Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **1**, No. 6, pp. 729-749

DOI: 10.4208/aamm.09-m09S03
December 2009

Efficient Reconstruction Methods for Nonlinear Elliptic Cauchy Problems with Piecewise Constant Solutions

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Received 15 March 2009; Accepted (in revised version) 25 August 2009
Available online 18 November 2009

Abstract. In this article, a level-set approach for solving nonlinear elliptic Cauchy problems with piecewise constant solutions is proposed, which allows the definition of a Tikhonov functional on a space of level-set functions. We provide convergence analysis for the Tikhonov approach, including stability and convergence results. Moreover, a numerical investigation of the proposed Tikhonov regularization method is presented. Newton-type methods are used for the solution of the optimality systems, which can be interpreted as stabilized versions of algorithms in a previous work and yield a substantial improvement in performance. The whole approach is focused on three dimensional models, better suited for real life applications.

AMS subject classifications: 65J20, 35J60
Key words: Nonlinear Cauchy problems, Elliptic operators, Level-set methods.

1 Introduction

We start by introducing the inverse problem under consideration. Let $\Omega \subset \mathbb{R}^3$, be an open bounded set with piecewise Lipschitz boundary $\partial\Omega$. Further, we assume that

$$\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2,$$

where Γ_i are two open disjoint parts of $\partial\Omega$. Given the function $q : \mathbb{R} \to \mathbb{R}^+$, we define the second order elliptic operator

$$\mathcal{P}(u) := -\nabla \cdot (q(u)\nabla u)\,. \tag{1.1}$$

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We denote by *nonlinear elliptic Cauchy problem* the following problem

$$\begin{cases} \mathcal{P}(u) = f, & \text{in } \Omega, \\ u = g_1, & \text{on } \Gamma_1, \\ q(u)u_\nu = g_2, & \text{on } \Gamma_1, \end{cases} \tag{CP_{nl}}$$

where the pair of functions $(g_1, g_2) \in H^{1/2}(\Gamma_1) \times H^{1/2}_{00}(\Gamma_1)'$ are given *Cauchy data* and $f \in L^2(\Omega)$ is a known source term in the model (see [32, p. 66] or [14] for a definition of the Sobolev spaces).

A solution of (CP_{nl}) is a distribution in $H^1(\Omega)$, which solves the weak formulation of the nonlinear elliptic equation $\mathcal{P}(u){=}f$ in Ω and further satisfies the Cauchy data on Γ_1 in the sense of the trace operators. Notice that, if we know the Neumann (or Dirichlet) trace of u on Γ_2, say $q(u)\, u_\nu|_{\Gamma_2}{=}\varphi$, then u can be computed as the solution of a nonlinear mixed boundary value problem (BVP) in a stable way, namely

$$\begin{cases} \mathcal{P}(u) = f, & \text{in } \Omega, \\ u = g_1, & \text{on } \Gamma_1, \\ q(u)u_\nu = \varphi, & \text{on } \Gamma_2, \end{cases} \tag{BVP}$$

Therefore, in order to solve (CP_{nl}), it is enough to consider the task of determining the Neumann trace of u on Γ_2 (a distribution in $H^{1/2}_{00}(\Gamma_2)'$).

Remark 1.1. For simplicity of the presentation the boundary parts Γ_i are assumed to be connected. Using standard elliptic theory one can prove that the results in this article also hold without this assumption. Moreover, the theory derived here extends naturally to Cauchy problems defined on domains with $\partial\Omega{=}\overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, where Γ_i are disjoint and some extra boundary condition (Dirichlet, Neumann, Robin, ...) is prescribed on Γ_3.

Remark 1.2. Let P be the linear elliptic operator defined in Ω by

$$P\,u := -\sum_{i,j=1}^{3} D_i(a_{i,j} D_j u),$$

where the real functions $a_{i,j} \in L^\infty(\Omega)$ are such that the matrix $A(x) := (a_{i,j})_{i,j=1}^d$ satisfies $\xi^t A(x)\xi > \alpha||\xi||^2$, for all $\xi \in \mathbb{R}^3$ and for a.e. $x \in \Omega$. Here α is some positive constant. The *linear elliptic Cauchy problem* corresponds to the problem (CP_{nl}) obtained when the operator $\mathcal{P}$ is substituted by P and the Neumann boundary condition is substituted by $u_{\nu_A}|_{\Gamma_1}{=}g_2$. The linear version of (CP_{nl}) has been intensively investigated over the last years [5–8, 11, 13, 17, 19, 23, 25, 28, 30, 31].

Linear elliptic Cauchy problems were used by Hadamard in the 1920's as an example of (exponentially) ill-posed problem [22]. For linear elliptic operators with analytical coefficients, uniqueness of solutions is known for over half a century [10, 12]. Moreover, as a straightforward argumentation with the Schwarz reflection principle shows, existence of solutions for arbitrary Cauchy data cannot be guaranteed [21,30]. More recently, analytical features of linear elliptic Cauchy problems were investigated in [5,6,11,19,30] (see also [24,26] and the references therein) and several numerical approaches were considered in [5,7,8,13,17,19,23,25,27,28,30,31,36].

What concerns nonlinear elliptic Cauchy problems, analytical and numerical approaches can be found in [29]. Uniqueness of $H^1(\Omega)$ solutions for (CP_{nl}) was proved in [29, Theorem 2]. The result in [5, Lemma 2.1], which guarantees existence of solutions of linear Cauchy problems for a dense subset of Cauchy data in $H^{1/2}(\Gamma_1) \times H_{00}^{1/2}(\Gamma_1)'$, can be extended for (CP_{nl}) (see Appendix).

Elliptic Cauchy problems arise in many industrial, engineering and biomedical applications including (A) Expansion of measured surface fields inside a body from partial boundary measurements [5]; (B) A classical thermostatics problem, which consists in recovering the temperature in a given domain when its distribution and the heat flux are known over the accessible region of the boundary [17]; (C) The analogous electrostatics case encountered in electric impedance tomography [5]; (D) An inverse problem related to corrosion detection [4,31].

The level-set approach proposed in this article for solving nonlinear elliptic Cauchy problems with piecewise constant solutions is motivated by application (D) above. Indeed, the inverse problem in corrosion detection consists in determining information about corrosion occurring on the inaccessible boundary part (Γ_2) of a specimen. The data for this inverse problem correspond to prescribed current flux (g_2) and voltage measurements (g_1) on the accessible boundary part (Γ_1) and the model is the Laplace equation with no source term $(P{=}\Delta,\ f{=}0)$. For simplicity one assumes the specimen to be a thin plate $(\Omega\subset\mathbb{R}^2)$ and $\partial\Omega{=}\Gamma_1 \cup \Gamma_2$. Moreover, the unknown corrosion damage γ is assumed to be the characteristic function of some $D\subset\Gamma_2$, corresponding to the boundary condition $u_\nu + \gamma u{=}0$ on Γ_2.

We pursue two main goals in this article:

- The first one is to use a level-set approach [9,33,34] in order to derive a functional analytical formulation for (CP_{nl}). Then, based on [20,31] we define a Tikhonov functional on a space of level-set functions, and prove stability and convergence results for our Tikhonov approach, characterizing it as a regularization method [16].

- The second main goal is to numerically investigate the application of an efficient method (Gauss-Newton) for solving the equations arising from the optimality conditions for our Tikhonov functional. The numerical approach is focused on three dimensional models, better suited for real a life applications, where $\Gamma_2\subset\partial\Omega$ is a 2D-manifold.

The manuscript is organized as follows: In Section 2, we write (CP_{nl}) in the functional analytical framework of an (ill-posed) operator equation in appropriate Hilbert spaces. In Section 3, we propose and analyze a Tikhonov regularization method for (CP_{nl}), which is based on a level-set approach [20,31]. Convergence and stability results are proven. Section 4 is devoted to numerical tests. A Newton-type method for the iterative solution of the optimality systems is proposed. Furthermore, some

3D numerical experiments are provided, in order to illustrate the effectiveness of our approach. Section 5 contains final remarks and conclusions.

2 Functional analytical formulation

In this section we write the elliptic Cauchy problem (CP_{nl}) in the form of an operator equation in Hilbert spaces. In what follows we assume the coefficient function q in (1.1) to satisfy the following assumptions:

A1) $q \in C^\infty(\mathbb{R})$;

A2) $q(t) \in [q_{min}, q_{max}]$ for all $t \in \mathbb{R}$, where $0 < q_{min} < q_{max} < \infty$.

2.1 A linearization step

In the sequel the nonlinear problem (CP_{nl}) is transformed into a linear elliptic Cauchy problem, which is then reduced to a linear operator equation. The first step is to define the primitive of function q

$$Q(t) := \int_0^t q(s)\, ds,$$

(Q is strictly monotone increasing and therefore invertible). Notice that, given $u \in L^2(\Omega)$ the function $U := Q(u)$ is also in $L^2(\Omega)$ and satisfies

$$-\Delta U = -\nabla \cdot (\nabla Q(u)) = -\nabla \cdot (q(u)\nabla u) = \mathcal{P}(u)\,.$$

Moreover, $U_\nu = q(u)u_\nu$ holds on $\partial\Omega$. Thus, if u is the solution of (CP_{nl}) then U solves the linear Cauchy problem

$$\begin{cases} -\Delta U = f, & \text{in } \Omega, \\ U = Q(g_1), & \text{on } \Gamma_1, \\ U_\nu = g_2, & \text{on } \Gamma_1. \end{cases} \tag{CP_l}$$

Reciprocally, if problem (CP_l) admits a solution, say U, for the Cauchy data $(Q(g_1), g_2)$, it follows from $Q' = q > 0$ (cf. assumption *A2)*), that

$$u := Q^{-1}(U) \in H^1(\Omega),$$

solves problem (CP_{nl}). Summarizing, in order to obtain a solution for (CP_{nl}) it is necessary and sufficient to solve the linearized problem (CP_l).

Next we consider the auxiliary problems

$$-\Delta w_a = 0, \quad \text{in } \Omega, \quad w_a = 0, \quad \text{on } \Gamma_1, \quad (w_a)_\nu = \varphi, \quad \text{on } \Gamma_2, \tag{2.1}$$

$$-\Delta w_b = f, \quad \text{in } \Omega, \quad w_b = Q(g_1), \quad \text{on } \Gamma_1, \quad (w_b)_\nu = 0, \quad \text{on } \Gamma_2, \tag{2.2}$$

in order to define the function $z := (w_b)_\nu|_{\Gamma_1}$, and the operator

$$L : \varphi \mapsto (w_a)_\nu|_{\Gamma_1}\,. \tag{2.3}$$

It is straightforward to check that $\varphi = q(u)\, u_\nu|_{\Gamma_2}$ $(= U_\nu|_{\Gamma_2})$ is the solution of (CP_{nl}) iff it is a solution of the operator equation

$$L\,\varphi \;=\; g_2 - z\,. \tag{2.4}$$

Notice that $z = z(g_1, f, q)$ can be computed *a priori*.

A more precise definition of L as well as some regularity properties are investigated in Subsection 2.2.

2.2 Abstract functional analytical framework

We consider (CP_{nl}) in the form of the operator equation (2.4). Further we assume the Cauchy data to satisfy

$$(g_1, g_2) \;\in\; H^{1/2}(\Gamma_1) \times H^{1/2}_{00}(\Gamma_1)', \tag{2.5}$$

and the source term f to be a distribution in $L^2(\Omega)$. Here, $H^{1/2}_{00}(\Gamma_1)'$ denotes the dual space of $H^{1/2}_{00}(\Gamma_1)$, which consists of functions in $H^{1/2}(\partial\Omega)$, vanishing on Γ_2. For details on these spaces we refer to [2, 14]. Due to the choice of g_1, f and q, it follows from the elliptic theory [14, Ch. VII.2] that the mixed BVP (2.2) has a unique solution $v_b \in H^1(\Omega)$. Moreover, $z := (v_b)_\nu|_{\Gamma_1} \in H^{1/2}_{00}(\Gamma_1)'$ (cf. [30, Theorem A.4]) and the term $g_2 - z$ on the right hand side of (2.4) is in $H^{1/2}_{00}(\Gamma_1)'$.

The next result ensures that the linear operator L in (2.3) is well defined for an appropriate choice of spaces.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^3$ and Γ_i ($i = 1, 2$), be defined as in Section 1. The operator defined in* (2.3) *is a linear injective bounded map*

$$L : L^{3/2}(\Gamma_2) \to H^{1/2}_{00}(\Gamma_1)'.$$

Sketch of the proof. The linearity of L is obvious. Since the boundary part Γ_2 is a 2D-manifold, it follows from the Sobolev embedding theorem [2] that the embedding $H^s(\Gamma_2) \subset L^p(\Gamma_2)$ is compact for $p < 2(1-s)^{-1}$. In particular we have $H^{1/2}(\Gamma_2) \hookrightarrow L^p(\Gamma_2)$ for $p < 4$, from what follows

$$H^{1/2}_{00} \subset H^{1/2} \subset L^3 \quad \text{and} \quad L^{3/2} = [L^3]' \subset H^{-1/2} \subset [H^{1/2}_{00}]',$$

(recall that $H^s = H^s_0$ for $s \le 1/2$). Then, given $\varphi \in L^{3/2}(\Gamma_2)$, it follows from elliptic theory [14, Ch. VII.2] that the mixed BVP in (2.1) has a unique solution $w_a \in H^1(\Omega)$ satisfying the *a priori* estimate

$$\|v_a\|_{H^1(\Omega)} \;\le\; C_1 \|\varphi\|_{H^{1/2}_{00}(\Gamma_2)'},$$

where $C_1 = C_1(\Omega, \Gamma_2) > 0$. Moreover, from the continuity of the Neumann trace operator

$$\gamma_{N,1} : H^1(\Omega) \ni v \mapsto v_\nu|_{\Gamma_1} \in H^{1/2}_{00}(\Gamma_1)',$$

it follows that

$$\|L\varphi\|_{H^{1/2}_{00}(\Gamma_1)'} \;\leq\; C_2\|v_a\|_{H^1(\Omega)} \;\leq\; C_3\|\varphi\|_{L^{3/2}(\Gamma_2)}\,,$$

proving the boundedness of L. It remains to prove the injectivity. Notice that, if $L\varphi{=}0$ then w_a in (2.1) satisfies: $\Delta w_a{=}0$ in Ω, $w_a{=}(w_a)_\nu{=}0$ on Γ_1. Thus, $\varphi{=}0$ follows from the uniqueness of weak solution for (CP_l). □

Summarizing, if the Cauchy data is given as in (2.5) and assumptions $A1)$, $A2)$ hold, then problem (CP_{nl}) can be stated in the form of equation (2.4), where L is the linear continuous and injective operator

$$L : L^{3/2}(\Gamma_2) \to H^{1/2}_{00}(\Gamma_1)', \tag{2.6}$$

defined in (2.3).

Remark 2.1. The choice of the space $L^{3/2}(\Gamma_2)$ in Proposition 2.1 is non standard. More natural would be the choice $H^{1/2}_{00}(\Gamma_2)'$. This point will become clear when we introduce the level-set method in Section 3 (Lemma 3.1).

2.3 A remark on noisy Cauchy data

If only corrupted noisy data (g_1^δ, g_2^δ) are available for problem (CP_{nl}), we assume the existence of a consistent Cauchy data (g_1, g_2) satisfying (2.5) such that

$$\|g_1 - g_1^\delta\|_{L^2(\Gamma_1)} + \|g_2 - g_2^\delta\|_{L^2(\Gamma_1)} \;\leq\; \delta\,. \tag{2.7}$$

Since z in (2.4) depends continuously on g_1 in the $H^{1/2}(\Gamma_1)$ topology, a natural question arises:

(Q) Is it possible to obtain from measured data (g_1^δ, g_2^δ) satisfying (2.7), a corresponding $z^\delta{\in}H^{1/2}_{00}(\Gamma_1)'$ such that $\|z - z^\delta\|_{H^{1/2}_{00}(\Gamma_1)'} \to 0$ as $\delta \to 0$?

The next Lemma gives a positive answer to this question.

Lemma 2.1. *Let the noisy Cauchy data be given as in* (2.7), *where* $g_1{\in}H^s(\Gamma_1)$ *for some* $s{>}1/2$. *Then* (CP_{nl}) *reduces to the operator equation*

$$L\,\varphi \;=\; g_2^\delta - z^\delta\,,$$

where the right hand side satisfies

$$\|(g_2 - z) - (g_2^\delta - z^\delta)\|_{H^{1/2}_{00}(\Gamma_1)'} \;\leq\; h(\delta)\,. \tag{2.8}$$

Here $h : \mathbb{R}^+ \to \mathbb{R}^+$ *is a function satisfying* $\lim_{\delta\to 0} h(\delta) = 0$.

Sketch of the proof. Notice that $\|g_2 - g_2^\delta\|_{H^{1/2}_{00}(\Gamma_1)'}{\leq}\|g_2 - g_2^\delta\|_{L^2(\Gamma_1)}$. The key argument to construct z^δ and the function h is the existence of a continuous smoothing operator $S : L^2(\Gamma_1) \to H^{1/2}(\Gamma_1)$ and of a function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{\delta\to 0}\mu(\delta){=}0$, such that $\|g_1 - S(g_1^\delta)\|{\leq}h(\delta)$. For details see [17, Section 4.2]. □

Lemma 2.1 will be used in Section 3 for the proof of classical results from regularization theory.

3 Level-set approximations

In this section, we investigate a level-set approach for (CP_{nl}). In what follows we shall consider the functional analytical framework for (CP_{nl}) discussed in Subsection 2.2.

3.1 Level-set approach and constrained optimization

The standard level-set approach uses the assumption that the solution $\overline{\varphi}$ of (2.4) is piecewise constant, taking only one of two possible values. For simplicity, we assume that $\overline{\varphi}$ is the characteristic function χ_D of a sub-domain $D \subset\subset \Gamma_2$. Next we introduce a function $\phi : \Gamma_2 \to \mathbb{R}$, in such a way that $\overline{\varphi}$ can be represented by a level-set of ϕ

$$\overline{\varphi}(x) = \chi_D(x) = 1 \iff x \in D = \{x \in \Gamma_2;\ \phi(x) \geq 0\}\,.$$

Under this assumption, the Cauchy problem (2.4) can be stated in the form of the least-square problem

$$\min_{\phi \in H^1(\Gamma_2)} \|L(H(\phi)) - (g_2^\delta - z^\delta)\|_Y^2\,, \tag{3.1}$$

where $Y := H_{00}^{1/2}(\Gamma_1)'$ and H the Heaviside projector.

The level-set method discussed here corresponds to a continuous evolution of the *level-set function* ϕ for an artificial time t. This evolution is motivated by the minimization of the Tikhonov functional

$$\mathcal{F}_\alpha(\phi) := \|L(H(\phi)) - (g_2^\delta - z^\delta)\|_Y^2 + \alpha\Big[\beta|H(\phi)|_{BV} + \|\phi - \phi_0\|_{H^1}^2\Big]\,, \tag{3.2}$$

based on TV-H^1-penalization for the least-square functional in (3.1). Here $\alpha > 0$ plays the role of a regularization parameter and $\beta > 0$ is a scaling factor.

Since H is discontinuous (considered as an operator from H^1 to $L^{3/2}$), one cannot prove that the Tikhonov functional in (3.2) attains a minimizer. In order to guarantee existence of minimizers for $\mathcal{F}_\alpha$, it is necessary to use a generalized minimizer concept. With this in mind we define

Definition 3.1. *Let the boundary part* $\Gamma_2 \subset \partial\Omega$ *be defined as in Section* 1.
i) *A pair of functions*

$$(\psi, \phi) \in L^\infty(\Gamma_2) \times H^1(\Gamma_2),$$

is called admissible if there exists a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ *in* $H^1(\Gamma_2)$ *such that* $\phi_k \to \phi$ *with respect to the* $L^2(\Gamma_2)$*-norm, and there exists a sequence* $\{\varepsilon_k\}_{k\in\mathbb{N}}$ *of positive numbers converging to zero such that* $H_{\varepsilon_k}(\phi_k) \to \psi$ *in* $L^{3/2}(\Gamma_2)$.[†]

[†]Given $\varepsilon > 0$, the functions H_ε are defined by

$$H_\varepsilon(t) := \begin{cases} 0, & \text{for} \quad t < -\varepsilon\,, \\ 1 + \frac{t}{\varepsilon}, & \text{for} \quad -\varepsilon \leq t \leq 0\,. \\ 1, & \text{for} \quad t \geq 0\,. \end{cases}$$

ii) *The set of admissible pairs is defined by*

$$Ad := \Big\{(\psi,\phi) \in L^\infty(\Gamma_2) \times H^1(\Gamma_2); \quad \exists \ \{\phi_k\} \in H^1 \text{ and } \{\varepsilon_k\} \in \mathbb{R}^+, \ \text{ s.t. :} \\ \lim_{k\to\infty} \varepsilon_k = 0, \ \lim_{k\to\infty} \|\phi_k - \phi\|_{L^2} = 0, \ \lim_{k\to\infty} \|H_{\varepsilon_k}(\phi_k) - \psi\|_{L^{3/2}} = 0\Big\},$$

iii) *The functional $\mathcal{F}_\alpha(\psi,\phi)$ is defined on Ad by*

$$\mathcal{F}_\alpha(\psi,\phi) := \|L\psi - (g_2^\delta - z^\delta)\|_Y^2 + \alpha\rho(\psi,\phi)\,, \tag{3.3}$$

where

$$\rho(\psi,\phi) := \inf_{\{\phi_k\},\{\varepsilon_k\}} \liminf_{k\to\infty} \big\{2\beta|H_{\varepsilon_k}(\phi_k)|_{BV} + \|\phi_k - \phi_0\|_{H^1}^2\big\}\,,$$

the infimum being taken with respect to all sequences $\{\phi_k\}_{k\in\mathbb{N}}$ and $\{\varepsilon_k\}_{k\in\mathbb{N}}$ characterizing (ψ,ϕ) as an element of Ad.

iv) *A generalized minimizer of $\mathcal{F}_\alpha(\phi)$ is a minimizer of $\mathcal{F}_\alpha(\psi,\phi)$ on Ad.*

Remark 3.1. A consequence of the definition above is the fact that $\mathcal{F}_\alpha$ is no longer considered as a functional on H^1, but as a functional defined on the closure of the graph of H, contained in $BV \times H^1$, w.r.t. the topology of $L^1 \times L^2$.

Another consequence is that the penalization term in (3.2) can now be interpreted as a functional $\rho : Ad \to \mathbb{R}^+$.

3.2 Convergence analysis

In order to prove coerciveness and weak lower semi-continuity of ρ, the assumption that L is a continuous operator on a $L^{3/2}$-space is crucial (see Proposition 2.1). These properties of ρ are the main arguments needed to prove existence of a generalized minimizer $(\overline{\psi}_\alpha, \overline{\phi}_\alpha)$ of $\mathcal{F}_\alpha$ in Ad, as we shall see next:

Lemma 3.1. *Let the boundary part $\Gamma_2 \subset \partial\Omega$ be defined as in Section* 1*. The following assertions hold true:*
i) *The semi-norm $|\cdot|_{BV}$ is weakly lower semi-continuous with respect to $L^{3/2}$-convergence;*
ii) *$BV(\Gamma_2)$ is compactly embedded in $L^{3/2}(\Gamma_2)$.*

Proof. For (i) see [3, Section 2.3.2]. For (ii) see [18, Section 5.2.1]. □

Theorem 3.2. *Let the functionals ρ, $\mathcal{F}_\alpha$ and the set Ad be defined as above. The following assertions hold true:*
i) *The functional $\rho(\psi,\phi)$ is coercive on Ad;*
ii) *The functional $\rho(\psi,\phi)$ is weakly lower semi-continuous on Ad;*
iii) *The functional $\mathcal{F}_\alpha(\psi,\phi)$ attains a minimizer on Ad.*

Proof. (i) Let $(\psi,\phi)\in Ad$. Then, there exist sequences $\{\phi_k\}_{k\in\mathbb{N}}$ and $\{\varepsilon_k\}_{k\in\mathbb{N}}$ as in Definition 3.1 (i). Thus, $\|\phi-\phi_0\|^2_{H^1} \le \liminf_k \|\phi_k-\phi_0\|^2_{H^1}$. Moreover, Lemma 3.1 implies $|\psi|_{BV} \le \liminf_k |H_{\varepsilon_k}(\phi_k)|_{BV}$. Therefore,

$$2\beta|\psi|_{BV(\Gamma_2)} + \|\phi-\phi_0\|^2_{H^1(\Gamma_2)} \le \rho(\psi,\phi)\,.$$

(ii) Follows from Lemma 3.1 and the weak lower semi-continuity of the H^1-norm.
(iii) Since $(0,-1)\in Ad$, then $Ad\neq\emptyset$ and $\inf\mathcal{F}_\alpha<\infty$. Let $(\psi_k,\phi_k)\in Ad$ be a minimizing sequence for $\mathcal{F}_\alpha$, i.e., $\mathcal{F}_\alpha(\psi_k,\phi_k)\to\inf\mathcal{F}_\alpha$ as $k\to\infty$. This fact implies the boundedness of $\rho(\psi_k,\phi_k)$. Item (i) above implies the boundedness of both sequences $\|\phi_k-\phi_0\|^2_{H^1}$ and $|\psi_k|_{BV}$. From the compactness of the embedding $H^1\hookrightarrow L^2$ and Lemma 3.1 (ii) we can extract subsequences (again denoted by $\{\psi_k\}$ and $\{\phi_k\}$) such that

$$\psi_k \rightharpoonup \psi \text{ in } BV, \qquad \psi_k\to\psi \text{ in } L^{3/2}, \qquad \phi_k\rightharpoonup\phi \text{ in } H^1, \qquad \phi_k\to\phi \text{ in } L^2,$$

for some $(\psi,\phi)\in BV(\Gamma_2)\times H^1(\Gamma_2)$. Now, arguing with (2.6) and item (ii) above, one obtains

$$\begin{aligned}\inf\mathcal{F}_\alpha &= \lim_{k\to\infty}\mathcal{F}_\alpha(\psi_k,\phi_k) = \lim_{k\to\infty}\{\|L\psi_k-(g_2^\delta-z^\delta)\|^2_Y+\alpha\rho(\psi_k,\phi_k)\}\\ &\ge \liminf_{k\to\infty}\{\|L\psi_k-(g_2^\delta-z^\delta)\|^2_Y\}+\liminf_{k\to\infty}\{\alpha\rho(\psi_k,\phi_k)\}\\ &\ge \|L\psi-(g_2^\delta-z^\delta)\|^2_Y+\alpha\rho(\psi,\phi) = \mathcal{F}_\alpha(\psi,\phi)\,.\end{aligned}$$

It remains to prove that $(\psi,\phi)\in Ad$. This is done analogously as in the final part of the proof of [20, Theorem 2.9]. ∎

Remark 3.2. If the Cauchy data (g_1,g_2) is consistent, i.e., δ=0 in (2.7), the existence of a minimum norm solution $(\psi^\dagger,\phi^\dagger)$ can be proved, i.e., an element $(\psi^\dagger,\phi^\dagger)\in Ad$, such that $L(\psi^\dagger)=g_2-z$, and

$$\rho(\psi^\dagger,\phi^\dagger)=\inf\{\rho(\psi,\phi);\ (\psi,\phi)\in Ad \text{ and } L(\psi)=g_2-z\}\,.$$

The proof of this result follows the lines of the proof of [20, Theorem 2.10].

The classical analysis of Tikhonov type regularization methods [16] can be applied to the functional $\mathcal{F}_\alpha$, as we shall see next.

Theorem 3.3 (Convergence)**.** *Let the Cauchy data (g_1,g_2) be consistent. Moreover, Let $\alpha:\mathbb{R}^+\to\mathbb{R}^+$ be a function satisfying $\lim_{\delta\to0}\alpha(\delta)=0$ and $\lim_{\delta\to0}\delta^2\alpha^{-1}(\delta)=0$. Given a sequence $\delta_k\to0$ and $\{(g_1^{\delta_k},g_2^{\delta_k})\}_k$ corresponding noisy data satisfying* (2.7)*, the generalized minimizers (ψ_k,ϕ_k) of $\mathcal{F}_{\alpha(\delta_k)}$ converge in $L^{3/2}\times L^2$ to a generalized minimizer $(\overline{\psi}_\alpha,\overline{\phi}_\alpha)\in Ad$ of $\mathcal{F}_\alpha$.*

Proof. The proof uses classical techniques from the analysis of Tikhonov regularization methods [16] and thus omitted. ∎

3.3 Stabilized approximation

We conclude this section with a result that guarantees the efficiency of a numerical approximation scheme for solving (2.4). Indeed, we prove that the generalized minimizers of the functional $\mathcal{F}_\alpha$ defined in (3.3) can be approximated by minimizers of the stabilized functional

$$\mathcal{F}_{\alpha,\varepsilon}(\phi) := \|L(H_\varepsilon(\phi)) - (g_2 - z)\|_Y^2 + \alpha\Big[\beta|H_\varepsilon(\phi)|_{BV} + \|\phi - \phi_0\|_{H^1}^2\Big], \tag{3.4}$$

in the following sense:

Theorem 3.4. *If $\phi_{\alpha,\varepsilon}$ are minimizers of $\mathcal{F}_{\alpha,\varepsilon}$ then, given a sequence $\varepsilon_k \to 0^+$, there exists a subsequence $(H(\phi_{\alpha,\varepsilon}), \phi_{\alpha,\varepsilon})$ converging in $L^{3/2}(\Gamma_2) \times L^2(\Gamma_2)$ and the limit minimizes $\mathcal{F}_{\alpha,}$ in Ad.*

Sketch of the proof. The minimizers $\phi_{\alpha,k}$ of $\mathcal{F}_{\alpha,\varepsilon_k}$ are uniformly bounded in H^1. Moreover, $H_{\varepsilon_k}(\phi_{\alpha,k})$ is uniformly bounded in BV. Then (up to subsequences) these sequences converge strongly in $L^{3/2} \times L^2$ to a limit $(\widetilde{\psi}, \widetilde{\phi}) \in BV \times H^1$ (notice that from this convergence follows $(\widetilde{\psi}, \widetilde{\phi}) \in Ad$). In order to prove that $(\widetilde{\psi}, \widetilde{\phi})$ minimizes $\mathcal{F}_{\alpha,}$, one argues with (2.6) and Theorem 3.2. ∎

The existence of minimizers of $\mathcal{F}_{\alpha,\varepsilon}$ in $H^1(\Gamma_2)$ still has to be cleared:

Lemma 3.2. *For any $\phi_0 \in H^1(\Gamma_2)$ the functional $\mathcal{F}_{\alpha,\varepsilon}$ in* (3.4) *attains a minimizer.*

Proof. Notice that a minimizing sequence $\{\phi_k\}$ for $\mathcal{F}_{\alpha,\varepsilon}$ is bounded in $H^1(\Gamma_2)$. Therefore, up to a subsequence, we have

$$\phi_k \rightharpoonup \phi, \quad \text{in } H^1, \quad \text{and} \quad \phi_k \to \phi, \quad \text{in } L^2,$$

for some $\phi_{\alpha,\varepsilon} \in H^1(\Gamma_2)$. On the other hand,

$$\|H_\varepsilon(\phi_k) - H_\varepsilon(\phi_{\alpha,\varepsilon})\|_{L^{3/2}(\Gamma_2)} \ \le\ \varepsilon^{-1}\,\mathrm{meas}(\Gamma_2)^{1/6}\,\|\phi_k - \phi_{\alpha,\varepsilon}\|_{L^2(\Gamma_2)} \to 0\,,$$

and from Lemma 3.1 (i) follows

$$|H_\varepsilon(\phi_{\alpha,\varepsilon})|_{BV} \le \liminf_k |H_\varepsilon(\phi_k)|_{BV}.$$

The lemma follows now from (2.6) and the weak lower semi-continuity of the H^1-norm. ∎

This relation between the minimizers of $\mathcal{F}_\alpha$ and $\mathcal{F}_{\alpha,\varepsilon}$ is the starting point for the derivation of a numerical method. This is our main goal in the next section (see Subsection 4.2).

4 Numerical realization and experiments

In this section we illustrate the usability of our approach by numerical experiments. After introducing our model problem, we shortly discuss its discretization and then report on details concerning the implementation of the level-set method in Section 3. We conclude with presenting results of some numerical tests.

4.1 The model problem, its discretization and linearization

Let us consider the following nonlinear Cauchy problem: For $a>0$, let

$$\Omega := (0,1) \times (0,1) \times (0,a),$$

and the boundary $\partial\Omega$ be composed of three parts $\partial\Omega = \overline{\Gamma_0 \cup \Gamma_L \cup \Gamma_a}$, with

$$\Gamma_0 := (0,1)^2 \times \{0\}, \quad \Gamma_a := (0,1)^2 \times \{a\}, \quad \text{and} \quad \Gamma_L := \partial\Omega \setminus \overline{\Gamma_0 \cup \Gamma_a}\,.$$

We consider the solution of the Cauchy problem $\mathcal{L}\varphi = g^\delta$ with forward operator $\mathcal{L}$ defined by $\mathcal{L}(\varphi) := q(u)u_\nu|_{\Gamma_0}$ and u being the solution of the nonlinear BVP

$$-\nabla \cdot (q(u)\nabla u) = 0 \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \Gamma_0 \cup \Gamma_L, \qquad q(u)u_\nu = \varphi \ \text{ on } \Gamma_a, \tag{4.1}$$

with nonlinear coefficient $q(u) = 1 + u^2$.

The choice u=0 on the lateral boundary Γ_L above is done to simplify the calculations (see Remark 1.1). For solution of the nonlinear mixed BVP (4.1) we consider a finite difference discretization. In order to cope with the nonlinearity, we propose a simple fix-point iteration: let $P(u)$ denote the stiffness Matrix of the operator $-\nabla \cdot (q(u)\nabla)$ and b denote the right hand side of the discretization resembling the non-homogeneous Neumann data. For computing a sequence of iterates we use the schema

$$P(u_n)u_{n+1} = b\,, \tag{4.2}$$

which is stopped as soon as the norm of the residual

$$r_k = b - P(u_n)u_n,$$

has reached a required tolerance of 10^{-8}. In each step of the iteration, the linearized systems (4.2) are solved by a preconditioned conjugate gradient method. Throughout our numerical tests, the fix-point iteration converged within less then 10 iteration to the required tolerance.

Following the linearization procedure outlined in Subsection 2.1 we first transform the nonlinear Cauchy problem (4.1) into a linear one by setting

$$U(x,y,z) = Q(u(x,y,z)),$$

with

$$Q(u) = \int_0^u q(v)dv = u + \frac{1}{3}u^3,$$

being the primitive of q. The linearized Cauchy problem then reads

$$L\varphi = g^\delta, \tag{4.3}$$

(notice that z in (2.4) vanishes due to the particular choice of g_1=f=0 and $q(u)$=$1+u^2$) where the operator L is now defined by

$$L\varphi = U_\nu,$$

and U solves the system

$$-\Delta U = 0 \ \text{ in } \Omega, \qquad U = 0 \ \text{ on } \Gamma_0 \cup \Gamma_L, \qquad U_\nu = \varphi \ \text{ on } \Gamma_a.$$

4.2 Implementation of the level-set approach

Let us shortly discuss how minimizers of the functional $\mathcal{F}_\alpha$ can be found numerically. As shown in Subsection 3.3, generalized minimizers of (3.3) can be approximated by minimizers of the stabilized functional $\mathcal{F}_{\alpha,\varepsilon}$ defined in (3.4). In our numerical experiments, we choose

$$H_\varepsilon(x) = 2^{-1}\left[\mathrm{erf}(\frac{x}{\varepsilon}) + 1\right],$$

where

$$\mathrm{erf}(x) := 2/\sqrt{\pi}\int_0^x \exp(-t^2)dt,$$

denotes the error function.‡ Let $\phi_{\alpha,\varepsilon_n}$ be a minimizer of $\mathcal{F}_{\alpha,\varepsilon_n}$ for a sequence $\varepsilon_n \to 0^+$. Then one can find a subsequence $(H_\epsilon(\phi_{\alpha,\varepsilon_n}), \phi_{\alpha,\varepsilon_n})$ converging in $L^{3/2}(\Gamma_2) \times L^2(\Gamma_2)$ and the limit minimizes $\mathcal{F}_\alpha$ in Ad. In the sequel we only discuss how to find minimizers of the stabilized functional.

For the derivation of a method, let us start from the necessary first order conditions for a minimum of (3.4)

$$\begin{aligned} 0 =& - H'_\varepsilon(\phi)L(H_\varepsilon(\phi))^*\left[L\big(H_\varepsilon(\phi)\big) - g^\delta\right] \\ &+ \alpha\Big[\beta H'_\varepsilon(\phi)\nabla\cdot\frac{\nabla H_\varepsilon(\phi)}{|\nabla H_\varepsilon(\phi)|} + (I-\Delta)(\phi-\phi_0)\Big] \\ =:& \,\mathcal{R}_{\alpha,\varepsilon}(\phi)\,. \end{aligned} \tag{4.4}$$

Here L^* denotes the adjoint of the operator L with respect to the $H^{1/2} - H^{-1/2}$ duality pairing.

For finding a solution of (4.4) a simple fixed point iteration was proposed in [20]. Here, we use a different approach based on the ideas of the Gauß-Newton method. For β=0, we define the update $\delta\phi_k = \phi_{k+1} - \phi_k$ by

$$[H'_\varepsilon(\phi_k)L(H_\varepsilon(\phi_k))^*L(H_\varepsilon(\phi_k))H'_\varepsilon(\phi_k) + \alpha(I-\Delta)]\delta\phi_k = -\mathcal{R}_{\alpha,\varepsilon}(\phi_k). \tag{4.5}$$

In case $\beta \neq 0$, one can add an additional term accounting for the BV regularization to the Gauß-Newton matrix. In our numerical experiments however, we could always set β=0. After space discretization, the linear systems (4.5) can be solved, e.g., by the conjugate gradient method.

The iteration (4.5) can be stopped as soon as the norm of $\mathcal{R}_{\alpha,\varepsilon}$ is sufficiently small. The parameter α can then be chosen by a discrepancy principle, i.e., one solves the minimization problem (3.4) for a sequence of decreasing values α_n. Arguing that the minima for different values of α will be close together and that the residual $\mathcal{R}_{\alpha_{n+1},\varepsilon}$ will be reduced sufficiently by only one step of the iteration when started at the minimizer for α_n, one can alternatively replace the two nested iterations of determining the

‡This definition of the operator H_ε is slightly different from the one given in Subsection 3.1. It is worth noticing that the theoretical results derived in Section 3 also hold with this definition.

optimal α and finding the minimizer by a single iteration. The resulting scheme corresponds to the iteratively regularized Gauß-Newton method [16]. For our test problem, we obtain the following algorithm:

Algorithm

Step 1: choose $\phi_0 \in H^1(\Gamma_2)$ and set $k=0$;
Step 2: loop

- evaluate the residual $r_k := L(H_\varepsilon(\phi_k)) - g^\delta$; the action of L on $H_\varepsilon(\phi_k)$ is given by $L(H_\varepsilon(\phi_k)) = (U_k)_\nu|_{\Gamma_0}$, where U_k solves

$$\Delta U_k = 0 \text{ in } \Omega \qquad (U_k)|_{\Gamma_0} = 0, \qquad (U_k)_\nu|_{\Gamma_a} = H_\varepsilon(\phi_k), \qquad U_k|_{\Gamma_L} = 0;$$

- if $\|r_k\| < \tau\delta$ stop;
- evaluate $W_k := L(H_\varepsilon(\phi_k))^* r_k$; where $L(H_\varepsilon(\phi_k))^* r_k = -V_k|_{\Gamma_a}$, where V_k solves the problem

$$\Delta V_k = 0 \text{ in } \Omega \qquad V_k|_{\Gamma_0} = r_k, \qquad (V_k)_\nu|_{\Gamma_a} = 0, \qquad V_k|_{\Gamma_L} = 0;$$

- set $Z_k := H'_\varepsilon(\phi_k) W_k + \alpha_k (I - \Delta)\phi_k$;
- compute the update $\delta\phi_k$ by solving the linearized system

$$[A_k^* A_k + \alpha_k (I - \Delta)]\delta\phi_k = -Z_k, \qquad A_k := H'_\varepsilon(\phi_k) L(H_\varepsilon(\phi_k)),$$

  with the method of conjugate gradients, where the action of the operators L and L^* is defined as above;
- update the level-set function $\phi_{k+1} = \phi_k + \delta\phi_k$ and set $k = k+1$.

Note that calculating $Z_k = \mathcal{R}_{\alpha,\varepsilon}(\phi_k)$ as well as each step of the conjugate gradient method requires the solution of two mixed BVPs.

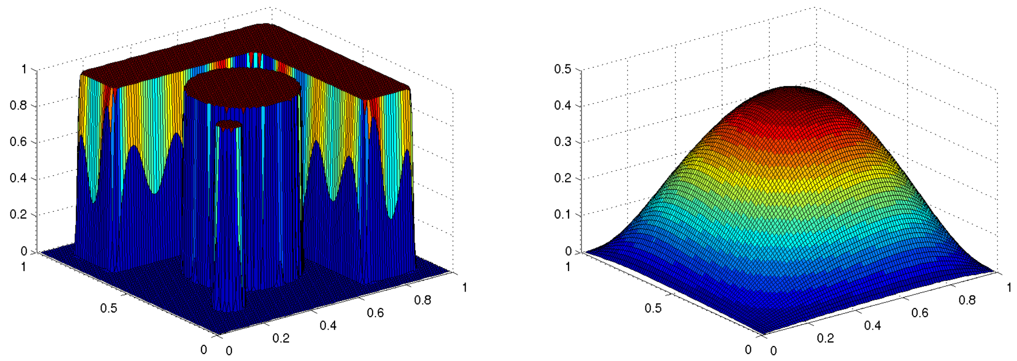
4.3 First numerical experiment: "almost" exact data

As data for the linearized Cauchy problem we consider the ones simulated by the finite difference method for the nonlinear problem. This justifies the title of this subsection.

In order to estimate the discretization error, we solve the linearized forward problem with the finite difference and the Fourier transform based method and take $\delta := \|g_{DFT} - g_{FD}\|$ as a measure for the noise level. Here, g_{DFT} and g_{FD} denote the data generated by the two different methods. We also compared the data simulated by the finite difference method with the ones generated by the DFT method with N replaced by $2N$, which gave similar values for the noise level δ. The data are then additionally perturbed by random noise of size δ.

In this numerical test, we set $a = 1/\pi$ in the model problem of Subsection 4.1 and aim to reconstruct a binary valued coefficient (the unknown Neumann data) possessing several – large/small, convex/concave, round/edgy – features, cf. Fig. 1 (a). The corresponding Cauchy data is $(0, g)$, where g is the measured heat flux (Neumann data on Γ_0) depicted in Fig. 1 (b).

The data g are generated by a finite difference solution of the nonlinear Cauchy problem (4.1) on a $100 \times 100 \times 100$ grid. For solution of the inverse problem, we apply the Gauß-Newton level-set method introduced in Subsection 4.2.



(a) (b)

Figure 1: Setup of the first numerical experiment: (a) True solution (Neumann data on Γ_a); (b) Measured heat flux (Neumann data on Γ_0).

Throughout our numerical experiments we use ε=0.1. As initial level-set function we choose the parabola

$$\phi_0(x, y) = 0.1^2 - (x - 0.5)^2 + (y - 0.5)^2.$$

The initial zero level-set hence is a circle with center $(0.5, 0.5)$ and radius 0.1 (see top-left picture in Fig. 2). The evolution of the 0.5-level-sets of the iterates $H_\varepsilon(\phi_k)$ is shown in Fig. 2. The corresponding evolution of the level-set functions ϕ_k is depicted in Fig. 3.

We conclude this first experiment presenting a comparison between our method and the iterated Tikhonov method. Its worth noticing that the iterated Tikhonov method corresponds to the choice $H_\varepsilon(x) = x$, i.e. no projection. The approximation obtained after 100 iterative steps and its corresponding 0.5-level-set is shown in Fig. 4. Comparing the results in the bottom-right picture in Fig. 2 with the results in Fig. 4 one notices that method is clearly more advantageous.

4.4 Second numerical experiment: noisy data

In this second experiment we consider the same basic setup as in Subsection 4.3. This time however, we artificially introduce noise to the Cauchy data g shown in Fig. 1 (b) and aim to solve (4.3) with Cauchy data $(0, g^\delta)$.

In a first test, 1% random noise is added to the 'exact' data g and the iterative scheme described in Subsection 4.2 is applied. In a second test, the data is perturbed

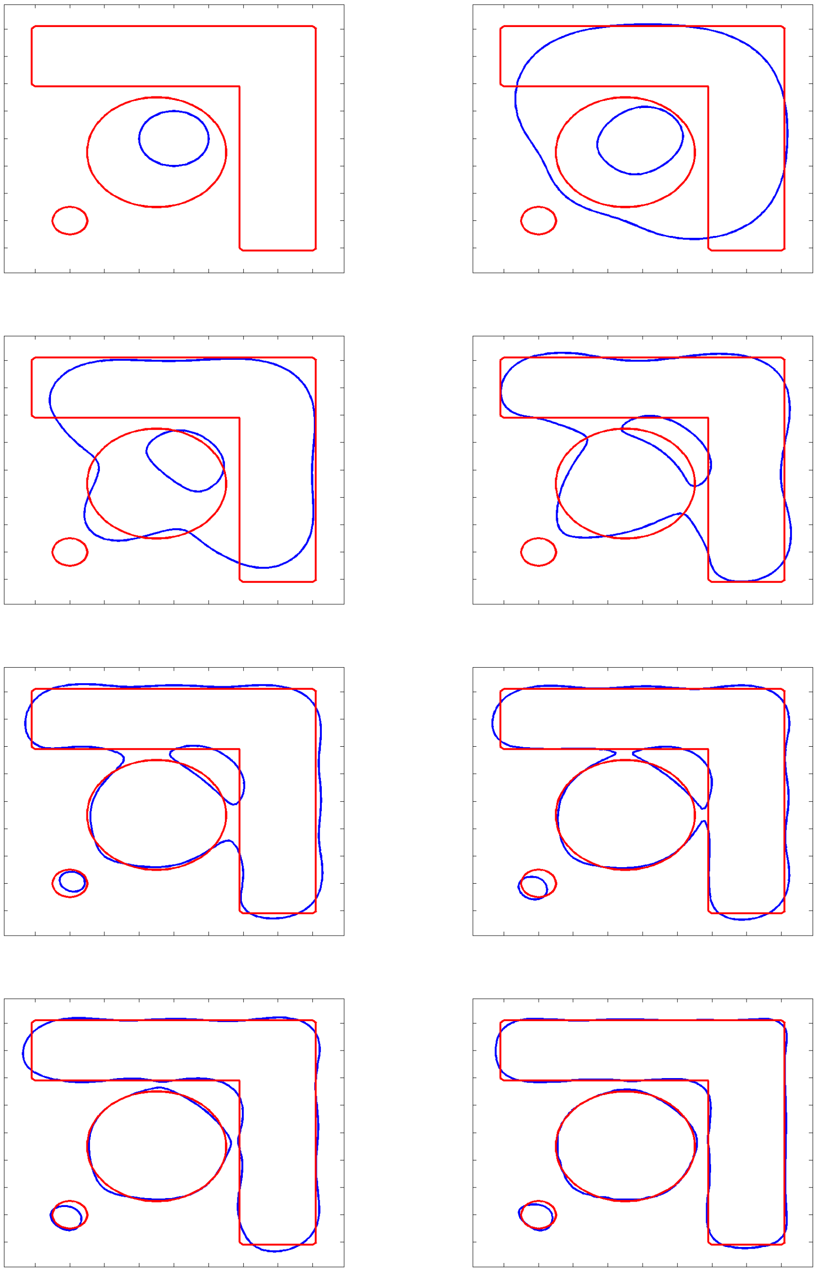


Figure 2: First numerical experiment: 0.5-level-sets of the iterates $H_\varepsilon(\phi_k)$ for k=0,1,2,4,6,8,10,13 (top-left to bottom-right).

with 5% random noise. The stop criteria is reached after a small number of iterations. The obtained results are depicted in Fig. 5. As expected, the reconstruction becomes more and more unstable as the noise level increases. Fine structures can no longer

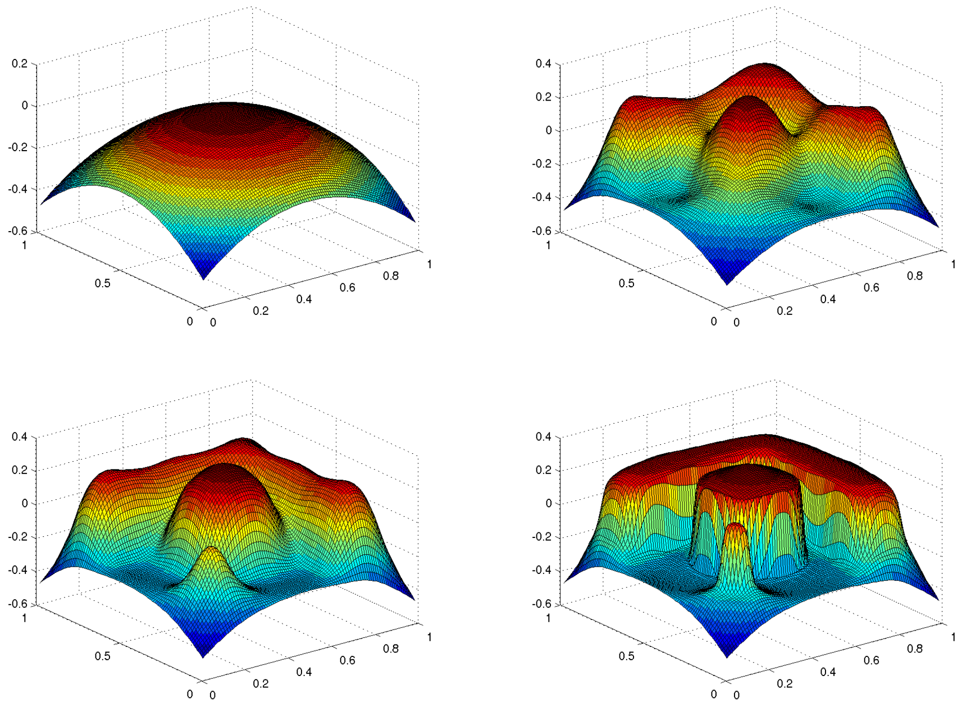


Figure 3: First numerical experiment: Level-set function ϕ_k for k=0,4,8,13.

be identified as the noise level increases. However, some large structures and basic features of the solution can still be recovered even in the presence of high levels of noise, what is unusual for exponentially ill-posed problems of this kind.

It is worth noticing that the preconditioning strategy tremendously improves the performance of the level-set method introduced in [20]. In [20,31], this level-set method was implemented for exponentially ill-posed problems and several hundreds of iterations were needed to reach the stop criteria, while here only a few iterations are required. For further discussion on this issue we refer to [15].

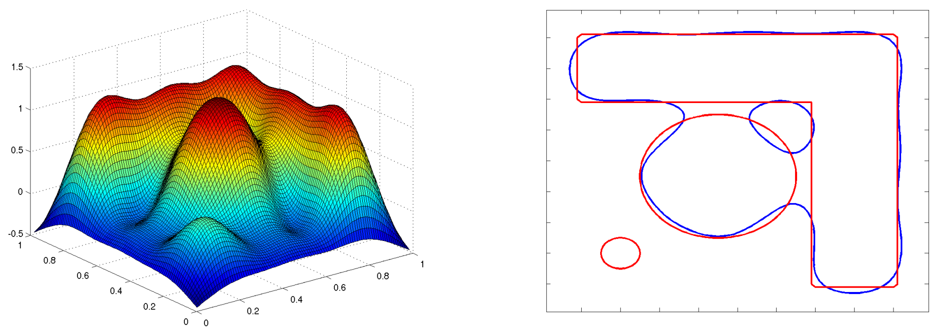


Figure 4: First numerical experiment: Results obtained with the iterated Tikhonov-Morosov method after 100 iterations.

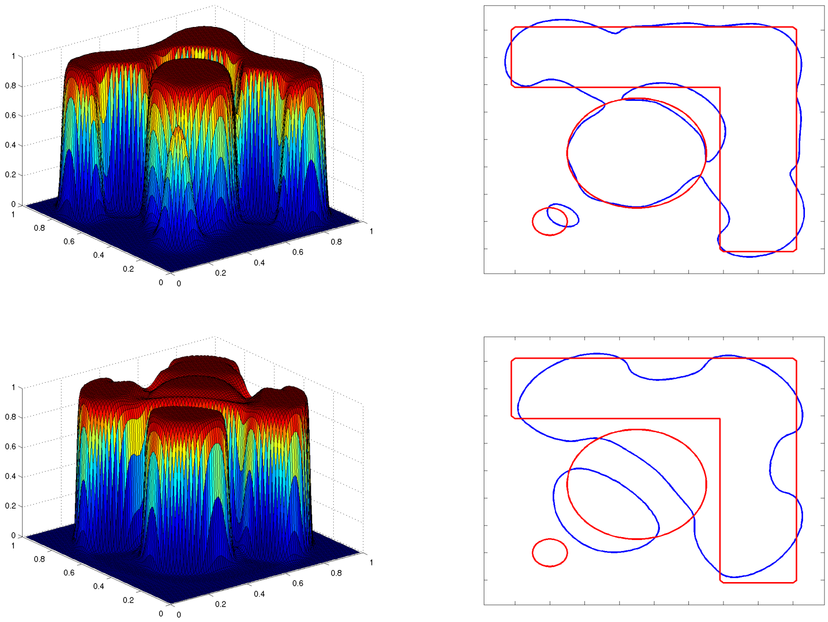


Figure 5: Second numerical experiment: Top pictures show reconstruction results for 1% random white noise and bottom pictures for 5% noise.

5 Conclusions

In this article a Tikhonov regularization method based on a level-set approach for solving nonlinear elliptic Cauchy problems in 3D is considered. A framework for the level-set approach is established and convergence analysis for the Tikhonov method is developed (convergence and stability results).

Further we discuss the numerical realization of a related level-set method. Different numerical experiments illustrate relevant features of the method, like: rates of convergence, adaptability to identify non-connected inclusions, robustness with respect to noise.

The numerical method analyzed in this article can be extended in a straightforward way to arbitrary elliptic Cauchy problems possessing a solution with similar structure, i.e. whenever the assumption that $q(u)u_\nu|_{\Gamma_2}$ is a piecewise constant function assuming one of only two possible values (not necessarily zero and one) is valid. The proposed method relates to evolution flows of Hamilton-Jacobi type.

The method derived in this article can be extended to the case where the unknown parameter is a piecewise constant function assuming a finite number of constant values. This is possible in different ways, e.g., by utilizing multiple levelset functions [35] or by using a more general function H_ε.

If more general than piecewise constant solutions are sought for, one can omit the H^1 regularization of the levelset function and search for $x = H_\varepsilon(\phi)$ directly; in this case the method collapses with Tikhonov's method with BV regularization [1]. The use of the reparametrization however allows us to incorporate the additional a-priori knowledge here, that the solution is piecewise constant. In case the solution of the inverse problem is unique, which was the case in our numerical example, both methods converge to the same solution.

Most of the analysis presented in Section 3 was formulated in [20] for operators L continuous in the L^1-topology. The results in [20] cannot be directly applied to (CP_l), since L^1 is not embedded in $[H_{00}^{1/2}]'$ (see Subsection 2.2). We used Lemma 3.1 to improve the convergence results in [20] to operators that are continuous in the $L^{3/2}$-topology. This allowed the application and analysis of this level-set type method for (CP_{nl}). It is worth noticing that Lemma 3.1 still holds for L^p with $1\le p<2$. Therefore, the analytical results in Section 3 can be extended to any linear inverse problem modeled by operators continuous in the L^p-topology.

Acknowledgments

A. Leitão acknowledges support from the Brazilian National Research Council (CNPq), grants 306020/2006-8, 474593/2007-0 and from the Alexander von Humboldt Foundation (AvH).

Appendix: An existence result for (CP_{nl})

Proposition 5.1. *Let $\Omega\subset\mathbb{R}^3$ and Γ_i be defined as in Section 1. Moreover, let the operator $\mathcal{P}$ be defined as in* (1.1)*, where $q:\mathbb{R}\to[q_{min},q_{max}]\subset(0,\infty)$ is a C^∞–function. There exists a dense subset $M\subset H^{1/2}(\Gamma_1)\times H_{00}^{1/2}(\Gamma_1)'$ such that the nonlinear problem*

$$\begin{cases} \mathcal{P}(u)=0, & in\ \Omega,\\ u=g_1, & on\ \Gamma_1,\\ q(u)u_\nu=g_2, & on\ \Gamma_1, \end{cases} \tag{CP_{nl}}$$

is consistent for $(g_1,g_2)\in M$. Moreover, the complement of M is also a dense subset of $H^{1/2}(\Gamma_1)\times H_{00}^{1/2}(\Gamma_1)'$.

Proof. From Subsection 2.1 we know that problem (CP_{nl}) admits a solution u for the Cauchy data (g_1,g_2) iff the BVP

$$\begin{cases} -\Delta U=0, & \text{in } \Omega,\\ U=Q(g_1), & \text{on } \Gamma_1,\\ U_\nu=g_2, & \text{on } \Gamma_1, \end{cases} \tag{CP_l}$$

admits a solution U for the Cauchy data $(Q(g_1),g_2)$ and, in this case, $U=Q(u)$. Since Q is continuous and invertible, it is enough to prove that (CP_l) is solvable for a dense subset of $H^{1/2}(\Gamma_1)\times H_{00}^{1/2}(\Gamma_1)'$. Moreover, due to the superposition principle, it is enough to prove that:

i) For a fixed $g_1 \in H^{1/2}(\Gamma_1)$, the set of data g_2 for which (CP_l) admits a solution is dense in $[H_{00}^{1/2}(\Gamma_1)]'$;

ii) For a fixed $g_2 \in [H_{00}^{1/2}(\Gamma_1)]'$, the set of data g_1 for which (CP_l) admits a solution is dense in $H^{1/2}(\Gamma_1)$.

We prove only first case, the proof of the second one being analogous. Let us assume (without loss of generality) that $g_1 = 0$. We define

$$\mathcal{M} \ := \ \{g_2 \in [H_{00}^{1/2}(\Gamma_1)]';\ (0, g_2) \text{ is consistent for } (CP_l)\}\,. \tag{5.1}$$

If $\mathcal{M}$ were not dense in $[H_{00}^{1/2}(\Gamma_1)]'$, the Hahn-Banach's theorem would guarantee the existence a nonzero continuous linear functional $\Lambda \in H_{00}^{1/2}(\Gamma_1)$ such that $\langle \Lambda, g_2 \rangle = 0$ for all $g_2 \in \mathcal{M}$.[§] Therefore, the mixed BVP

$$\begin{cases} -\Delta v = 0, & \text{in } \Omega, \\ v = \Lambda, & \text{on } \Gamma_1, \\ v_\nu = 0, & \text{on } \Gamma_2, \end{cases}$$

has a unique solution $v \in H^1(\Omega)$. Likewise, given an arbitrary test function $\vartheta \in C_0^\infty(\Gamma_2)$, the mixed BVP

$$\begin{cases} -\Delta w = 0, & \text{in } \Omega, \\ w = 0, & \text{on } \Gamma_1, \\ w_\nu = \vartheta, & \text{on } \Gamma_2, \end{cases}$$

has a unique solution $w \in H^1(\Omega)$. Since $w_\nu|_{\Gamma_1} \in \mathcal{M}$, it follows from integration by parts

$$0 \ = \ \int_\Omega \Delta w\, v - \int_\Omega w\, \Delta v \ = \ \int_{\Gamma_1 \cup \Gamma_2} w_\nu\, v \ = \ \int_{\Gamma_1} \Lambda\, w_\nu + \int_{\Gamma_2} \vartheta\, v \ = \ \int_{\Gamma_2} \vartheta\, v\,.$$

Since $\vartheta \in C_0^\infty(\Gamma_2)$ is arbitrary, $v|_{\Gamma_2} = 0$ follows. Therefore, $v|_{\Gamma_2} = v_\nu|_{\Gamma_2} = 0$ and $-\Delta v = 0$ in Ω. From the uniqueness of solutions of (linear) Cauchy problems we conclude that $v = 0$ in Ω, contradicting the choice of Λ.

It remains to prove that the complement of M is also a dense subset of $H^{1/2}(\Gamma_1) \times H_{00}^{1/2}(\Gamma_1)'$. It is enough to consider two cases:

i) For a fixed $g_1 \in H^{1/2}(\Gamma_1)$, the set of data g_2 for which (CP_l) does not admit a solution is dense in $[H_{00}^{1/2}(\Gamma_1)]'$;

ii) For a fixed $g_2 \in [H_{00}^{1/2}(\Gamma_1)]'$, the set of data g_1 for which (CP_l) does not admit a solution is dense in $H^{1/2}(\Gamma_1)$.

As before, we prove only the first case. Let us assume (without loss of generality) that $g_1 = 0$, and let $\mathcal{M}$ be the set defined in (5.1). Notice that $\mathcal{M}$ is the range of the linear continuous trace operator

$$\gamma_n : \mathcal{H} = \{w \in H_0^1(\Omega \cup \Gamma_2)\ ,\ \Delta w = 0\} \to [H_{00}^{1/2}(\Gamma_1)]',$$

[§]Here $\langle \cdot, \cdot \rangle$ denotes the canonical duality paring between $H_{00}^{1/2}(\Gamma_1)$ and $[H_{00}^{1/2}(\Gamma_1)]'$.

defined by $\gamma_n(w) := w_\nu|_{\Gamma_1}$. Notice that $\mathcal{H}$ is a Hilbert space when considered with the H^1 inner product.

Assume by contradiction that $\mathcal{M}^C$ is not dense in $[H_{00}^{1/2}(\Gamma_1)]'$. Thus, there must exist a $\rho>0$ such that $B_\rho(0)\subset\mathcal{M}$. Since $\mathcal{M}$ is a linear space, we conclude that $\mathcal{M}$ is open. However, since $\mathcal{M}$ is dense, we must have $\mathcal{M} = [H_{00}^{1/2}(\Gamma_1)]'$, and γ_n is onto. However, γ_n is injective (due to the uniqueness of solutions for (CPl)). Therefore, it follows from the open mapping theorem that γ_n^{-1} is bounded, contradicting the ill-posedness of (CPl) [6]. $\square$

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