b, 2\pi\epsilon k/L) : j,k\in\mathbb{Z}\}$.

Now it is ready to present the CSA method based on the WFT technique, which will be referred to as WFT-CSA in the later.

WFT-CSA A. Given a wave function $f(x)\in L^2(\mathbb{R})$ and an error tolerance $\mathscr{E}$. Set the parameters a, b and L appropriately, and set $q_j = j\sqrt{\epsilon a}/b$.

WFT-CSA B. For each spatial point q_j, compute

$$m_j = ||f\chi_j h_j^{\frac{1}{2}}||^2.$$

Sort $\{m_j\}_{j\in\mathbb{Z}}$, and find a subset of $\{m_j\}_{j\in\mathbb{Z}}$ with least number from large to small, so that the summation of these quantities is larger than $(1-\mathscr{E})\|f\|^2$. Denote by $\mathbb{I}$ the set of associated indices.

WFT-CSA C. If $|\mathbb{I}|=0$, i.e., the number of elements in $\mathbb{I}$ equals 0, then the algorithm terminates. Otherwise, for each $j\in\mathbb{I}$, compute the Fourier coefficients

$$C_{jk} = \int_{\mathbb{R}} f(x)\chi_j(x)h_j^{\frac{1}{2}}(x)\exp\left(-\frac{2i\pi k(x-q_j)}{L}\right)dx,\ k\in\mathbb{Z}.$$

WFT-CSA D. Set $m_{jk}=|C_{jk}|^2/L$. Sort $\{m_{jk}\}_{j\in\mathbb{I},k\in\mathbb{Z}}$, and find a subset of $\{m_{jk}\}_{j\in\mathbb{I},k\in\mathbb{Z}}$ with least number from large to small, so that the summation of these quantities is larger than $(1-\mathscr{E})||f||^2$. Denote by $\mathbb{J}$ the set of associated double indices.

WFT-CSA E. Each (j,k) with $j\in\mathbb{I}$ corresponds to a coherent state of the form

$$\mathrm{CS}_{jk}(x) = \frac{C_{jk}}{L}\exp\left(\frac{2i\pi k(x-q_j)}{L}\right)h_j^{\frac{1}{2}}(x). \tag{2.5}$$

A CSA is formed by setting

$$f_{\mathrm{CS}}(x) = \sum_{(j,k)\in\mathbb{J}} \mathrm{CS}_{jk}(x).$$

Remark 2.3. The basic idea of WFT-CSA is as follows:

- decompose the wave function f into a sequence of windowed functions fh_j;
- approximate $fh_j^{\frac{1}{2}}$ with a set of harmonics. Since $fh_j = fh_j^{\frac{1}{2}}h_j^{\frac{1}{2}}$, the windowed function fh_j is then approximated by a set of coherent states as (2.5);
- the overall wave function is approximated by summing up these coherent states together.

Theorem 2.1. *Given an error tolerance $\mathscr{E}$. Suppose a CSA is given by the above WFT-CSA method. Set a, b and L such that*

$$\exp(-\pi^2 b^2) \le \mathscr{E}, \qquad (2\pi b^2)^{-\frac{1}{4}}\exp\left(-\frac{L^2}{16\epsilon a}\right) \le \mathscr{E}. \tag{2.6}$$

Then there exists a constant c, independent of $\mathscr{E}$ and ϵ, such that

$$||f - f_{\mathrm{CS}}|| \le c\sqrt{\mathscr{E}}\|f\|.$$

Proof. Let us introduce the following functions

$$f_1(x)=\sum_{j\in\mathbb{Z}} f(x)\chi_j(x)h_j(x), \qquad f_2(x)=\sum_{j\in\mathbb{I}} f(x)\chi_j(x)h_j(x),$$
$$f_3(x)=\sum_{(j,k)\in\mathbb{J}} \chi_j(x)\mathrm{CS}_{jk}(x).$$

Obviously, f_1 is an approximation of f by the localized approximate partition of unity, f_2 removes those terms in f_1 with small energies, and f_3 is the approximation of f_{CS} by truncation. We intend to show the differences among all these terms are $\mathcal{O}(\sqrt{\mathscr{E}})$ under the assumptions on the parameters a, b and L.

By Lemma A.2, Lemma 2.1 and Lemma 2.2, it holds

$$\begin{aligned}
||f-f_1|| &\le \Big\| f-\sum_{j\in\mathbb{Z}} fh_j\Big\| + \Big\|\sum_{j\in\mathbb{Z}} f(1-\chi_j)h_j\Big\| \\
&\le \Big\| f-\sum_{j\in\mathbb{Z}} fh_j\Big\| + \Big\|\sum_{j\in\mathbb{Z}} fh_j^{\frac{1}{2}}(1-\chi_j)h_j^{\frac{1}{2}}\Big\| \\
&\le \Big(C_0(b)+\sqrt{C_5(a,b,L)(1+C_0(b))}\Big)||f||.
\end{aligned} \tag{2.7}$$

By Lemma A.2, Lemma 2.2 and WFT-CSA B, we have

$$\begin{aligned}
||f_1-f_2|| &= \Big\|\sum_{j\notin\mathbb{I}} f\chi_j h_j\Big\| = \Big\|\sum_{j\notin\mathbb{I}} f(x)\chi_j h_j^{\frac{1}{2}}(x)h_j^{\frac{1}{2}}(x)\Big\| \\
&\le \sqrt{1+C_0(b)}\sqrt{\sum_{j\notin\mathbb{I}} m_j} = \sqrt{1+C_0(b)}\sqrt{\sum_{j\in\mathbb{Z}} m_j-\sum_{j\in\mathbb{I}} m_j} \\
&\le \sqrt{1+C_0(b)}\sqrt{\sum_{j\in\mathbb{Z}} ||fh_j^{\frac{1}{2}}||_2^2-(1-\mathscr{E})||f||^2} \\
&\le \sqrt{1+C_0(b)}\sqrt{C_0+\mathscr{E}}||f||.
\end{aligned} \tag{2.8}$$

Since

$$f_2(x)=\frac{1}{L}\sum_{j\in\mathbb{I}}\sum_{k\in\mathbb{Z}} C_{jk}\exp\left(\frac{2i\pi k(x-q_j)}{L}\right)\chi_j(x)h_j^{\frac{1}{2}}(x),$$

we have

$$f_2(x)-f_3(x)=\frac{1}{L}\sum_{j\in\mathbb{I},(j,k)\notin\mathbb{J}} C_{jk}\exp\left(\frac{2i\pi k(x-q_j)}{L}\right)\chi_j(x)h_j^{\frac{1}{2}}(x).$$

By Lemma 2.2 and WFT-CSA D, we have

$$\begin{aligned}||f_2(x)-f_3(x)||^2 &\le (1+C_0(b))\sum_{j\in\mathbb{I},(j,k)\notin\mathbb{J}}\frac{|C_{jk}|^2}{L}=(1+C_0(b))\sum_{j\in\mathbb{I},(j,k)\notin\mathbb{J}}m_{jk}\\&=(1+C_0(b))\left(\sum_{j\in\mathbb{I}}\sum_{k\in\mathbb{Z}}m_{jk}-\sum_{(j,k)\in\mathbb{J}}m_{jk}\right)\\&\le(1+C_0(b))\left(\sum_{j\in\mathbb{Z}}||f(x)h_j^{\frac{1}{2}}(x)||^2-(1-\mathscr{E})||f||^2\right)\\&\le(1+C_0(b))(C_0(b)+\mathscr{E})||f||^2.\end{aligned}\tag{2.9}$$

Finally, by Lemma 2.2 we have

$$\begin{aligned}||f_3(x)-f_{\rm CS}(x)||^2&\le C_6(a,b,L)\sum_{(j,k)\in\mathbb{J}}m_{jk}\le C_6(a,b,L)\sum_{j\in\mathbb{I}}m_j\\&\le C_6(a,b,L)\left(\sum_{j\in\mathbb{Z}}||f(x)h_j^{\frac{1}{2}}(x)||^2\right)\\&\le C_6(a,b,L)[1+C_0(b)]||f||^2.\end{aligned}\tag{2.10}$$

Combining (2.7)-(2.10) together gives

$$\begin{aligned}||f(x)-f_{\rm CS}(x)||\le C_0(b)||f||+\left[\sqrt{C_5(a,b,L)}+\sqrt{C_6(a,b,L)}\right]\sqrt{1+C_0(b)}||f||\\+2\sqrt{1+C_0(b)}\sqrt{C_0(b)+\mathscr{E}}||f||.\end{aligned}$$

This finishes the proof by applying the condition (2.6). □

Remark 2.4. In our computations, we set

$$a=1,\quad b=1.4,\quad L=6\sqrt{4\epsilon a}.$$

In this case, we have

$$C_0<6.31\times10^{-17},\quad C_5<4.37\times10^{-16},\quad C_6<8.74\times10^{-16}.$$

Thus at least for $\mathscr{E}\ge10^{-14}$, the relative L^2 error is approximately bounded by $2\sqrt{\mathscr{E}}$. This implies that in principle the WFT-CSA method can give a CSA with *arbitrary accuracy* for *any given wave function*, if the parameters involved are set appropriately. This explains the universality property of WFT-CSA. However, the number of coherent states can be on the scale of $\mathcal{O}(\epsilon^{-1})$, and in general this bound cannot be further reduced.

3 Error analysis on a coherent states approximation

In this section, we make an error analysis on a specific CSA (see Theorem 3.1) for a single-phased wave function. The basic idea is essentially the same as that in [12]. It will be shown that this CSA has an $\mathcal{O}(\epsilon)$ error. We also explain that a small deviation of the phase parameters will not deteriorate the approximating accuracy. We need the following lemmas.

Lemma 3.1. *The following holds*

$$|\exp(i\theta)-1|\le|\theta|,\quad |\exp(i\theta)-1-i\theta|\le\frac{\theta^2}{\sqrt{3}},\quad \forall\theta\in\mathbb{R}.$$

Lemma 3.2. *For any nonnegative integer m, there exists a universal constant $c_m>0$ such that*

$$\sum_{j\in\mathbb{Z}}|x-q_j|^m h_j(x)\le c_m(\epsilon a)^{\frac{m}{2}},\ \ \forall x\in\mathbb{R}.$$

Proof. Since

$$\begin{aligned}|x-q_j|^m h_j^{\frac12}(x)&=|x-q_j|^m(2\pi b^2)^{-\frac14}\exp\left(-\frac{(x-q_j)^2)}{4\epsilon a}\right)\\&\le(2\pi b^2)^{-\frac14}\max_{x\in\mathbb{R}}\left[|x|^m\exp\left(-\frac{x^2}{4\epsilon a}\right)\right]\\&=(2\pi b^2)^{-\frac14}(\epsilon a)^{\frac{m}{2}}\max_{x\in\mathbb{R}}\left[|x|^m\exp\left(-x^2/4\right)\right],\end{aligned}$$

by (2.1) we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|x-q_j|^m h_j(x)=&\sum_{j\in\mathbb{Z}}|x-q_j|^m h_j^{\frac12}(x)h_j^{\frac12}(x)\\&\le(2\pi b^2)^{-\frac14}(\epsilon a)^{\frac{m}{2}}\max_{x\in\mathbb{R}}\left[|x|^m\exp\left(-x^2/4\right)\right]\sum_{j\in\mathbb{Z}}h_j^{\frac12}(x)\\&\le\sqrt{2}(\epsilon a)^{\frac{m}{2}}\max_{x\in\mathbb{R}}\left[|x|^m\exp\left(-x^2/4\right)\right]\left(1+4\exp(-4\pi^2b^2)\right).\end{aligned}$$

This ends the proof. □

Theorem 3.1. *Given a wave function of the form*

$$u_\epsilon(x)=A_\epsilon(x)\exp\left(\frac{iS_\epsilon(x)}{\epsilon}\right),$$

where $A_\epsilon(x)\in C^2(\mathbb{R})$ is the amplitude, and $S_\epsilon(x)\in C^4(\mathbb{R})$ is the real phase function. Set

$$u_{\epsilon,CS}(x)=\sum_{j\in\mathbb{Z}} A_\epsilon(q_j)\exp\left(\frac{i}{\epsilon}\left(S_\epsilon(q_j)+S'_\epsilon(q_j)(x-q_j)+\frac{S''_\epsilon(q_j)}{2}(x-q_j)^2\right)\right)h_j(a,b,x).$$

Suppose A_ϵ is uniformly bounded in $C_b^2(\mathbb{R})$ and $S'''_\epsilon(x)$ is uniformly bounded in $C_b^1(\mathbb{R})$, i.e., there exists a constant $M>0$ independent of ϵ such that

$$||A_\epsilon||_\infty+||A'_\epsilon||_\infty+||A''_\epsilon||_\infty+||S'''_\epsilon||_\infty+||S''''_\epsilon||_\infty\le M,\ \ \forall\epsilon>0.$$

Then there exists a constant c depending only on a and M, such that for sufficiently small ϵ, by setting b with $\exp\left(-2\pi^2b^2\right)\le\epsilon$, the following holds

$$|u_\epsilon(x)-u_{\epsilon,CS}(x)|\le c\epsilon,\ \ \forall x\in\mathbb{R}.$$

Remark 3.1. If we perform local Taylor expansion for the amplitude A_ϵ and the phase $S_\epsilon(x)$, and set

$$\begin{aligned}\hat{u}_{\epsilon,j}(x)=&\left[A_\epsilon(q_j)+A'_\epsilon(q_j)(x-q_j)\right]\\&\times\exp\left(\frac{i}{\epsilon}\left(S_\epsilon(q_j)+S'_\epsilon(q_j)(x-q_j)+\frac{S''_\epsilon(q_j)}{2}(x-q_j)^2+\frac{S'''_\epsilon(q_j)}{6}(x-q_j)^3\right)\right),\end{aligned}$$

it is easy to prove that $\hat{u}_{\epsilon,j}h_j^{\frac{1}{2}}$ is a first order approximation of $u_\epsilon h_j^{\frac{1}{2}}$. Thus by Lemma 2.2 and (A.2), we have

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\hat{u}_{\epsilon,j}(x)h_j(x)-u_\epsilon(x)\\=&\sum_{j\in\mathbb{Z}}\hat{u}_{\epsilon,j}(x)h_j(x)-\sum_{j\in\mathbb{Z}}u_\epsilon(x)h_j(x)+\sum_{j\in\mathbb{Z}}u_\epsilon(x)h_j(x)-u_\epsilon(x)\\=&\sum_{j\in\mathbb{Z}}(\hat{u}_{\epsilon,j}(x)h_j^{\frac{1}{2}}(x)-u_\epsilon(x)h_j^{\frac{1}{2}}(x))h_j^{\frac{1}{2}}(x)+u_\epsilon(x)\left(\sum_{j\in\mathbb{Z}}h_j(x)-1\right)=\mathcal{O}(\epsilon).\end{aligned}$$

Thus the point for Theorem 3.1 is to prove that the effects of the linear term of A_ϵ and the cubic term of S_ϵ are actually on the level of $\mathcal{O}(\epsilon)$, though seemingly they are on the level of $\mathcal{O}(\sqrt{\epsilon})$. This is exactly the spirit of *error cancelation effect*.

Proof. For brevity of notations, we introduce

$$\begin{aligned}S_j^{(2)}(x)&=S_\epsilon(q_j)+S'_\epsilon(q_j)(x-q_j)+\frac{S''_\epsilon(q_j)}{2}(x-q_j)^2,\\S_j^{(3)}(x)&=S_\epsilon(q_j)+S'_\epsilon(q_j)(x-q_j)+\frac{S''_\epsilon(q_j)}{2}(x-q_j)^2+\frac{S'''_\epsilon(x)}{6}(x-q_j)^3.\end{aligned}$$

It is easy to verify that

$$|S_j^{(2)}(x)-S_\epsilon(x)|\le\frac{||S_\epsilon'''||_\infty}{6}|x-q_j|^3, \tag{3.1}$$

$$|S_j^{(3)}(x)-S_\epsilon(x)|\le\frac{5||S_\epsilon''''||_\infty}{24}|x-q_j|^4, \tag{3.2}$$

$$|A_\epsilon(q_j)+A_\epsilon'(x)(x-q_j)-A_\epsilon(x)|\le\frac{3||A_\epsilon''||_\infty}{2}|x-q_j|^2. \tag{3.3}$$

Let us introduce the following functions

$$u_{\epsilon,1}(x)=\sum_{j\in\mathbb{Z}}A_\epsilon(x)\exp\Big(iS_\epsilon(x)/\epsilon\Big)h_j(x), \tag{3.4}$$

$$u_{\epsilon,2}(x)=\sum_{j\in\mathbb{Z}}A_\epsilon(x)\exp\Big(iS_j^{(3)}(x)/\epsilon\Big)h_j(x), \tag{3.5}$$

$$u_{\epsilon,3}(x)=\sum_{j\in\mathbb{Z}}A_\epsilon(x)\exp\Big(iS_j^{(2)}(x)/\epsilon\Big)\left(1+\frac{iS_\epsilon'''(x)}{6\epsilon}(x-q_j)^3\right)h_j(x), \tag{3.6}$$

$$u_{\epsilon,31}(x)=\sum_{j\in\mathbb{Z}}A_\epsilon(x)\exp\Big(iS_j^{(2)}(x)/\epsilon\Big)h_j(x), \tag{3.7}$$

$$u_{\epsilon,32}(x)=\sum_{j\in\mathbb{Z}}A_\epsilon(x)\exp\Big(iS_j^{(2)}(x)/\epsilon\Big)\frac{iS_\epsilon'''(x)}{6\epsilon}(x-q_j)^3h_j(x), \tag{3.8}$$

$$u_{\epsilon,41}(x)=\sum_{j\in\mathbb{Z}}(A_\epsilon(q_j)+A_\epsilon'(x)(x-q_j))\exp\Big(iS_j^{(2)}(x)/\epsilon\Big)h_j(x), \tag{3.9}$$

$$u_{\epsilon,42}(x)=\sum_{j\in\mathbb{Z}}A_\epsilon(x)\exp\Big(iS_\epsilon(x)/\epsilon\Big)\frac{iS_\epsilon'''(x)}{6\epsilon}(x-q_j)^3h_j(x), \tag{3.10}$$

$$u_{\epsilon,51}(x)=\sum_{j\in\mathbb{Z}}A_\epsilon'(x)(x-q_j)\exp\Big(iS_j^{(2)}(x)/\epsilon\Big)h_j(x), \tag{3.11}$$

$$u_{\epsilon,61}(x)=\sum_{j\in\mathbb{Z}}A_\epsilon'(x)(x-q_j)\exp\Big(iS_\epsilon(x)/\epsilon\Big)h_j(x). \tag{3.12}$$

Obviously, we have

$$u_{\epsilon,3}(x)=u_{\epsilon,31}(x)+u_{\epsilon,32}(x),\quad u_{\epsilon,41}(x)=u_{\epsilon,\mathrm{CS}}(x)+u_{\epsilon,51}(x).$$

In addition, we have the following estimates

$$|u_{\epsilon,1}(x)-u_\epsilon(x)|\le||A_\epsilon||_\infty C_0(b), \tag{3.13}$$

$$|u_{\epsilon,2}(x)-u_{\epsilon,1}(x)|\le\frac{5c_4\epsilon a^2||S_\epsilon''''||_\infty||A_\epsilon||_\infty}{24}, \tag{3.14}$$

$$|u_{\epsilon,3}(x)-u_{\epsilon,2}(x)|\le\frac{c_6\epsilon a^3||S_\epsilon'''||_\infty^2||A_\epsilon||_\infty}{36\sqrt{3}}, \tag{3.15}$$

$$|u_{\epsilon,31}(x)-u_{\epsilon,41}(x)|\le\frac{3c_2\epsilon a||A''_\epsilon||_\infty}{2},\tag{3.16}$$

$$|u_{\epsilon,32}(x)-u_{\epsilon,42}(x)|\le\frac{c_6\epsilon a^3||S'''_\epsilon||^2_\infty||A_\epsilon||_\infty}{36},\tag{3.17}$$

$$|u_{\epsilon,51}(x)-u_{\epsilon,61}(x)|\le\frac{c_4\epsilon a^2||S'''_\epsilon||_\infty||A'_\epsilon||_\infty}{6},\tag{3.18}$$

$$|u_{\epsilon,42}(x)|\le\frac{||S'''_\epsilon||_\infty||A_\epsilon||_\infty}{6\epsilon}C_3(b)(\epsilon a)^{\frac{3}{2}},\tag{3.19}$$

$$|u_{\epsilon,61}(x)|\le||A'_\epsilon||_\infty C_1(b)\sqrt{\epsilon a},\tag{3.20}$$

where C_0, C_1 and C_3 are defined as in Lemma A.2. The estimates (3.13), (3.19) and (3.20) are obvious. Let us prove (3.14). Since

$$\begin{aligned}|u_{\epsilon,2}(x)-u_{\epsilon,1}(x)|&=\left|\sum_{j\in\mathbb{Z}}A_\epsilon(x)\left[\exp\left(\frac{iS_j^{(3)}(x)}{\epsilon}\right)-\exp\left(\frac{iS_\epsilon(x)}{\epsilon}\right)\right]h_j(x)\right|\\&\le||A_\epsilon||_\infty\sum_{j\in\mathbb{Z}}\left|\exp\left(\frac{i(S_j^{(3)}(x)-S_\epsilon(x))}{\epsilon}\right)-1\right|h_j(x),\end{aligned}$$

by Lemma 3.1 and formula (3.2), we have

$$|u_{\epsilon,2}(x)-u_{\epsilon,1}(x)|\le\frac{5||S''''_\epsilon||_\infty||A_\epsilon||_\infty}{24\epsilon}\sum_{j\in\mathbb{Z}}|x-q_j|^4h_j(x).$$

Applying Lemma 3.2 then yields (3.14). The proof of (3.15)-(3.18) is analogous.

If $\exp(-2\pi^2b^2)\le\epsilon$, then by Lemma A.2 we have

$$C_0(b)\le4\epsilon,\quad C_1(b)\le c\sqrt{\epsilon},\quad C_3(b)\le c\sqrt{\epsilon}.\tag{3.21}$$

Since

$$\begin{aligned}|u_\epsilon(x)-u_{\epsilon,CS}(x)|\le&|u_\epsilon(x)-u_{\epsilon,1}(x)|+|u_{\epsilon,1}(x)-u_{\epsilon,2}(x)|+|u_{\epsilon,2}(x)-u_{\epsilon,3}(x)|\\&+|u_{\epsilon,31}(x)-u_{\epsilon,41}(x)|+|u_{\epsilon,51}(x)-u_{\epsilon,61}(x)|+|u_{\epsilon,61}(x)|\\&+|u_{\epsilon,32}(x)-u_{\epsilon,42}(x)|+|u_{\epsilon,42}(x)|,\end{aligned}$$

using (3.13)-(3.20) and (3.21) finishes the proof. □

Remark 3.2. The above proof also reveals that the coherent state *width controlling parameter a* has a first order influence on the approximating accuracy of the coherent states summation. This is because the error is dominated by (3.14)-(3.18), considering $C_0(b)$, $C_1(b)$ and $C_3(b)$ decays extremely fast with respect to the parameter b.

Lemma 3.3. *Suppose there exist two positive numbers M and c such that*

$$|\tilde{A}_{1,j}-\tilde{A}_{2,j}|\le c\epsilon,\quad |p_{1,j}-p_{2,j}|\le c\epsilon^{\frac{3}{2}},\quad |\gamma_{1,j}-\gamma_{2,j}|\le c\epsilon,\quad |\tilde{A}_{1,j}|+|\tilde{A}_{2,j}|\le M,\quad \forall j\in\mathbb{Z},$$

where $p_{l,j}$ and $\gamma_{l,j}$ ($l=1,2$) are real. Set

$$\begin{aligned} S_{l,j}(x)&=p_{l,j}(x-q_j)+\frac{\gamma_{l,j}}{2}(x-q_j)^2,\\ f_l(x)&=\sum_{j\in\mathbb{Z}}\tilde{A}_{l,j}\exp\left(\frac{iS_{l,j}(x)}{\epsilon}\right)h_j(a,b,x). \end{aligned}$$

Then there exists a constant c^* depending only on a and M, such that for sufficiently small ϵ, the following holds

$$|f_1(x)-f_2(x)|\le c^*\epsilon,\ \ \forall x\in\mathbb{R}.$$

Proof. To estimate the difference of f_1 and f_2, we separate f_1-f_2 into two parts, namely,

$$\begin{aligned} f_1(x)-f_2(x)=&\sum_{j\in\mathbb{Z}}(\tilde{A}_{1,j}-\tilde{A}_{2,j})\exp\left(\frac{iS_{1,j}(x)}{\epsilon}\right)h_j(x)\\ &+\sum_{j\in\mathbb{Z}}\tilde{A}_{2,j}\left(\exp\left(\frac{iS_{1,j}(x)}{\epsilon}\right)-\exp\left(\frac{iS_{2,j}(x)}{\epsilon}\right)\right)h_j(x). \end{aligned}$$

Denote these two parts by I_1 and I_2 respectively. Then using Lemma A.2 gives

$$|I_1|\le c\epsilon\sum_{j\in\mathbb{Z}}h_j(x)\le c\epsilon(1+C_0)\le 5c\epsilon. \tag{3.22}$$

Since

$$\left|S_{1,j}(x)-S_{2,j}(x)\right|\le c\left(\epsilon^{\frac{3}{2}}|x-q_j|+\frac{\epsilon|x-q_j|^2}{2}\right),$$

by Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned} |I_2|&=\left|\sum_{j\in\mathbb{Z}}\tilde{A}_{2,j}\left(\exp\left(\frac{iS_{1,j}(x)}{\epsilon}\right)-\exp\left(\frac{iS_{2,j}(x)}{\epsilon}\right)\right)h_j(x)\right|\\ &\le M\sum_{j\in\mathbb{Z}}\left|\exp\left(\frac{i(S_{1,j}(x)-S_{2,j}(x))}{\epsilon}\right)-1\right|h_j(x)\\ &\le cM\sum_{j\in\mathbb{Z}}\left(\epsilon^{\frac{1}{2}}|x-q_j|+\frac{1}{2}|x-q_j|^2\right)h_j(x)\le cM\epsilon\left(c_1\sqrt{a}+\frac{c_2a}{2}\right). \end{aligned}$$

Thus then,

$$|f_1(x)-f_2(x)|\le|I_1|+|I_2|\le 5c\epsilon+cM\epsilon\left(c_1\sqrt{a}+\frac{c_2a}{2}\right).$$

The proof thus finishes. □

Combining Theorem 3.1 and Lemma 3.3 we get the following result.

Theorem 3.2. *Given the conditions as those in Theorem* 3.1*. Suppose there exist three sequences* $\{\tilde{A}_j\}_{j\in\mathbb{Z}}$*,* $\{p_j\}_{j\in\mathbb{Z}}$ *and* $\{\gamma_j\}_{j\in\mathbb{Z}}$ *(*p_j *and* γ_j *are real), and a constant* c *such that*

$$\left|\tilde{A}_j - A_\epsilon(q_j)\exp\left(\frac{iS_\epsilon(q_j)}{\epsilon}\right)\right| \le c\epsilon, \quad |p_j - S'_\epsilon(q_j)| \le c\epsilon^{\frac{3}{2}}, \quad |\gamma_j - S''_\epsilon(q_j)| \le c\epsilon. \tag{3.23}$$

Set

$$\tilde{u}_{\epsilon,CS}(x) = \sum_{j\in\mathbb{Z}} \tilde{A}_j \exp\left(\frac{i}{\epsilon}\left(p_j(x-q_j) + \frac{\gamma_j}{2}(x-q_j)^2\right)\right) h_j(x).$$

Then there exists a constant c^* *depending only on* a *and* M*, such that*

$$|u_\epsilon(x) - \tilde{u}_{\epsilon,CS}(x)| \le c^*\epsilon, \ \ \forall x \in \mathbb{R}.$$

Remark 3.3. Theorem 3.2 reveals that given a single-phased WKB wave function $u_\epsilon(x)$ WITHOUT knowing the analytical expressions of the amplitude A_ϵ and the phase S_ϵ, if one manages to derive a first order approximation of $A_\epsilon(q)\exp\left(\frac{iS_\epsilon(q)}{\epsilon}\right)$ and $S''_\epsilon(q)$, and a 1.5th order approximation of $S'_\epsilon(q)$ at any $q=q_j$, a CSA with the accuracy to $\mathcal{O}(\epsilon)$ can still be formed. In the sequel, we will term

$$\left\{A_\epsilon(q)\exp\left(\frac{iS_\epsilon(q)}{\epsilon}\right),\ S'_\epsilon(q),\ S''_\epsilon(q)\right\}$$

the *phase parameters* of $u_\epsilon(x)$.

4 Parameter recovery for single-phased data

This section explains how to recover the phase parameters associated with a single-phased corrupted wave function, which fulfills the accuracy requirements (3.23). The underlying idea is the foundation of the composite coherent states approximation method which will be proposed in the next section.

Suppose we have a corrupted single-phased wave function

$$u(x) = u_e(x) + \epsilon R_\epsilon(x), \quad u_e(x) = A_\epsilon(x)\exp\left(\frac{iS_\epsilon(x)}{\epsilon}\right),$$

where the noise function $R_\epsilon(x)$ satisfies the following assumption.

Assumption 4.1. There exists a *density function* $r_\epsilon(\xi) \in L^1(\mathbb{R})$ which is compactly supported in an interval $[-M,M]$ independent of ϵ, such that

$$||r_\epsilon(\xi)||_{L^1(\mathbb{R})} = \mathcal{O}(1), \quad R_\epsilon(x) = \int_{\mathbb{R}} \exp\left(\frac{i\xi x}{\epsilon}\right) r_\epsilon(\xi) d\xi.$$

This implies that $R_\epsilon \in C^\infty(\mathbb{R})$ and $R_\epsilon(x) = \mathcal{O}(1)$, *but* $R'_\epsilon(x) = \mathcal{O}(\epsilon^{-1})$.

For any $a>0$, we set

$$\phi_a(x)=\exp\left(-\frac{x^2}{2\epsilon a}\right).$$

In particular, we indicate $\phi(x)=\phi_1(x)$. It is straightforward to verify that

$$\mathcal{F}_\epsilon[\phi_a]=\sqrt{2\pi\epsilon a}\,\phi_{\frac{1}{a}},\quad \mathcal{F}_\epsilon^{-1}[\phi_a]=\sqrt{\frac{a}{2\pi\epsilon}}\,\phi_{\frac{1}{a}}.$$

For any sufficiently smooth function f we have

$$\frac{1}{2\pi\epsilon}\int_{\mathbb{R}}\mathcal{F}_\epsilon[f](\xi)d\xi=f(0),\quad \frac{1}{2\pi\epsilon}\int_{\mathbb{R}}\xi\mathcal{F}_\epsilon[f](\xi)d\xi=-i\epsilon f'(0).$$

Applying these to the windowed wave function $u\phi_{2a}$ gives

$$\frac{1}{2\pi\epsilon}\int_{\mathbb{R}}\mathcal{F}_\epsilon[u\phi_{2a}](\xi)d\xi=A_\epsilon(0)\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right)+\epsilon R_\epsilon(0), \tag{4.1}$$

$$\frac{1}{2\pi\epsilon}\int_{\mathbb{R}}\xi\mathcal{F}_\epsilon[u\phi_{2a}](\xi)d\xi=[S'_\epsilon(0)A_\epsilon(0)-i\epsilon A'_\epsilon(0)]\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right)-i\epsilon^2R'_\epsilon(0). \tag{4.2}$$

Set

$$g_0=\frac{1}{2\pi\epsilon}\int_{\mathbb{R}}\mathcal{F}_\epsilon[u\phi_{2a}](\xi)d\xi,\quad g_1=\frac{1}{2\pi\epsilon}\int_{\mathbb{R}}\xi\mathcal{F}_\epsilon[u\phi_{2a}](\xi)d\xi.$$

We then have

$$\tilde{A}\overset{def}{=}g_0=A_\epsilon(0)\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right)+\mathcal{O}(\epsilon), \tag{4.3}$$

$$p^*\overset{def}{=}\Re\left(\frac{g_1}{g_0}\right)=S'_\epsilon(0)+\mathcal{O}(\epsilon). \tag{4.4}$$

These formulae imply that $\tilde{A}$ and p^* are the first order approximations of the phase parameters $A_\epsilon(0)\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right)$ and $S'_\epsilon(0)$.

Next, let us seek an approximation $S''_\epsilon(0)$. For any smooth function f such that $f\phi_{2a}$ is of Schwartz type, it is straightforward to verify that

$$\begin{aligned}\frac{1}{2\pi\epsilon}(\mathcal{F}_\epsilon[f\phi_{2a}],\phi(\xi-p^*))&=\frac{1}{\sqrt{2\pi\epsilon}}\left(f\phi_{2a},\exp\left(\frac{ip^*x}{\epsilon}\right)\phi\right)=\frac{1}{\sqrt{2\pi\epsilon}}\left(f\phi_{2a}\phi,\exp\left(\frac{ip^*x}{\epsilon}\right)\right)\\&=\frac{1}{\sqrt{2\pi\epsilon}}\left(f\phi_c,\exp\left(\frac{ip^*x}{\epsilon}\right)\right)=\frac{1}{\sqrt{2\pi\epsilon}}\mathcal{F}_\epsilon[f\phi_c](p^*).\end{aligned}$$

Here $c=\frac{2a}{2a+1}$. If $f=\mathcal{O}(\epsilon)$, then

$$|\mathcal{F}_\epsilon[f\phi_c](p^*)|\le||f||_\infty\int_{\mathbb{R}}\phi_c(x)dx=\mathcal{O}(\epsilon^{\frac{3}{2}}),$$

which implies that

$$\frac{1}{2\pi\epsilon}(\mathcal{F}_\epsilon[f\phi_{2a}],\phi(\xi-p^*))=\mathcal{O}(\epsilon). \tag{4.5}$$

If f is given as

$$f(x)=\left(\tilde{A}+Bx+\frac{irx^3}{6\epsilon}\right)\exp\left(\frac{i}{\epsilon}\left(px+\frac{\gamma}{2}x^2\right)\right),$$

where p and γ are real, a direct computation shows that

$$\begin{aligned}&\frac{1}{2\pi\epsilon}(\mathcal{F}_\epsilon[f\phi_{2a}],\phi(\xi-p^*))\\ =&\left(\tilde{A}+\frac{Bci(p-p^*)}{1-ic\gamma}-\frac{c^2r(p-p^*)}{2(1-ic\gamma)^2}+\frac{c^3r(p-p^*)^3}{6\epsilon(1-ic\gamma)^3}\right)\sqrt{\frac{c}{1-ic\gamma}}\exp\left(-\frac{c(p-p^*)^2}{2\epsilon(1-ic\gamma)}\right).\end{aligned}$$

If $p-p^*=\mathcal{O}(\epsilon)$, then

$$\frac{1}{2\pi\epsilon}(\mathcal{F}_\epsilon[f\phi_{2a}],\phi(\xi-p^*))=\tilde{A}\sqrt{\frac{c}{1-ic\gamma}}+\mathcal{O}(\epsilon). \tag{4.6}$$

Now let us indicate

$$\hat{u}_e(x)=\left(A_\epsilon(0)+A'_\epsilon(0)x+\frac{iA_\epsilon(0)S'''_\epsilon(0)x^3}{6\epsilon}\right)\exp\left(\frac{i}{\epsilon}\left(S'_\epsilon(0)+S'_\epsilon(0)x+\frac{S''_\epsilon(0)}{2}x^2\right)\right),$$

and

$$\tilde{u}_e(x)=u_e(x)-\hat{u}_e(x).$$

$\hat{u}_e$ is a local approximation of u_e with the accuracy to $\mathcal{O}(\epsilon)$ by performing Taylor expansion for the amplitude function $A_\epsilon(x)$ and the phase function $S_\epsilon(x)$. Due to the modulation of a fast decaying Gaussian function, it is not hard to prove that

$$\tilde{u}_e(x)\phi_{4a}(x)=\mathcal{O}(\epsilon). \tag{4.7}$$

We omit the proof here. Thus then, by Assumption 4.1, (4.4) and (4.5)-(4.7) we have

$$\begin{aligned}&\frac{1}{2\pi\epsilon}(\mathcal{F}_\epsilon[u\phi_{2a}],\phi(\xi-p^*))\\ =&\frac{1}{2\pi\epsilon}(\mathcal{F}_\epsilon[\hat{u}\phi_{2a}],\phi(\xi-p^*))+\frac{1}{2\pi\epsilon}(\mathcal{F}_\epsilon[\tilde{u}\phi_{2a}],\phi(\xi-p^*))+\frac{1}{2\pi\epsilon}(\mathcal{F}_\epsilon[\epsilon R_\epsilon\phi_{2a}],\phi(\xi-p^*))\\ =&A_\epsilon(0)\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right)\sqrt{\frac{c}{1-icS''_\epsilon(0)}}+\frac{1}{2\pi\epsilon}(\mathcal{F}_\epsilon[\tilde{u}\phi_{4a}\phi_{4a}],\phi(\xi-p^*))+\mathcal{O}(\epsilon)\\ =&A_\epsilon(0)\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right)\sqrt{\frac{c}{1-icS''_\epsilon(0)}}+\mathcal{O}(\epsilon).\end{aligned}$$

By setting

$$g=\frac{1}{2\pi\epsilon}(\mathcal{F}_\epsilon[u\phi_{2a}],\phi(\xi-p^*)),$$

we have

$$\frac{g}{g_0} = \sqrt{\frac{c}{1 - icS''_\epsilon(0)}} + \mathcal{O}(\epsilon).$$

This implies that

$$\gamma \stackrel{def}{=} -\Im\left(\frac{g_0}{g}\right)^2 = S''_\epsilon(0) + \mathcal{O}(\epsilon). \tag{4.8}$$

So far, we have succeeded in finding first order approximations of the phase parameters $A_\epsilon(0)\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right)$, $S'_\epsilon(0)$ and $S''_\epsilon(0)$. The requirements for the first and third quantities have been met as in (3.23). We need to improve the approximating accuracy half-order higher for the second quantity. However, without some representation constraint specified, this mission is generally impossible. To illustrate this point, let us rewrite $u_\epsilon(x)$ into

$$u_\epsilon(x) = A_\epsilon(x)\exp(-i\arg(A_\epsilon(x)))\exp\left(\frac{i(S_\epsilon(x) + \epsilon\arg(A_\epsilon(x)))}{\epsilon}\right).$$

Here $\arg(A_\epsilon(x))$ is the angle function of $A_\epsilon(x)$. If $A_\epsilon(x)$ is smooth enough, so can $\arg(A_\epsilon(x))$ be made, and the new amplitude function $\tilde{A}_\epsilon(x) = A_\epsilon(x)\exp(-i\arg(A_\epsilon(x)))$ is thus real. We call this form of WKB data *canonical*. Note that $\tilde{S}''_\epsilon(x)$, the second order derivative of the new phase function $\tilde{S}_\epsilon(x) = S_\epsilon(x) + \epsilon\arg(A_\epsilon(x))$, approximates $S''_\epsilon(x)$ within the first order accuracy, which means that any first order approximation of $S''_\epsilon(x)$ is still a first order approximation of $\tilde{S}''_\epsilon(x)$. However, since $S'_\epsilon(x)$ and $\tilde{S}'_\epsilon(x)$ have an $\mathcal{O}(\epsilon)$ difference, compared with $\tilde{S}'_\epsilon(x)$, the accuracy of any approximation of $S'_\epsilon(x)$ cannot exceed first order. Considering this point, from now on, *we always assume the WKB data is expressed in the canonical form*.

Let us reexamine the approximation of $S'_\epsilon(0)$. A direct computation shows that

$$\frac{g_1}{g_0} = S'_\epsilon(0) - \frac{i\epsilon A'_\epsilon(0)}{A_\epsilon(0)} - \frac{i\epsilon^2 A_\epsilon(0)\left(R'_\epsilon(0) - \frac{iS'_\epsilon(0)R_\epsilon(0)}{\epsilon}\right) - i\epsilon^2 A'_\epsilon(0)R_\epsilon(0)}{A_\epsilon(0)\left[A_\epsilon(0)\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right) + \epsilon R_\epsilon(0)\right]}. \tag{4.9}$$

A remarkable fact is that if the data is locally noise-free, i.e., $R_\epsilon(0) = R'_\epsilon(0) = 0$, $\Re(g_1/g_0)$ would recover $S'(0)$ exactly since $A_\epsilon(x)$ is real by canonical. If the density function $r_\epsilon(\xi)$ of $R_\epsilon(x)$ (see Assumption 4.1) is supported in an interval of length $\mathcal{O}(\sqrt{\epsilon|\ln\epsilon|})$ centered at $\xi = S'_\epsilon(0)$, then we have

$$\left|R'_\epsilon(0) - \frac{iS'_\epsilon(0)R_\epsilon(0)}{\epsilon}\right| = \left|\int_{\mathbb{R}} \frac{i(\xi - S'_\epsilon(0))}{\epsilon} r_\epsilon(\xi)d\xi\right| \le \frac{\mathcal{O}(\sqrt{\epsilon|\ln\epsilon|})}{\epsilon}||r_\epsilon(\xi)||_{L^1(\mathbb{R})}. \tag{4.10}$$

Recalling (4.9) we have

$$\Re\left(\frac{g_1}{g_0}\right) = S'_\epsilon(0) + \mathcal{O}(\epsilon^{\frac{3}{2}}|\ln\epsilon|^{\frac{1}{2}}).$$

In the general case, we can still follow the same line by a simple filtering technique. In the Fourier space the function $\mathcal{F}_\epsilon[u_\epsilon\phi_{2a}](\xi)$ is essentially supported in an interval of length $\mathcal{O}(\sqrt{\epsilon})$ centered at $\xi = S'_\epsilon(0)$. For example, in the special case of $A(x)\equiv A$ and $S(x)=px+\frac{\gamma}{2}x^2$, we have

$$\mathcal{F}_\epsilon[u_e\phi_{2a}](\xi)=A\sqrt{\frac{4\pi\epsilon a}{1-2ia\gamma}}\exp\left(-\frac{a(p-\xi)^2}{\epsilon(1-2ia\gamma)}\right). \tag{4.11}$$

The profile of this function is Gaussian with the characteristic width $\sqrt{\frac{\epsilon(1+4a^2\gamma^2)}{2a}}$. By setting

$$I=p+\left[-\sqrt{\frac{3\epsilon|\ln\epsilon|(1+4a^2\gamma^2)}{2a}},\sqrt{\frac{3\epsilon|\ln\epsilon|(1+4a^2\gamma^2)}{2a}}\right],$$

a direct computation shows that

$$\frac{1}{2\pi\epsilon}\int_{\mathbb{R}\backslash I}\mathcal{F}_\epsilon[u_e\phi_{2a}](\xi)d\xi=\mathcal{O}(\epsilon^{\frac{3}{2}}),\quad \frac{1}{2\pi\epsilon}\int_{\mathbb{R}\backslash I}\xi\mathcal{F}_\epsilon[u_e\phi_{2a}](\xi)d\xi=\mathcal{O}(\epsilon^{\frac{3}{2}}).$$

In the general case, we expect the above formulae also hold by resetting

$$I=p^*+\left[-\sqrt{\frac{3\epsilon|\ln\epsilon|(1+4a^2\gamma^2)}{2a}},\sqrt{\frac{3\epsilon|\ln\epsilon|(1+4a^2\gamma^2)}{2a}}\right],$$

where p^* and γ are the first order approximation of $S'_\epsilon(0)$ and $S''_\epsilon(0)$ determined by (4.4) and (4.8) (the proof is open). Combining these with (4.1)-(4.2) in case that $R_\epsilon=0$ gives

$$\begin{aligned}
&\frac{1}{2\pi\epsilon}\int_I\mathcal{F}_\epsilon[u_e\phi_{2a}](\xi)d\xi=A_\epsilon(0)\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right)+\mathcal{O}(\epsilon^{\frac{3}{2}}),\\
&\frac{1}{2\pi\epsilon}\int_I\xi\mathcal{F}_\epsilon[u_e\phi_{2a}](\xi)d\xi=[S'_\epsilon(0)A_\epsilon(0)-i\epsilon A'_\epsilon(0)]\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right)+\mathcal{O}(\epsilon^{\frac{3}{2}}).
\end{aligned}$$

On the other hand, by setting

$$\tilde{r}_\epsilon(\xi)=\frac{1}{2\pi\epsilon}\mathcal{F}_\epsilon[R_\epsilon\phi_{2a}](\xi)\chi_I(\xi),\quad \tilde{R}_\epsilon(x)=\int_{\mathbb{R}}\exp\left(\frac{i\xi x}{\epsilon}\right)\tilde{r}_\epsilon(\xi)d\xi,$$

where $\chi_I(\xi)$ is the indicator function of I, we have

$$||\tilde{r}_\epsilon||_{L^1(\mathbb{R})}\le||r_\epsilon||_{L^1(\mathbb{R})}\left\|\frac{1}{2\pi\epsilon}\mathcal{F}_\epsilon[\phi_{2a}]\right\|_{L^1(\mathbb{R})}=||r_\epsilon||_{L^1(\mathbb{R})}=\mathcal{O}(1).$$

This implies that $\tilde{R}_\epsilon(x)=\mathcal{O}(1)$. Thus then,

$$\begin{aligned}
&\tilde{g}_0\overset{def}{=}\frac{1}{2\pi\epsilon}\int_I\mathcal{F}_\epsilon[u\phi_{2a}](\xi)d\xi=A_\epsilon(0)\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right)+\mathcal{O}(\epsilon^{\frac{3}{2}})+\epsilon\tilde{R}_\epsilon(0),\\
&\tilde{g}_1\overset{def}{=}\frac{1}{2\pi\epsilon}\int_I\xi\mathcal{F}_\epsilon[u\phi_{2a}](\xi)d\xi=[S'_\epsilon(0)A_\epsilon(0)-i\epsilon A'_\epsilon(0)]\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right)+\mathcal{O}(\epsilon^{\frac{3}{2}})-i\epsilon^2\tilde{R}'_\epsilon(0),
\end{aligned}$$

and we have

$$\frac{\tilde{g}_1}{\tilde{g}_0}=S'_\epsilon(0)-\frac{i\epsilon A'_\epsilon(0)}{A_\epsilon(0)}-\frac{i\epsilon^2 A_\epsilon(0)\left(\tilde{R}'_\epsilon(0)-\frac{iS'_\epsilon(0)\tilde{R}_\epsilon(0)}{\epsilon}\right)-i\epsilon^2 A'_\epsilon(0)\tilde{R}_\epsilon(0)+\mathcal{O}(\epsilon^{\frac{3}{2}})}{A_\epsilon(0)\left[A_\epsilon(0)\exp\left(\frac{iS_\epsilon(0)}{\epsilon}\right)+\epsilon\tilde{R}_\epsilon(0)\right]+\mathcal{O}(\epsilon^{\frac{3}{2}})}. \tag{4.12}$$

Since $\tilde{R}_\epsilon(x)$ has a density function $\tilde{r}_\epsilon(\xi)$ which is compactly supported in I, by (4.10) we have

$$p\overset{def}{=}\Re\left(\frac{\tilde{g}_1}{\tilde{g}_0}\right)=S'_\epsilon(0)+\mathcal{O}(\epsilon^{\frac{3}{2}}|\ln\epsilon|^{\frac{1}{2}}).$$

For the moment we do not know how to remove the $|\ln\epsilon|^{\frac{1}{2}}$ factor by a more sophisticated filtering technique. However, this factor increases very slowly with respect to ϵ. For example, when ϵ is as small as 2^{-23}, this quantity is still less than 4.

We summarize the above parameter recovery algorithm in the following. The window function ϕ_{2a} is replaced with $h_0^{\frac{1}{2}}=(2\pi b^2)^{-\frac{1}{4}}\phi_{2a}$. This treatment is purely for the ease of reference in the later.

PR A. Given a spatial point q. Multiply $u(x+q)$ with $h_0^{\frac{1}{2}}(x)$ and compute the Fourier transform

$$D(\xi)\overset{def}{=}\mathcal{F}_\epsilon[u(x+q)h_0^{\frac{1}{2}}(x)](\xi).$$

PR B. Compute

$$g_0=\frac{(2\pi b^2)^{\frac{1}{4}}}{2\pi\epsilon}\int_{\mathbb{R}}D(\xi)d\xi,\ g_1=\frac{(2\pi b^2)^{\frac{1}{4}}}{2\pi\epsilon}\int_{\mathbb{R}}\xi D(\xi)d\xi, \tag{4.13}$$

and set

$$\tilde{A}=g_0,\quad p^*=\Re\left(\frac{g_1}{g_0}\right). \tag{4.14}$$

PR C. Compute

$$g=\frac{(2\pi b^2)^{\frac{1}{4}}}{2\pi\epsilon}\int_{\mathbb{R}}D(\xi)\phi(\xi-p^*)d\xi, \tag{4.15}$$

and set

$$\gamma=-\Im\left(\frac{g_0}{g}\right)^2.$$

PR D. Set

$$I=p^*+\left[-\sqrt{\frac{3\epsilon|\ln\epsilon|(1+4a^2\gamma^2)}{2a}},\sqrt{\frac{3\epsilon|\ln\epsilon|(1+4a^2\gamma^2)}{2a}}\right], \tag{4.16}$$

and compute

$$\tilde{g}_0=\frac{(2\pi b^2)^{\frac{1}{4}}}{2\pi\epsilon}\int_I D(\xi)d\xi,\quad \tilde{g}_1=\frac{(2\pi b^2)^{\frac{1}{4}}}{2\pi\epsilon}\int_I\xi D(\xi)d\xi. \tag{4.17}$$

Set $p=\Re(\tilde{g}_1/\tilde{g}_0)$.

In the end, $\{\tilde{A},p,\gamma\}$ are the approximations of phase parameters of the un-corrupted single-phased wave function $u_e(x)=A_\epsilon(x)\exp\left(\frac{iS_\epsilon(x)}{\epsilon}\right)$ at $x=q$.

5 Composite coherent states approximation method

In the last section, we proposed the parameter recovery algorithm for the corrupted single-phased wave function. Now we study the possibility of generalizing this algorithm to the more complicated multi-phased wave functions.

By multi-phased we mean at least in some interval, the data can be written as

$$u(x)=\sum_{l=1}^{L}u_l(x)+\epsilon R_\epsilon(x),\quad u_l(x)=A_{l,\epsilon}(x)\exp\left(\frac{iS_{l,\epsilon}(x)}{\epsilon}\right), \tag{5.1}$$

where $R_\epsilon(x)$ satisfies Assumption 4.1. Each $u_l(x)$ is called a *phase branch*. Multi-phased structure may arise when the data is indeed a combination of single-phased wave functions in the whole definition domain (see Example 2), or it is derived by evolving a single-phased data based on some Hamiltonian flow (see Example 3). In the latter case, the caustics may appear, and away from caustics, the number of phases may be different at different regions.

Our basic idea is to *separate the different phase branches in the Fourier space*. Suppose in an open interval I, the first phase branch separates from the others, i.e., there exists a constant $C_{gap}>0$ such that for any $q\in I$, it holds that $S'_{1,\epsilon}(q)\neq S'_{j,\epsilon}(q)$, $\forall\, j\geq 2$, and

$$\begin{aligned}S'_{1,\epsilon}(q)&\geq \max_{S'_{j,\epsilon}(q)<S'_{1,\epsilon}(q),j\geq 2} S'_{j,\epsilon}(q)+C_{gap},\\ S'_{1,\epsilon}(q)&\leq \min_{S'_{j,\epsilon}(q)>S'_{1,\epsilon}(q),j\geq 2} S'_{j,\epsilon}(q)-C_{gap}.\end{aligned}$$

Then for any fixed $q\in I$, the Fourier transform of the windowed function $u(x+q)h_0^{\frac{1}{2}}(x)$ will contain a bump centered at $\xi=S'_{1,\epsilon}(q)$ and having a width of $\mathcal{O}(\sqrt{\epsilon})$. If ϵ is small enough, this bump contains most part of the significant energy spectra of the function $u_1(x+q)h_0^{\frac{1}{2}}(x)$, and separates from the other significant part of energy spectra of $u(x+q)h_0^{\frac{1}{2}}(x)$ almost with a distance of C_{gap}. Thus the phase parameters of $u_1(x)$ at $x=q$ can be extracted out with the recovery algorithm described in the last section.

To make through this idea, we need to locate all bumps in the energy spectra, and judge whether they are indeed associated with some phase branch. If so, suppose $\{\tilde{A}_{jl},p_{jl},\gamma_{jl}\}$ is the recovered phase parameters at $q=q_j$ associated with a phase branch u_l, then the following coherent state

$$\tilde{A}_{jl}\exp\left(\frac{i}{\epsilon}\left(p_{jl}(x-q_j)+\frac{\gamma_{jl}}{2}(x-q_j)^2\right)\right)h_j^{\frac{1}{2}}(x)$$

is a half-order approximation of $u_l(x)h_j^{\frac{1}{2}}$. Otherwise, the difference between the above coherent state and $u_l(x)h_j^{\frac{1}{2}}$ will be large. This implies that the difference is a good candidate which judges whether an energy bump is related to a specific phase branch.

Our CSA method based on the parameter recovery is described in the following, which will be referred to as PR-CSA in the later.

PR-CSA A. Given a wave function $u(x)\in L^2(\mathbb{R})$ and an *energy ratio* $r=o(\epsilon^2)$. Set the parameters a, b and L appropriately, and set $q_j=j\sqrt{\epsilon a}/b$.

PR-CSA B. At each spatial point q_j, compute

$$m_j=||u(x+q_j)\chi(x)h_0^{\frac{1}{2}}(x)||^2.$$

Sort $\{m_j\}_{j\in\mathbb{Z}}$, and find a subset of $\{m_j\}_{j\in\mathbb{Z}}$ with least number from large to small, so that

$$\sum_{j\in\mathbb{I}}m_j\geq(1-r)\|u\|^2.$$

Denote the set of associated indices by $\mathbb{I}$.

PR-CSA C. Specify two error tolerances $\mathscr{E}_1=o(\epsilon^2)$, $\mathscr{E}_2=\mathcal{O}(\epsilon)$. For each index $j\in\mathbb{I}$, compute the Fourier coefficients $\{C_{jk}\}_{k\in\mathbb{Z}}$ of $u(x+q_j)\chi(x)h_0^{\frac{1}{2}}(x)$. Set $m_{jk}=|C_{jk}|^2/L$. Sort $\{m_{jk}\}_{k\in\mathbb{Z}}$, and find a subset of $\{m_{jk}\}_{k\in\mathbb{Z}}$ with least number from large to small, the associated indices denoted by $\mathbb{I}_j$, so that

$$\sum_{k\in\mathbb{I}_j}m_{jk}\geq(1-\mathscr{E}_1)m_j.$$

Decompose $\mathbb{I}_j$ into a union of segments, i.e.,

$$\mathbb{I}_j=\cup_{l=1}^{N_j}\mathbb{I}_{jl},\quad |\mathbb{I}_{jl}|=b_l-a_l+1,\quad \mathbb{I}_{jl}=[a_l,b_l],\quad a_1<b_1<\cdots<a_{N_j}<b_{N_j}.$$

Apply the parameter recovery algorithm at q_j by replacing $\mathbb{R}$ in (4.13) and (4.15) with $\frac{2\pi\epsilon}{L}\mathbb{I}_{jl}$, I in (4.16) with $I\cap\frac{2\pi\epsilon}{L}\mathbb{I}_{jl}$, and all the continuous integrals with their discrete counterparts. The derived results are denoted by $\{\tilde{A}_{jl},p_{jl},\gamma_{jl}\}$. Set

$$u_{jl}(x)=\tilde{A}_{jl}\exp\left(\frac{i}{\epsilon}\left(p_{jl}(x-q_j)+\frac{\gamma_{jl}}{2}(x-q_j)^2\right)\right).$$

The exact Fourier transform of $u_{jl}(x+q_j)h_0^{\frac{1}{2}}(x)$ is

$$\tilde{C}(\xi)=(2\pi b^2)^{-\frac{1}{4}}\tilde{A}_{jl}\sqrt{\frac{4\pi\epsilon a}{1-2ia\gamma_{jl}}}\exp\left(-\frac{a(p_{jl}-\xi)^2}{\epsilon(1-2ia\gamma_{jl})}\right).$$

If

$$\frac{\sum_{k\in\mathbb{I}_{jl}}|C_{jk}-\tilde{C}(\frac{2\pi\epsilon k}{L})|^2}{L\sum_{k\in\mathbb{I}_{jl}}m_{jk}}=\frac{\sum_{k\in\mathbb{I}_{jl}}|C_{jk}-\tilde{C}(\frac{2\pi\epsilon k}{L})|^2}{\sum_{k\in\mathbb{I}_{jl}}|C_{jk}|^2}\leq\mathscr{E}_2, \tag{5.2}$$

we say the energy segment $\mathbb{I}_{jl}$ is a *coherent state segment*, or $\mathbb{I}_{jl}$ *admits a coherent state approximation*, and the computed parameters $\{\tilde{A}_{jl},p_{jl},\gamma_{jl}\}$ are *acceptable*. The corresponding coherent state function is denoted by

$$\mathrm{CS}_{jl}(x)=\tilde{A}_{jl}\exp\left(\frac{i}{\epsilon}\left(p_{jl}(x-q_j)+\frac{\gamma_{jl}}{2}(x-q_j)^2\right)\right)h_j(x).$$

Otherwise, we say $\mathbb{I}_{jl}$ is a *non-coherent state segment*. If all $\mathbb{I}_{jl}$ associated with q_j are coherent state segments, we say the point q_j is a *regular point*. Otherwise, it is *irregular*.

PR-CSA D. A CSA is formed by setting

$$u_{\mathrm{CS}}(x)=\sum_{j\in\mathbb{I}}\sum_{\mathbb{I}_{jl}:\text{coherent state segment}}\mathrm{CS}_{jl}(x).$$

Remark 5.1. In the above algorithm, PR-CSA B intends to remove some part of the wave function which has a negligible energy. The left hand side of (5.2) in PR-CSA C is the square of L^2 error between a *possible* phase branch and its local half-order approximation windowed by the Gaussian function $h_0^{\frac{1}{2}}$.

Generally, the parameter recovery algorithm in PR-CSA C would fail to present a set of acceptable parameters at a specific energy segment $\mathbb{I}_{jl}$ in three cases:

- The noise is relatively strong compared with the specific phase branch;
- At least two phase branches approach each other, which leads to an enlarged bump not resembling a Gaussian at all;
- A caustic point is developing around q_j, so that the detected spread γ_{jl} has a large magnitude.

When either of these cases happens, the accuracy of the derived CSA by PR-CSA becomes problematic in a small region containing the irregular point q_j. To solve this problem, we can apply WFT-CSA to further reduce the approximating error. The point is how to specify the error tolerance $\mathscr{E}$ in the WFT-CSA algorithm appropriately. If $\mathscr{E}$ is set too large, the approximating accuracy around the irregular points does not match the CSA accuracy away from the irregular points. On the other hand, if $\mathscr{E}$ is set too small, there exists a risk that in the regular region, which only contains regular points, more coherent states will be looked for, so that the benefit of PR-CSA is destroyed completely, and the overall number of coherent states will be comparative to that by applying WFT-CSA straightforwardly.

If q_j is a regular point, then due to the error cancelation effect, $u_{\mathrm{CS}}(x)h_j^{\frac{1}{2}}(x)$ is a first order approximation of $u(x)h_j^{\frac{1}{2}}(x)$ on any energy segment associated with q_j. However, if q_j is irregular, and $\mathbb{I}_{jl}$ is one of the associated non-coherent state energy segments, then $u_{\mathrm{CS}}(x)h_j^{\frac{1}{2}}(x)$ does not approximate $u(x)h_j^{\frac{1}{2}}(x)$ at all on $\mathbb{I}_{jl}$. Even worse, the same thing will happen to any coherent state segment $\mathbb{I}_{j'l'}$ which is adjacent to $\mathbb{I}_{jl}$ in the phase space. This is because only one-sided coherent state approximation is constructed at $q_{j'}$ by PR-CSA and *the error cancelation effect does not take place around* $q_{j'}$.

Based on the above analysis, we can separate all the energy segments into two groups: a bad group $\mathbb{B}$ which needs further treatment and the remaining good group. The bad group includes all non-coherent state segments and a part of coherent state segments which are close to any one of the non-coherent state segments. Given a non-coherent state segment $\mathbb{I}_{jl}$ and a coherent state segment $\mathbb{I}_{j'l'}$, the criterions set empirically in our numerical experiments for this *closeness* are

- The spatial distance between these two energy segments is small so that

$$|\mathrm{CS}_{j'l'}(q_j)| \geq 10^{-14}. \tag{5.3}$$

- The two segments should overlap in the momentum space, which means

$$\mathbb{I}_{jl} \cap \mathbb{I}_{j'l'} \neq \emptyset.$$

The error tolerance $\mathscr{E}$ used in WFT-CSA is then computed by

$$\mathscr{E} = \|u - u_{\mathrm{CS}}\|_2^2 - \sum_{\mathbb{I}_{jl} \in \mathbb{B}, k \in \mathbb{I}_{jl}} L|C_{jk}|^2, \tag{5.4}$$

where C_{jk} are the Fourier coefficients of $(u(x+q_j) - u_{\mathrm{CS}}(x+q_j))\chi(x)h_0^{\frac{1}{2}}(x)$. After a set of coherent states are determined by WFT-CSA, the overall CSA is then set as their summation together with $u_{\mathrm{CS}}(x)$.

The above CSA method, which merges the ideas of PR-CSA and WFT-CSA, is termed *composite coherent states approximation* (CCSA). For the ease of reference, we formulate the CCSA method in the following.

CCSA A. Given a wave function $u(x) \in L^2(\mathbb{R})$. Set the parameters a and b appropriately, and set $q_j = j\sqrt{\epsilon a}/b$.

CCSA B. Set the energy ratio $r = o(\epsilon^2)$. Perform the PR-CSA algorithm to derive a first CSA $u_{1,\mathrm{CS}}(x)$.

CCSA C. Determine the bad group of energy segments $\mathbb{B}$. For each $\mathbb{I}_{jl} \in \mathbb{B}$, compute the Fourier coefficients C_{jk} of the function $(u(x+q_j) - u_{1,\mathrm{CS}}(x+q_j))\chi(x)h_0^{\frac{1}{2}}(x)$.

CCSA D. Compute the error tolerance $\mathscr{E}$ with (5.4) by replacing u_{CS} in (5.4) with $u_{1,\mathrm{CS}}$.

CCSA E. Set $m_{jk} = |C_{jk}^2|/L$. Sort m_{jk} with $\mathbb{I}_{jl} \in \mathbb{B}$ and $k \in \mathbb{I}_{jl}$. Find a set of m_{jk} with least number from large to small, so that their summation is larger than $\sum_{\mathbb{I}_{jl} \in \mathbb{B}, k \in \mathbb{I}_{jl}} m_{jk} - \mathscr{E}$. Denote by $\mathbb{J}$ the set of associated double indices.

CCSA F. Each $(j,k) \in \mathbb{J}$ corresponds to a coherent state of the form

$$\mathrm{CS}_{jk}(x) = \frac{C_{jk}}{L} \exp\left(\frac{2i\pi k(x - q_j)}{L}\right) h_j^{\frac{1}{2}}(x).$$

A CSA is formed by setting

$$u_{2,\mathrm{CS}}(x) = \sum_{(j,k) \in \mathbb{J}} \mathrm{CS}_{jk}(x).$$

CCSA G. The CSA derived by CCSA is finally set as $u_{\mathrm{CS}}(x) = u_{1,\mathrm{CS}}(x) + u_{2,\mathrm{CS}}(x)$.

6 Numerical experiments

In all the numerical tests of this section, we set $a=1$, $b=1.4$ and $L=6\sqrt{4\epsilon a}$. In the PR-CSA, we set $r=\epsilon^3$, $\mathscr{E}_1=\epsilon^3$ and $\mathscr{E}_2=100\epsilon$.

As a first numerical test, we validate the asymptotic accuracy of the phase parameter recovery algorithm in Section 3. The corrupted wave function is set as

$$u(x)=u_e(x)+\epsilon R_\epsilon(x),$$

where

$$u_e(x)=\exp\left(-\frac{x^2}{2}\right)\exp\left(\frac{i(\sin x+x^2)}{\epsilon}\right).$$

and

$$R_\epsilon(x)=\exp\left(-\frac{x^2}{2}\right)\left[\exp\left(\frac{i(1+\sqrt{\epsilon})x}{\epsilon}\right)+\exp\left(\frac{2ix}{\epsilon}\right)\right].$$

In Fig. 1 we demonstrate the absolute error of $\tilde{A}$, p^*, p and γ at $q=0$, derived by the parameter recovery algorithm. Linear regression shows that the asymptotic accuracies of these quantities are 1.000, 1.009, 1.494 and 0.985 respectively, which matches our analysis in Section 3 very well.

In the following, we present three examples to demonstrate the performance of the proposed composite coherent states approximation (CCSA) method.

Example 6.1. Consider the single-phased WKB wave function

$$u_\epsilon(x)=\exp\left(-\frac{x^2}{2}\right)\exp\left(\frac{ix^2}{\epsilon}\right).$$

Table 1 compares the WFT-CSA method and the CCSA method, where the relative L^2 errors and some other statistics are listed. Num1 stands for the number of coherent states determined by PR-CSA embedded in CCSA, while Num2 stands for the number by the local WFT-CSA method in CCSA. For this example, Num2 is always zero. One can see that the beam number reduction by CCSA is tremendous. For example, when $\epsilon=2^{-13}$, the number of coherent states by CCSA is less than one sixtieth of that by WFT-CSA, but their approximating accuracies remain on the same level. In Fig. 2 we show the coherent

Table 1: Example 6.1.

$1/\epsilon$		2^9	2^{10}	2^{11}	2^{12}	2^{13}
WFT-CSA	Num	15176	23556	36450	55890	85276
	Errors	7.56E-4	3.98E-4	2.03E-4	1.03E-4	5.18E-5
	Rate	-	1.90	1.96	1.97	1.99
CCSA	Num1	261	389	579	857	1266
	Num2	0	0	0	0	0
	Errors	8.35E-4	4.27E-4	2.12E-4	1.05E-4	5.30E-5
	Rate	-	1.96	2.01	2.02	1.98

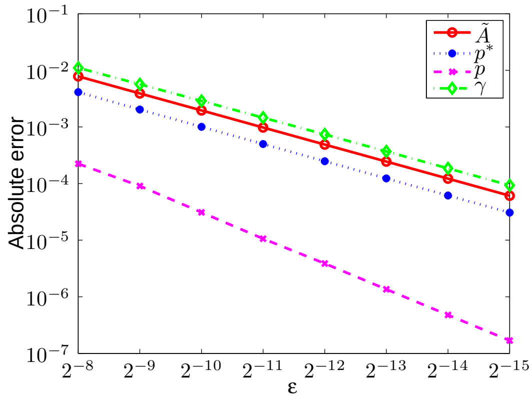

Figure 1: The decaying rates are 1.000, 1.009, 1.494 and 0.985, respectively.

state distribution in the phase space for both CCSA and WFT-CSA in a small region when $\epsilon = 2^{-10}$. The black line depicts the function $p = 2q$, which is the central curve of the energy distribution. Compared with that by CCSA, the number of coherent states by WFT-CSA is more than seventy times larger.

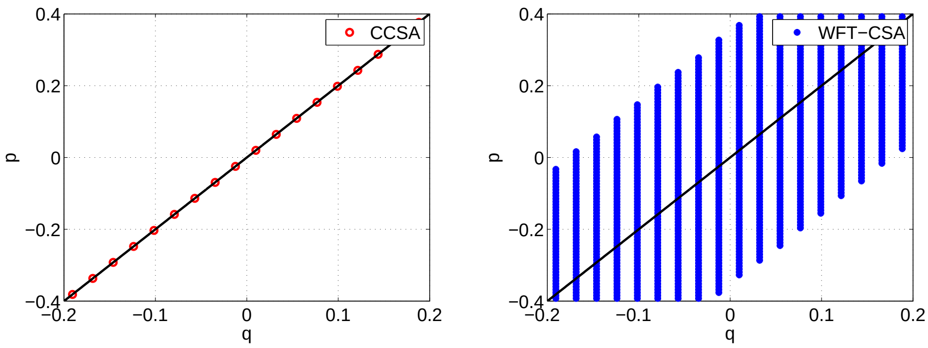

Figure 2: Example 6.1. $\epsilon = 2^{-10}$. Left: beam distribution for CCSA. Right: beam distribution for WFT-CSA.

Example 6.2. Consider the two-phased WKB wave function

$$u_\epsilon(x) = \exp\left(-\frac{x^2}{2}\right)\left[\exp\left(\frac{ix^2}{2\epsilon}\right) + \exp\left(-\frac{ix^2}{2\epsilon}\right)\right].$$

Since the wave function has two branches which cross at point (0,0) in the phase space, the coherent states approximation method, essentially based on the single-phased parameter recovery, fails in presenting an acceptable set of phase parameters around this point. The local WFT-CSA part of CCSA method handles this problem, and acts like a patch which connects the four broken phase branches, see Fig. 3. Table 2 lists some statistics. Again, CCSA reduces the number of coherent states significantly, and meanwhile, it maintains the accuracy on the same level as WFT-CSA.

Table 2: Example 6.2.

$1/\epsilon$		2^9	2^{10}	2^{11}	2^{12}	2^{13}
WFT-CSA	Num	17023	26155	40168	61173	93077
	Errors	4.89E-4	2.88E-4	1.60E-4	8.69E-5	4.56E-5
	Rate	-	1.70	1.80	1.84	1.91
CCSA	Num1	484	740	1118	1672	2486
	Num2	2022	2217	2471	2581	2893
	Errors	5.49E-4	3.10E-4	1.67E-4	8.98E-5	4.70E-5
	Rate	-	1.77	1.86	1.86	1.91

Example 6.3. The wave function u_ϵ is obtained by evolving the Schrödinger equation

$$\begin{aligned} &i\epsilon\Phi_{\epsilon,t}=-\frac{\epsilon^2}{2}\Phi_{\epsilon,xx},\\ &\Phi_\epsilon(x,0)=\exp\left(-\frac{x^2}{2}\right)\exp\left(\frac{i\exp(-x^2)}{\epsilon}\right)\end{aligned}$$

to $t=2$, i.e., $u_\epsilon(x)=\Phi_\epsilon(x,2)$. The numerical method used is the Fourier transform method by truncating the domain to $[-10,10]$ and imposing periodic boundary conditions. At the initial time, the central energy curve, actually the Lagrangian manifold associated with the WKB initial data, is

$$\{(s,\tau)|\tau=-2s\exp(-s^2),\ s\in\mathbb{R}\}.$$

Then, this curve evolves in the phase space obeying the Hamiltonian system

$$\dot{q}=p,\qquad \dot{p}=0.$$

At $t=2$, the displaced curve is

$$\{(s+2\tau,\tau)|\tau=-2s\exp(-s^2),\ s\in\mathbb{R}\}.$$

Note that though the wave function is single-phased at the initial time, it develops two caustic points at $q\approx\pm1.0775$ when $t=2$.

In Fig. 4 we show the coherent states distribution by the CCSA method. The local WFT part of CCSA presents a set of coherent states which stick the broken phase branch together. Table 3 lists the beam number and some other information. One can see that the coherent states reduction by CCSA, though still obvious, is not as significant as that for the last two examples. In particular, unlike the last two examples, the relative error does not degenerate with first order. The reason for this is that the detected errors $\mathscr{E}$ do not decrease with second order, see Table 3. In fact, as ϵ changes from 2^{-12} to 2^{-13}, this quantity even increases. To understand why this happens, in Fig. 5 we plot the error function $u_\epsilon(x)-u_{\epsilon,\mathrm{CS}}(x)$, and the local coherent states distribution. We see that the coherent states approximation error by PR-CSA increases very fast when the spatial point approaches the caustic point (See also the zoom-in plots of the error function in Fig. 6).

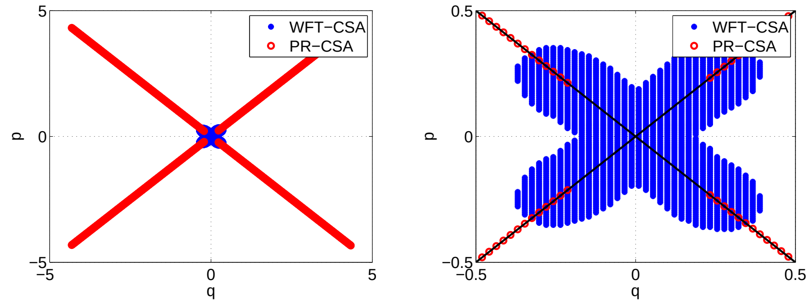

Figure 3: Example 6.2. $\epsilon=2^{-10}$. Left: beam distribution for CCSA. Right: zoom-in plot.

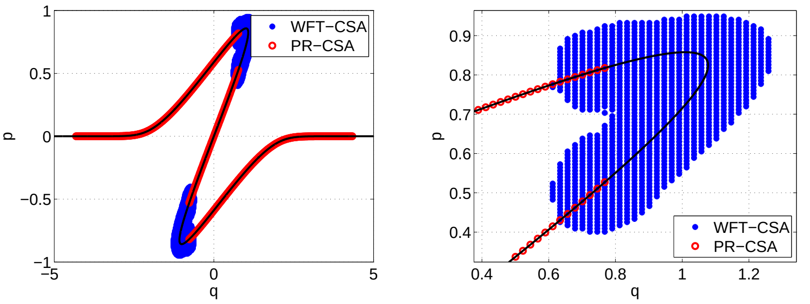

Figure 4: Example 6.3. $\epsilon=2^{-10}$. Left: beam distribution for CCSA. Right: zoom-in plot.

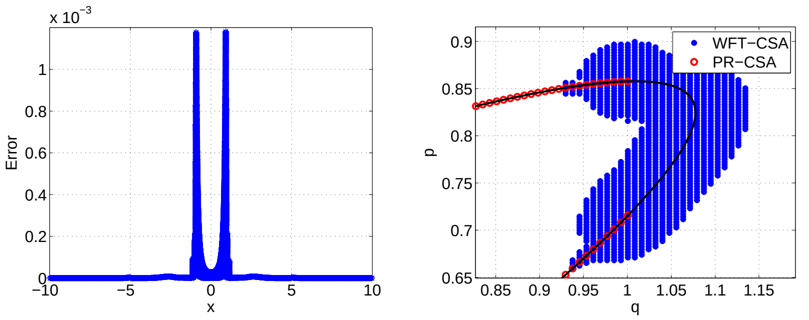

Figure 5: Example 6.3. $\epsilon=2^{-13}$. Left: error plot. Right: local beam distribution.

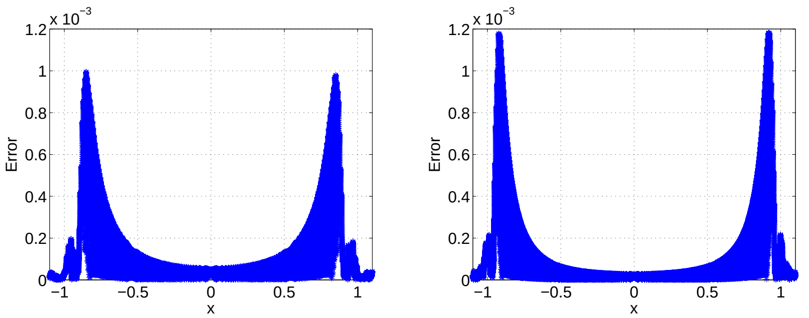

Figure 6: Example 6.3. Error plot. Left: $\epsilon=2^{-12}$. Right: $\epsilon=2^{-13}$.

Table 3: Example 6.3.

$1/\epsilon$		2^9	2^{10}	2^{11}	2^{12}	2^{13}
CCSA	Num1	337	529	805	1203	1776
	Num2	2789	2661	2688	2621	2682
	Errors	2.51E-4	2.32E-4	2.27E-4	2.43E-4	2.45E-4
	Rate	-	1.08	1.02	0.93	0.99
	$\mathscr{E}$ by (5.4)	2.79E-7	1.66E-7	1.09E-7	1.09E-7	1.13E-7
WFT-CSA	Num	8975	12822	18158	25450	35950
	Errors	2.16E-4	2.09E-4	2.11E-4	2.30E-4	2.37E-4

Though the approximating error decreases at the regular region when ϵ gets smaller, *the regular region itself is becoming larger*. This explains the abnormal phenomenon observed from Table 3.

7 Conclusions and future works

Function approximation for highly oscillatory functions is an interesting and important issue in the computational science. In this paper we considered this problem by utilizing Gaussian coherent state functions and proposed a composite approximating method. The traditional windowed Fourier transform method, though powerful, is not efficient when the energy distribution of the data function is focused around some low-dimensional manifolds of the phase space. An important case of such kind is the multi-phased WKB wave function.

It is known that if an analytical expression of the WKB function is available, the Taylor expansion method can present a coherent states approximation with the accuracy to $\mathcal{O}(\epsilon)$. However, generally we cannot expect this a priori information, especially when the wave function is corrupted with noise. We proposed a parameter recovery algorithm based on the asymptotic expansion of some moments associated with the wave function. Unlike the optimization-based methods [3, 18], our algorithm is a direct method, which avoids the fragile stability problem on the initial guess selection.

A simple but important fact about multi-phased wave functions is such that the different phase branches will mostly separate in the Fourier space. Based on this fact, we proposed the coherent states approximation algorithm based on the phase parameter recovery. If neither cross-points nor caustics appear, this algorithm will present a coherent states approximation with the accuracy to $\mathcal{O}(\epsilon)$. Otherwise, the coherent states approximation algorithm will fail in some local regions. In this case, we resort to the windowed Fourier transform technique. The essence of composite coherent states approximation method is to handle these two cases by different means, but maintain as much as possible the benefit of the coherent states approximation algorithm based on the parameter recovery.

Our numerical experiments showed that the reduction of coherent states is tremendous wherever the parameter recovery is performed successfully. However, the proposed composite coherent states approximation method can still be improved in several aspects:

- Coherent states approximation around cross-points and caustics. Numerical experiments showed that the number of coherent states by the local windowed Fourier transform method is still large. A rough idea for solving the caustic problem is to utilize the fractional Fourier transform [1].
- Generalize the method into two dimensions. This work might be more tricky, but intuitively, the idea of parameter recovery algorithm proposed in this paper is still applicable for two-dimensional single-phased wave functions.

These are the topics of our current interest. The progress will be reported in a forthcoming paper.

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A Partition of unity by Gaussians

Set

$$h_j(a,b,x)=\frac{1}{\sqrt{2\pi b^2}}\exp\left(-\frac{(x-q_j)^2}{2\epsilon a}\right),$$

where $q_j=j\sqrt{\epsilon a}/b$. We would like to compute the summations

$$\mathfrak{S}_m(a,b,x)\overset{def}{=}\sum_{j\in\mathbb{Z}}(x-q_j)^m h_j(a,b,x). \tag{A.1}$$

Since $\mathfrak{S}_m$ is an $\sqrt{\epsilon a}/b$-periodic function, we can expand $\mathfrak{S}_m$ into the Fourier series form

$$\mathfrak{S}_m(a,b,x)=\sum_{n\in\mathbb{Z}}a_{n,m}\exp\left(\frac{2\pi nibx}{\sqrt{\epsilon a}}\right),$$

where the n-th Fourier coefficient $a_{n,m}$ satisfies the relation

$$\begin{aligned}
a_{n,m}&=\frac{b}{\sqrt{\epsilon a}}\int_0^{\sqrt{\epsilon a}/b}\mathfrak{S}_m(a,b,x)\exp\left(-\frac{2\pi nbix}{\sqrt{\epsilon a}}\right)dx\\
&=\frac{1}{\sqrt{2\pi\epsilon a}}\sum_{j\in\mathbb{Z}}\int_0^{\sqrt{\epsilon a}/b}(x-q_j)^m\exp\left(-\frac{(x-q_j)^2}{2\epsilon a}\right)\exp\left(-\frac{2\pi nibx}{\sqrt{\epsilon a}}\right)dx\\
&=\frac{1}{\sqrt{2\pi\epsilon a}}\sum_{j\in\mathbb{Z}}\int_0^{\sqrt{\epsilon a}/b}(x-q_j)^m\exp\left(-\frac{(x-q_j)^2}{2\epsilon a}\right)\exp\left(-\frac{2\pi nib(x-q_j)}{\sqrt{\epsilon a}}\right)dx
\end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{\sqrt{2\pi\epsilon a}}\int_{\mathbb{R}} x^m \exp\left(-\frac{x^2}{2\epsilon a}\right)\exp\left(-\frac{2\pi nibx}{\sqrt{\epsilon a}}\right)dx\\
&=\frac{(\epsilon a)^{\frac{m}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} x^m \exp\left(-\frac{x^2}{2}\right)\exp(-2\pi nibx)dx.
\end{aligned}
$$

Setting

$$\xi_n=2\pi nb,$$

we have

$$
a_{n,m}=\begin{cases}
\exp\left(-\frac{\xi_n^2}{2}\right), & m=0,\\
-i\xi_n\sqrt{\epsilon a}\exp\left(-\frac{\xi_n^2}{2}\right), & m=1,\\
(1-\xi_n^2)\epsilon a\exp\left(-\frac{\xi_n^2}{2}\right), & m=2,\\
i\xi_n(\xi_n^2-3)(\epsilon a)^{\frac{3}{2}}\exp\left(-\frac{\xi_n^2}{2}\right), & m=3.
\end{cases}
$$

Lemma A.1. *Set*

$$\mathcal{R}_m(\epsilon)=\sum_{n=1}^{\infty} n^m\epsilon^{n^2}.$$

If $\epsilon\le 2^{-(m+1)/3}$, *then*

$$\mathcal{R}_m(\epsilon)\le 2\epsilon.$$

Proof. Since

$$
\begin{aligned}
\mathcal{R}_m(\epsilon)&=\epsilon+\sum_{n=2}^{\infty} n^m\epsilon^{n^2}=\epsilon+\sum_{n=1}^{\infty}(n+1)^m\epsilon^{(n+1)^2}\\
&\le\epsilon+\sum_{n=1}^{\infty}2^m\epsilon^3 n^m\epsilon^{n^2}=\epsilon+2^m\epsilon^3\mathcal{R}_m(\epsilon),
\end{aligned}
$$

we have

$$\mathcal{R}_m(\epsilon)\le\frac{\epsilon}{1-2^m\epsilon^3}.$$

This ends the proof. □

Lemma A.2. *If* $\exp\left(-2\pi^2b^2\right)\le 2^{-4/3}$, *then for any* $x\in\mathbb{R}$ *we have*

$$|\mathfrak{S}_0(a,b,x)-1|\le 4\exp\left(-2\pi^2b^2\right)\equiv C_0(b), \tag{A.2}$$

$$|\mathfrak{S}_1(a,b,x)|\le 8\pi b\sqrt{\epsilon a}\exp\left(-2\pi^2b^2\right)\equiv C_1(b)\sqrt{\epsilon a}, \tag{A.3}$$

$$|\mathfrak{S}_2(a,b,x)-\epsilon a\mathfrak{S}_0(a,b,x)|\le 16\pi^2b^2\epsilon a\exp\left(-2\pi^2b^2\right)\equiv C_2(b)\epsilon a, \tag{A.4}$$

$$|\mathfrak{S}_3(a,b,x)|\le 8\pi b(4\pi^2b^2+3)(\epsilon a)^{\frac{3}{2}}\exp\left(-2\pi^2b^2\right)\equiv C_3(b)(\epsilon a)^{\frac{3}{2}}. \tag{A.5}$$

Furthermore, if $\exp\left(-2\pi^2b^2\right)\le\epsilon\le 2^{-4/3}$, *then there exists a universal constant c such that*

$$C_0(b)\le 4\epsilon,\quad C_1(b)\le c\sqrt{\epsilon},\quad C_3(b)\le c\sqrt{\epsilon}. \tag{A.6}$$

Proof. Since

$$|\mathfrak{S}_m(a,b,x)-a_{0,m}|\le \sum_{n\in\mathbb{Z}\setminus\{0\}}|a_{n,m}|,$$

noticing

$$\xi_n=2\pi bn,\quad \exp(-\xi_n^2/2)=\exp\left(-2\pi^2b^2\right)^{n^2},$$

and using Lemma A.1, the estimate (A.2)-(A.5) follows directly. Since

$$C_1(b)=8\pi b\exp\left(-2\pi^2b^2\right)\le\sqrt{\epsilon}\max_{b\in\mathbb{R}}\Big(8\pi b\exp\left(-\pi^2b^2\right)\Big),$$
$$C_3=8\pi b(4\pi^2b^2+3)\exp\left(-2\pi^2b^2\right)\le\sqrt{\epsilon}\max_{b\in\mathbb{R}}\Big(8\pi b(4\pi^2b^2+3)\exp\left(-\pi^2b^2\right)\Big),$$

the estimates (A.6) follows. □

B ϵ-scaled Fourier transforms

Given a function $f(x)\in L^2(\mathbb{R}_x)$, we denote by $\hat{f}(\xi)\in L^2(\mathbb{R}_\xi)$ the ϵ-scaled fourier transform

$$\hat{f}(\xi)=\mathcal{F}_\epsilon[f](\xi)=\int_{\mathbb{R}_x}f(x)\exp\left(-\frac{ix\xi}{\epsilon}\right)dx.$$

The inverse transform from $a(\xi)\in L^2(\mathbb{R}_\xi)$ to $\check{a}(x)\in L^2(\mathbb{R}_x)$ is

$$\check{a}(x)=\mathcal{F}_\epsilon^{-1}[a](x)=\frac{1}{2\pi\epsilon}\int_{\mathbb{R}_\xi}a(\xi)\exp\left(\frac{ix\xi}{\epsilon}\right)d\xi.$$

It is straightforward to verify that

$$(\mathcal{F}_\epsilon[f],\mathcal{F}_\epsilon[g])=2\pi\epsilon(f,g),\quad \mathcal{F}_\epsilon[fg]=\frac{1}{2\pi\epsilon}\mathcal{F}_\epsilon[f]*\mathcal{F}_\epsilon[g],\quad \mathcal{F}_\epsilon[f*g]=\mathcal{F}_\epsilon[f]\mathcal{F}_\epsilon[g].$$

Here $*$ denotes the convolution operator defined by

$$(f*g)(x)=\int_{\mathbb{R}}f(y)g(x-y)dy.$$

C Tight frame by coherent states

Denote

$$\varphi_{pq}(x)=\frac{1}{(\pi\epsilon a)^{\frac{d}{4}}(2\pi\epsilon)^{\frac{d}{2}}}\exp\left(\frac{i}{\epsilon}\left(p\cdot(x-q)+\frac{i}{2a}||x-q||^2\right)\right),$$

which is parameterized in the phase space $\mathbb{R}_p^d\times\mathbb{R}_q^d$. Then the set of coherent state functions $\{\varphi_{pq}\}$ forms a tight frame since

$$\begin{aligned}&\iint_{\mathbb{R}_p^d\times\mathbb{R}_q^d}\varphi_{pq}(x)\varphi_{pq}^*(y)dpdq\\ =&\frac{1}{(\pi\epsilon a)^{d/2}(2\pi\epsilon)^d}\iint_{\mathbb{R}_p^d\times\mathbb{R}_q^d}\exp\left(\frac{ip\cdot(x-y)}{\epsilon}-\frac{||x-q||^2+||y-q||^2}{2\epsilon a}\right)dpdq\end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{(\pi\epsilon a)^{d/2}}\int_{\mathbb{R}^d_q}\delta(x-y)\exp\left(-\frac{||x-q||^2+||y-q||^2}{2\epsilon a}\right)dq\\
&=\frac{1}{(\pi\epsilon a)^{d/2}}\int_{\mathbb{R}^d_q}\delta(x-y)\exp\left(-\frac{||x-q||^2}{\epsilon a}\right)dq\\
&=\delta(x-y).
\end{aligned}
$$

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