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CONVERGENCE ANALYSIS OF SPECTRAL METHODS FOR INTEGRO-DIFFERENTIAL EQUATIONS WITH VANISHING PROPORTIONAL DELAYS*

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Abstract

We describe the application of the spectral method to delay integro-differential equations with proportional delays. It is shown that the resulting numerical solutions exhibit the spectral convergence order. Extensions to equations with more general (nonlinear) vanishing delays are also discussed.

Mathematics subject classification: 65R20, 34K28.
Key words: Delay integro-differential equations, Proportional delays, Spectral methods, Convergence analysis.

1. Introduction

We consider the delay integro-differential equation of the form

$$\begin{aligned} y'(t) =& a(t)y(t) + b(t)y(qt) + \int_0^t K_0(t-s)y(s)ds \\ &+ \int_0^{qt} K_1(t-s)y(s)ds + g(t), \quad t \in I := [0,T], \qquad (1.1\text{a}) \\ y(0) =& y_0, \qquad (1.1\text{b}) \end{aligned}$$

where $0 < q < 1$, $a(t)$ and $b(t)$ are smooth functions on $I := [0,T]$ and K_0, $K_1 \in C(I)$. The special case corresponding to $K_0(t,s) \equiv 0$, $K_1(t,s) \equiv 0$, $g(t) = 0$, yields the (variable coefficient) *pantograph equation*. Results on the existence, uniqueness and regularity of solutions may be found in [3-6].

It has been shown in [6] that the approximation of the solution of (1.1) by collocation using piecewise polynomials of degree $m \geq 1$ and uniform meshes does not lead to the classical $\mathcal{O}(h^{2m})$- superconvergence at the mesh points when collocation is at the Gauss points; for $m \geq 2$ the optimal order is only $m+2$. Thus, it is of interest to investigate if the numerical solution of (1.1) by spectral methods leads to a higher (exponential) convergence order.

It will be shown that the results on the exponential order of convergence of the spectral method for the pantograph DDE [7] and for Volterra type integral equations [11, 12] remain valid for pantograph-type integro-differential equation (1.1).

In Section 2 we describe the spectral method for the integro-delay differential equation. This is followed, in Section 3, by corresponding results on the attainable order of convergence of these spectral methods and by remarks (Section 4) on their extension to equations with nonlinear vanishing delays. Section 5 is used to illustrate the convergence results by numerical examples.

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2. Spectral Method

Let $\{t_k\}_{k=0}^{N}$ be the set of the $(N+1)$ Gauss-Legendre, or Gauss-Radau, or Gauss-Lobatto points in $[-1,1]$ and denote by $\mathcal{P}_N$ the space of real polynomials of degree not exceeding N. Integrating (1.1a) from $[0,t_i]$ gives

$$\begin{aligned} y(t_i) =& y_0 + \int_0^{t_i} a(s)y(s)ds + \int_0^{t_i} b(s)y(qs)ds + \int_0^{t_i}\left(\int_0^s K_0(s-v)y(v)dv\right)ds \\ &+ \int_0^{t_i}\left(\int_0^{qs} K_1(s-v)y(v)dv\right)ds + \int_0^{t_i} g(s)ds. \end{aligned} \tag{2.1}$$

We will describe and analyzed spectral methods on the standard interval $[-1,1]$. Hence using for t_i $(i=1,\cdots,N)$ the linear transformation $s = \frac{t_i}{2}\theta + \frac{t_i}{2}$, we get

$$\begin{aligned} y(t_i) =& y_0 + \frac{t_i}{2}\int_{-1}^{1} a\Big(\frac{t_i}{2}(\theta+1)\Big)y\Big(\frac{t_i}{2}(\theta+1)\Big)d\theta + \frac{t_i}{2}\int_{-1}^{1} b\Big(\frac{t_i}{2}(\theta+1)\Big)y\Big(\frac{qt_i}{2}(\theta+1)\Big)d\theta \\ &+ \frac{t_i}{2}\int_{-1}^{1}\left(\int_0^{\frac{t_i}{2}(\theta+1)} K_0\Big(\frac{t_i}{2}(\theta+1)-v\Big)y(v)dv\right)d\theta \\ &+ \frac{t_i}{2}\int_{-1}^{1}\left(\int_0^{\frac{qt_i}{2}(\theta+1)} K_1\Big(\frac{t_i}{2}(\theta+1)-v\Big)y(v)dv\right)d\theta + G(t_i), \end{aligned} \tag{2.2}$$

where

$$G(t_i) := \int_0^{t_i} g(s)ds.$$

Using the $(N+1)$-point Gauss-Legendre, or Gauss-Radau, or Gauss-Lobatto quadrature formula relative to the Legendre weight leads to the semi-discretised spectral equations

$$\begin{aligned} y(t_i) \approx& y_0 + \frac{t_i}{2}\sum_{k=0}^{N} a(\tau_{ik})y(\tau_{ik})\omega_k + \frac{t_i}{2}\sum_{k=0}^{N} b(\tau_{ik})y(q\tau_{ik})\omega_k \\ &+ \frac{t_i}{2}\sum_{k=0}^{N}\left(\int_0^{\tau_{ik}} K_0(\tau_{ik}-v)y(v)dv\right)\omega_k \\ &+ \frac{t_i}{2}\sum_{k=0}^{N}\left(\int_0^{q\tau_{ik}} K_1(\tau_{ik}-v)y(v)dv\right)\omega_k + G(t_i), \end{aligned} \tag{2.3}$$

which we rewrite in the form

$$\begin{aligned} y(t_i) \approx& y_0 + \frac{t_i}{2}\sum_{k=0}^{N} a(\tau_{ik})y(\tau_{ik})\omega_k + \frac{t_i}{2}\sum_{k=0}^{N} b(\tau_{ik})y(q\tau_{ik})\omega_k \\ &+ \frac{t_i}{2}\sum_{k=0}^{N}\left(\frac{\tau_{ik}}{2}\int_{-1}^{1} K_0\Big(\tau_{ik}-\frac{\tau_{ik}}{2}(\theta+1)\Big)y\Big(\frac{\tau_{ik}}{2}(\theta+1)\Big)d\theta\right)\omega_k \\ &+ \frac{t_i}{2}\sum_{k=0}^{N}\left(\frac{q\tau_{ik}}{2}\int_{-1}^{1} K_1\Big(\tau_{ik}-\frac{q\tau_{ik}}{2}(\theta+1)\Big)y\Big(\frac{q\tau_{ik}}{2}(\theta+1)\Big)d\theta\right)\omega_k + G(t_i), \end{aligned} \tag{2.4}$$

where $\tau_{ik} := \frac{t_i}{2}(\theta_k+1)$ and $i=0,\cdots,N$.

Again using the $(N+1)$-point Gauss-Legendre, or Gauss-Radau, or Gauss-Lobatto quadrature formula relative to the Legendre weight for the integral term in (2.4) we get

$$\begin{aligned} y(t_i) \approx & y_0 + \frac{t_i}{2}\sum_{k=0}^{N} a(\tau_{ik})y(\tau_{ik})\omega_k + \frac{t_i}{2}\sum_{k=0}^{N} b(\tau_{ik})y(q\tau_{ik})\omega_k \\ & + \frac{t_i}{2}\sum_{k=0}^{N}\left(\frac{\tau_{ik}}{2}\sum_{n=0}^{N} K_0\Big(\frac{\tau_{ik}}{2}(2-\theta_n)\Big)y\Big(\frac{\tau_{ik}}{2}(\theta_n+1)\Big)\omega_n\right)\omega_k \\ & + \frac{t_i}{2}\sum_{k=0}^{N}\left(\frac{q\tau_{ik}}{2}\sum_{n=0}^{N} K_1\Big(\frac{\tau_{ik}}{2}(2-q\theta_n-q)\Big)y\Big(\frac{q\tau_{ik}}{2}(\theta_n+1)\Big)\omega_n\right)\omega_k + G(t_i). \end{aligned} \tag{2.5}$$

Let $Y_j \approx y(t_j)$ and suppose that the spectral approximation $Y(t)$ has the form

$$y(t) \approx Y(t) := \sum_{j=0}^{N} y(t_j)F_j(t), \qquad t \in [0,T], \tag{2.6}$$

where $F_j(t)$ is the standard Lagrange interpolation polynomial associated with the Gauss-Legendre, or Gauss-Radau, or Gauss-Lobatto points $\{t_k\}_{k=0}^N$. The efficient way to compute $F_j(s)$ is to express it in terms of the Legendre functions ([3, 9]). It follows from (2.3) that the numerical scheme for solving (1.1a) is given by

$$\begin{aligned} Y(t_i) = & y_0 + \frac{t_i}{2}\sum_{j=0}^{N} Y_j \left(\sum_{k=0}^{N} F_j(\tau_{ik})a(\tau_{ik})\omega_k\right) + \frac{t_i}{2}\sum_{j=0}^{N} Y_j \left(\sum_{k=0}^{N} F_j(q\tau_{ik})b(\tau_{ik})\omega_k\right) \\ & + \frac{t_i}{2}\sum_{j=0}^{N} Y_j \left(\sum_{n=0}^{N}\sum_{k=0}^{N}\frac{q\tau_{ik}}{2}K_1\Big(\frac{\tau_{ik}}{2}(2-q\theta_n-q)\Big)F_j\Big(\frac{q\tau_{ik}}{2}(\theta_n+1)\Big)\omega_n\omega_k\right) \\ & + \frac{t_i}{2}\sum_{j=0}^{N} Y_j \left(\sum_{k=0}^{N}\sum_{n=0}^{N}\frac{\tau_{ik}}{2}K_0\Big(\frac{\tau_{ik}}{2}(1-\theta_n)\Big)F_j\Big(\frac{\tau_{ik}}{2}(\theta_n+1)\Big)\omega_n\omega_k\right) + G(t_i). \end{aligned} \tag{2.7}$$

Setting $Y := [Y_0, \cdots, Y_N]^T$ and $F_N := [y_0, \cdots, y_0]^T + [G(t_0), \cdots, G(t_N)]^T$, we obtain a more compact form of (2.7),

$$Y - (A_1 + A_2 + A_3 + A_4)Y = F_N, \tag{2.8}$$

where the entries of the matrices $A_1, \ldots, A_4 \in \mathbb{R}^{(N+1)\times(N+1)}$ are given by:

$$\begin{aligned} A_1(i,j) &= \frac{t_i}{2}\sum_{k=0}^{N} F_j(\tau_{ik})a(\tau_{ik})\omega_k, \\ A_2(i,j) &= \frac{t_i}{2}\sum_{k=0}^{N} F_j(q\tau_{ik})b(\tau_{ik})\omega_k, \\ A_3(i,j) &= \frac{t_i}{2}\sum_{k=0}^{N}\sum_{n=0}^{N}\frac{\tau_{ik}}{2}K_0\left(\frac{\tau_{ik}}{2}(1-\theta_n)\right)F_j\left(\frac{\tau_{ik}}{2}(\theta_n+1)\right)\omega_n\omega_k, \\ A_4(i,j) &= \frac{t_i}{2}\sum_{n=0}^{N}\sum_{k=0}^{N}\frac{q\tau_{ik}}{2}K_1\left(\frac{\tau_{ik}}{2}(2-q\theta_n-q)\right)F_j\left(\frac{q\tau_{ik}}{2}(\theta_n+1)\right)\omega_n\omega_k. \end{aligned}$$

3. Convergence Analysis

To carry out the convergence analysis of our method we first introduce some useful lemmas.

Lemma 3.1. ([3]) **(Integration error for Gauss quadrature)** *Assume that a $(N+1)$-point Gauss-Legendre, or Gauss-Radau, or Gauss-Lobatto quadrature formula relative to the Legendre weights is used to integrate the product $y\phi$, where $y \in H^m(I)$ with $I := (-1,1)$ and some $m \geq 1$, and $\phi \in \mathcal{P}_N$. Then there exists a constant C not depending on N such that*

$$\left| \int_{-1}^{1} y(x)\phi(x)dx - (y,\phi)_N \right| \leq CN^{-m}|y|_{\tilde{H}_{m,N}(I)} \|\phi\|_{L^2(I)}, \tag{3.1}$$

where

$$\|\phi\|_{L^2(I)} = \sum_{k=0}^{N} \sqrt{\frac{2}{2k+1}} \tilde{\phi_k}, \tag{3.2a}$$

$$|y|_{\tilde{H}_{m,N}(I)} = \left(\sum_{k=\min(m,N+1)}^{m} \|y^{(k)}\|^2_{L^2(I)} \right)^{1/2}, \tag{3.2b}$$

$$(y,\phi)_N = \sum_{k=0}^{N} \omega_k y(x_k)\phi(x_k), \tag{3.2c}$$

where $\tilde{\phi_k}$ are discrete Legendre coefficients.

Lemma 3.2. (Estimates for interpolation error) *Assume that $y \in H^m(I)$ and denote by $I_N u$ the interpolation polynomial associated with the $(N+1)$ Gauss-Legendre, Gauss-Radau, or Gauss-Lobatto points $\{t_k\}_{k=0}^N$. Then*

$$\|y - I_N y\|_{L^2(I)} \leq CN^{-m}|y|_{\tilde{H}_{m,N}(I)}, \tag{3.3a}$$

$$\|y - I_N y\|_{L^\infty(I)} \leq CN^{1/2-m}|y|_{\tilde{H}_{m,N}(I)}. \tag{3.3b}$$

Proof. The estimate (3.3a) is given on p. 289 of [3]. The estimate

$$\|y - I_N y\|_{H^1(I)} \leq CN^{1-m}|y|_{\tilde{H}_{m,N}(I)}, \qquad 1 \leq s \leq m,$$

can also be found in [3]. Using the above estimate and the inequality

$$\|v\|_{L^\infty(a,b)} \leq \sqrt{\frac{1}{b-a} + 2} \|v\|^{1/2}_{L^2(a,b)} \|v\|^{1/2}_{H^1(a,b)}, \qquad \forall v \in H^1(a,b),$$

we readily obtain (3.3b). □

From [4], we have the following result on the Lebesgue constant for Lagrange interpolation based on the zeros of the Legendre polynomials.

Lemma 3.3. (Lebesgue constant for the Legendre series) *Assume that $\{F_j(x)\}_{j=0}^N$ are the Lagrange interpolation polynomials with respect to the Gauss-Legendre, Gauss-Radau, or Gauss-Lobatto points $\{x_j\}$. Then*

$$||I_N||_\infty := \max_{x\in(-1,1)} \sum_{j=0}^{N} |F_j(x)| = \mathcal{O}(\sqrt{N}). \tag{3.4}$$

Lemma 3.4 (Gronwall inequality) *Let $T > 0$ and $C_1, C_2, C_3, C_4 \geq 0$. If a non-negative continuous function $E(t)$ satisfies*

$$\begin{aligned} E(t) \leq & C_1 \int_0^t E(s)ds + C_2 \int_0^t E(qs)ds + C_3 \int_0^t \int_0^s E(v)dvds \\ & + C_4 \int_0^t \int_0^{qs} E(v)dvds + H(t), \qquad \forall t \in [0, T], \end{aligned} \tag{3.5}$$

where $0 < q < 1$ is a constant and $H(t)$ is a continuous function, then

$$\|E\|_{L^\infty(I)} \leq C\|H\|_{L^\infty(I)}. \tag{3.6}$$

Proof. For $E \geq 0$, we have

$$\begin{aligned} &\int_0^t E(qs)ds = \frac{1}{q}\int_0^{qt} E(s)ds \leq \frac{1}{q}\int_0^t E(s)ds, \\ &\int_0^t \int_0^s E(v)dvds = \int_0^t (t-v)E(v)dv \leq C\int_0^t E(v)dv, \\ &\int_0^t \int_0^{qs} E(v)dvds = \int_0^{qt} E(v)(t-qv)dv \leq C\int_0^t E(v)dv, \end{aligned}$$

which, together with (3.5), yield the standard Gronwall inequality,

$$E(t) \leq C\int_0^t E(v)dv + H(t).$$

Consequently, the desired estimate (3.6) is obtained. □

Theorem 3.1. *Consider the delay integro-differential equation* (1.1a) *and its spectral approximation method* (2.7)*. Then*

$$\begin{aligned} \|Y - y\|_{L^\infty(I)} \leq & CN^{1/2-m}(D_1 + D_2 + B_0 + B_1)\|y\|_{L^\infty} \\ & + CN^{-1/2-m}\Big(|y|_{\tilde{H}_{m,N}(0,\mathrm{T})} + |K_0 y|_{\tilde{H}_{m,N}(I)} + |K_1 y|_{\tilde{H}_{m,N}(I)}\Big), \end{aligned} \tag{3.7}$$

where Y is the polynomial of degree N associated with the spectral approximation (2.7)*, and C, D_1, D_2, B_0 B_1 are constants independent of N.*

Proof. Following the notations of (2.7), let

$$[Y]_{N,s} := \frac{t_i}{2}\sum_{k=0}^{N} Y(\tau_{ik})\omega_k.$$

Then

$$\Big[[Y]_{N,v}\Big]_{N,s} = \frac{t_i}{2}\sum_{k=0}^{N}\left(\frac{\tau_{ik}}{2}\sum_{n=0}^{N} Y\Big(\frac{\tau_{ik}}{2}(\theta_n + 1)\Big)\omega_n\right)\omega_k. \tag{3.8}$$

The second term on the right-hand side of (2.7) can be written as $[Y]_{N,qs}$. It follows from the numerical scheme (2.7) that

$$Y_i = y_0 + [aY]_{N,s} + [bY]_{N,qs} + \Big[[K_0(s-v)Y]_{N,v}\Big]_{N,s} + \Big[[K_1(s-v)Y]_{N,v}\Big]_{N,qs}, \tag{3.9}$$

which gives

$$
\begin{aligned}
Y(t_i) =& y_0 + \frac{t_i}{2}\int_{-1}^{1} a\Big(\frac{t_i}{2}(\theta+1)\Big)Y\Big(\frac{t_i}{2}(\theta+1)\Big)d\theta + \frac{t_i}{2}\int_{-1}^{1} b\Big(\frac{t_i}{2}(\theta+1)\Big)Y\Big(\frac{qt_i}{2}(\theta+1)\Big)d\theta \\
&+ \frac{t_i}{2}\int_{-1}^{1}\left(\int_0^{\frac{t_i}{2}(\theta+1)} K_0\Big(\frac{t_i}{2}(\theta+1)-v\Big)Y(v)dv\right)d\theta \\
&+ \frac{t_i}{2}\int_{-1}^{1}\left(\int_0^{\frac{qt_i}{2}(\theta+1)} K_1\Big(\frac{t_i}{2}(\theta+1)-v\Big)Y(v)dv\right)d\theta + G(t_i) \\
&- I_{i,1} - I_{i,2} - I_{i,3} - I_{i,4},
\end{aligned}
\tag{3.10}
$$

where

$$
\begin{aligned}
I_{i,1} &:= \frac{t_i}{2}\int_{-1}^{1} a\Big(\frac{t_i}{2}(\theta+1)\Big)Y\Big(\frac{t_i}{2}(\theta+1)\Big)d\theta - [aY]_{N,s}, \\
I_{i,2} &:= \frac{t_i}{2}\int_{-1}^{1} b\Big(\frac{t_i}{2}(\theta+1)\Big)Y\Big(\frac{qt_i}{2}(\theta+1)\Big)d\theta - [bY]_{N,qs}, \\
I_{i,3} &:= \frac{t_i}{2}\int_{-1}^{1}\left(\int_0^{\frac{t_i}{2}(\theta+1)} K_0\Big(\frac{t_i}{2}(\theta+1)-v\Big)Y(v)dv\right)d\theta - \Big[[K_0(s-v)Y]_{N,v}\Big]_{N,s}, \\
I_{i,4} &:= \frac{t_i}{2}\int_{-1}^{1}\left(\int_0^{\frac{qt_i}{2}(\theta+1)} K_1\Big(\frac{t_i}{2}(\theta+1)-v\Big)Y(v)dv\right)d\theta - \Big[[K_1(s-v)Y]_{N,v}\Big]_{N,qs}.
\end{aligned}
$$

It follows from (2.1) and (2.2) that

$$
\begin{aligned}
Y(t_i) =& Y_0 + \int_0^{t_i} a(s)Y(s)ds + \int_0^{t_i} b(s)Y(qs)ds + \int_0^{t_i}\int_0^{s} K_0(s-v)Y(v)dvds \\
&+ \int_0^{t_i}\int_0^{qs} K_1(s-v)Y(v)dvds - I_{i,1} - I_{i,2} - I_{i,3} - I_{i,4} + G(t_i).
\end{aligned}
\tag{3.11}
$$

Using Lemma 3.1 gives

$$
|I_{i,1}| \le CN^{-m}|a|_{\tilde{H}_{m,N}(I)}\|Y\|_{L^2(I)} \le CN^{-m}|a|_{\tilde{H}_{m,N}(I)}\Big(\|e\|_{L^\infty} + \|y\|_{L^2}\Big), \tag{3.12a}
$$

$$
|I_{i,2}| \le CN^{-m}|b|_{\tilde{H}_{m,N}(I)}\|Y\|_{L^2(I)} \le CN^{-m}|b|_{\tilde{H}_{m,N}(I)}\Big(\|e\|_{L^\infty} + \|y\|_{L^2}\Big). \tag{3.12b}
$$

For the other two terms, that is $|I_{i,3}|$ and $|I_{i,4}|$ we have,

$$
\begin{aligned}
|I_{i,3}| =& \int_0^{t_i}\left(\int_0^{s} K_0(s-v)Y(v)dv - [K_0(s-v)Y]_{N,v}\right)ds \\
&+ \int_0^{t_i}\Big[K_0(s-v)Y\Big]_{N,v}ds - \Big[[K_0(s-v)Y]_{N,v}\Big]_{N,s} \\
\le& \int_0^{t_i}\Big(CN^{-m}|K_0|_{H^m_{(0,s)}}\|Y\|_{L^2}\Big)ds \\
&+ CN^{-m}\|Y\|_{L^2}\max_{0\le n,j\le N}\left|\frac{s}{2}K_0\left(s-\frac{s}{2}(\theta_n+1)\right)F_j\left(\frac{s}{2}(\theta_n+1)\right)\right|_{H^m_{(0,t_i)}} \\
\le& CN^{-m}\|Y\|_{L^2}\left(\Big|K_0(s-v)\Big|_{H^m_{(0,s)}} + \max_{0\le n,j\le N}\left|\frac{s}{2}K_0\Big(s-\frac{s}{2}(\theta_n+1)\Big)F_j\Big(\frac{s}{2}(\theta_n+1)\Big)\right|_{H^m_{(0,t_i)}}\right)
\end{aligned}
$$

$$\begin{aligned}&\leq CN^{-m}\Big(\|e\|_{L^\infty}+\|y\|_{L^2}\Big)\\&\times\left(\Big|K_0(s-v)\Big|_{H^m_{(0,s)}}+\max_{0\leq n,j\leq N}\left|\frac{s}{2}K_0\left(s-\frac{s}{2}(\theta_n+1)\right)F_j\left(\frac{s}{2}(\theta_n+1)\right)\right|_{H^m_{(0,t_i)}}\right),\end{aligned}\tag{3.13}$$

where we have used Lemma 3.1, and

$$\begin{aligned}&\int_0^{t_i}\Big(K_0(s-v)Y\Big)_{N,v}ds-\Big((K_0(s-v)Y)_{N,v}\Big)_{N,s}\\&\leq CN^{-m}\Big|(K_0(s-v)Y)_{N,v}\Big|_{H^m_{(0,t_i)}}\\&=CN^{-m}\left|\sum_{n=0}^{N}\frac{s}{2}K_0\Big(s-\frac{s}{2}(\theta_n+1)\Big)Y\Big(\frac{s}{2}(\theta_n+1)\Big)\omega_n\right|_{H^m_{(0,t_i)}}\\&=CN^{-m}\left|\sum_{j=0}^{N}Y_j\sum_{n=0}^{N}\frac{s}{2}K_0\Big(s-\frac{s}{2}(\theta_n+1)\Big)F_j\Big(\frac{s}{2}(\theta_n+1)\Big)\omega_n\right|_{H^m_{(0,t_i)}}\\&\leq CN^{-m}\sum_{j=0}^{N}\sum_{n=0}^{N}\left|K_0\Big(s-\frac{s}{2}(\theta_n+1)\Big)F_j\Big(\frac{s}{2}(\theta_n+1)\Big)\right|_{H^m_{(0,t_i)}}Y_jw_n\\&\leq CN^{-m}\max_{0\leq j,n\leq N}\left|K_0\Big(s-\frac{s}{2}(\theta_n+1)\Big)F_j\Big(\frac{s}{2}(\theta_n+1)\Big)\right|_{H^m_{(0,t_i)}}\|Y\|_{L^2}.\end{aligned}\tag{3.14}$$

Similarly

$$\begin{aligned}|I_{i,4}|\leq&\left(\Big|K_1(s-qv)\Big|_{H^m_{(0,s)}}+\max_{0\leq n,j\leq N}\left|\frac{qs}{2}K_1(s-\frac{qs}{2}(\theta_n+1))F_j(\frac{s}{2}(\theta_n+1)\right|_{H^m_{(0,t_i)}}\right)\\&\times CN^{-m}\Big(\|e\|_{L^\infty}+\|y\|_{L^2}\Big).\end{aligned}\tag{3.15}$$

Multiplying $F_i(t)$ on both sides of (3.11) and summing up from 0 to N yield

$$\begin{aligned}Y(t)=&I_N\left(\int_0^t a(s)Y(s)ds\right)+I_N\left(\int_0^{qt}b(s)Y(qs)ds\right)\\&+I_N\left(\int_0^t\int_0^s K_0(s-v)Y(v)dvds\right)\\&+I_N\left(\int_0^t\int_0^{qs}K_1(s-v)Y(v)dvds\right)+y_0+G(t_i)+J_1(t)\\=&I_N\left(\int_0^t a(s)\Big(y(s)+e(s)\Big)ds\right)+I_N\left(\int_0^{qt}b(s)\Big(y(qs)+e(qs)\Big)ds\right)\\&+I_N\left(\int_0^t\int_0^s K_0(s-v)\Big(y(v)+e(v)\Big)dvds\right)+G(t_i)\\&+I_N\left(\int_0^t\int_0^{qs}K_1(s-v)\Big(y(v)+e(v)\Big)dvds\right)+y_0+J_1(t),\end{aligned}\tag{3.16}$$

where we have used $\sum_{i=0}^N F_i(t)\equiv 1$, and the notation J_1 means:

$$J_1(t):=\sum_{i=0}^{N}\Big(I_{i,1}+I_{i,2}+I_{i,3}+I_{i,4}\Big)F_i(t).$$

Defining:

$$e(t) := Y(t) - y(t), \qquad e(qt) := Y(qt) - y(qt),$$

and combining (3.16) with (2.1) gives

$$\begin{aligned} e(t) = &\int_0^t ae(s)ds + \int_0^t be(qs)ds + \int_0^t \int_0^s K_0(s-v)e(v)dvds \\ &+ \int_0^t \int_0^{qs} K_1(s-v)e(v)dvds + J_1(t) + J_2(t), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} J_2(t) :=&I_N\left(\int_0^t ay(s)ds\right) - \int_0^t ay(s)ds + I_N\left(\int_0^t ae(s)ds\right) - \int_0^t ae(s)ds \\ &+ I_N\left(\frac{1}{q}\int_0^{qt} by(qs)ds\right) - \int_0^t by(qs)ds + I_N\left(\frac{1}{q}\int_0^{qt} be(qs)ds\right) - \int_0^t be(qs)ds \\ &+ I_N\left(\int_0^t \int_0^s K_0(s-v)y(v)dvds\right) - \int_0^t \int_0^s K_0(s-v)y(v)dvds \\ &+ I_N\left(\int_0^t \int_0^s K_0(s-v)e(v)dvds\right) - \int_0^t \int_0^s K_0(s-v)e(v)dvds \\ &+ I_N\left(\int_0^t \int_0^{qs} K_1(s-v)y(v)dvds\right) - \int_0^t \int_0^{qs} K_1(s-v)y(v)dvds \\ &+ I_N\left(\int_0^t \int_0^{qs} K_1(s-v)e(v)dvds\right) - \int_0^t \int_0^{qs} K_1(s-v)e(v)dvds. \end{aligned} \tag{3.18}$$

It follows from the Gronwall inequality Lemma 3.4 that

$$\|e\|_{L^\infty(I)} \le C\Big(\|J_1\|_{L^\infty(I)} + \|J_2\|_{L^\infty(I)}\Big). \tag{3.19}$$

Next we will be concerned by the estimation of $\|J_1\|_{L^\infty(I)}$ and $\|J_2\|_{L^\infty(I)}$. First

$$\begin{aligned} &\|J_1\|_{L^\infty(I)} \\ \le&C\Big(\|I_{i,1}\|_{L^\infty(I)} + \|I_{i,2}\|_{L^\infty(I)} + \|I_{i,3}\|_{L^\infty(I)} + \|I_{i,4}\|_{L^\infty(I)}\Big)\sum_{i=0}^N \|F_i(t)\|_{L^\infty(I)} \\ \le&CN^{1/2-m}(\|e\|_{L^\infty(I)} + \|y\|_{L^\infty(I)})\Big(D_1 + D_2 + B_0 + B_1\Big), \end{aligned} \tag{3.20}$$

where we have used Lemma 3.3 and where D_1 , D_2 , B_0 and B_1 are constants depending on the given functions a, b, K_0 and K_1. Next, we have (using Lemma 3.2)

$$\begin{aligned} &\|J_2\|_{L^\infty(I)} \\ \le&CN^{-1/2-m}|y|_{\tilde{H}_{m,N}(I)} + CN^{-1/2-m}|y(qt)|_{\tilde{H}_{m,N}(I)} + CN^{-1/2-m}|K_0y|_{\tilde{H}_{m,N}(I)} \\ &+ CN^{-1/2-m}|K_1y|_{\tilde{H}_{m,N}(I)} + CN^{-1/2}\|e\|_{L^\infty} + CN^{-1/2}\|e(qs)\|_{L^\infty} \\ &+ CN^{-1/2}\|K_0e(s)\|_{L^\infty} + CN^{-1/2}\|K_1e(qs)\|_{L^\infty}. \end{aligned} \tag{3.21}$$

The above two estimates, together with (3.19), yields:

$$\begin{aligned} \|e\|_{L^\infty(I)} \le&CN^{1/2-m}(D_1 + D_2 + B_0 + B_1)\Big(\|e\|_{L^\infty(I)} + \|y\|_{L^\infty(I)}\Big) \\ &+ CN^{-1/2-m}|y|_{\tilde{H}_{m,N}(I)} + CN^{-1/2-m}|y(qt)|_{\tilde{H}_{m,N}(I)} \\ &+ CN^{-1/2-m}|K_0y|_{\tilde{H}_{m,N}(I)} + CN^{-1/2-m}|K_1y|_{\tilde{H}_{m,N}(I)}, \end{aligned}$$

which leads to the desired estimate (3.7), provided that N is sufficiently large. □

4. Nonlinear Vanishing Delays

In the preceding analysis we considered the delay integro-differential equation (1.1) with *linear* proportional delay $\theta(t) = qt$ $(0 < q < 1)$. A close look at the proof of the convergence theorem in Section 3 reveals that the analysis and hence the spectral convergence results remain valid for smooth *nonlinear* delay function θ that are subject to the following assumptions:

(i) $\theta(0) = 0$; θ is strictly increasing on $[0, T]$;
(ii) $\theta(t) \le q_1 t,\ t \in [0, T]$, for some $q_1 \in (0, 1)$;
(iii) θ is smooth on $[0, T]$.

The proof of Theorem 3.1 is then readily adapted to hold for such integro-differential equations with these vanishing nonlinear delays, since assumption (ii) leads to error inequalities that are classical Gronwall inequalities in which q is replaced by q_1. We leave the details of the proof to the reader.

5. Numerical Examples

In the following, we use two examples to illustrate the accuracy and efficiency of the spectral method (2.7). In our computations, we use $T = \pi$ and the Legendre-Gauss quadrature with weights

$$\omega_j = \frac{2}{(1 - x_j^2)[L'_{N+1}(x_j)]^2}, \qquad 0 < j \le N.$$

Example 5.1. Let $a(t) = \cos(t), b(t) = \sin(t)$, $K_0(t - s) = (t - s)^2$ and $K_1(t - s) = (t - s)$. Choose

$$\begin{aligned} g(t) =& \frac{1}{q}\cos(t/q) - \cos(t)\sin(t/q) - (\sin^2(t)) - 2q^3\cos(t/q) \\ & - qt^2 + 2q^3 + qt\cos(t) - q^2 t\cos(t) + q^2\sin(t) - qt, \end{aligned}$$

Table 5.1: Example 5.1: The point-wise error for $q = 0.01$ using (2.7).

N	$L^\infty(q = 0.01)$	N	$L^1(q = 0.01)$ error	N	$L^2(q = 0.01)$ error
6	8.552e-2	6	4.511e-2	6	5.440e-2
12	1.460e-2	12	5.306e-3	12	7.187e-3
14	1.429e-3	14	6.887e-4	14	8.623e-4
16	1.001e-4	16	4.568e-5	16	5.642e-5
18	8.686e-6	18	2.922e-6	18	3.651e-6
20	6.572e-7	20	2.148e-7	20	2.840e-7

Table 5.2: Example 5.1: The point-wise error for $q = 0.5$ using (2.7).

N	$L^\infty(q = 0.5)$	N	$L^1(q = 0.5)$ error	N	$L^2(q = 0.5)$ error
6	6.847e-03	6	1.924e-03	6	2.570e-03
12	1.506e-08	12	8.505e-09	12	9.903e-09
14	1.845e-10	14	7.941e-11	14	9.923e-11
16	1.376e-12	16	6.199e-13	16	7.521e-13
18	1.432e-14	18	5.876e-15	18	7.120e-15
20	4.465e-15	20	8.137e-16	20	1.357e-15

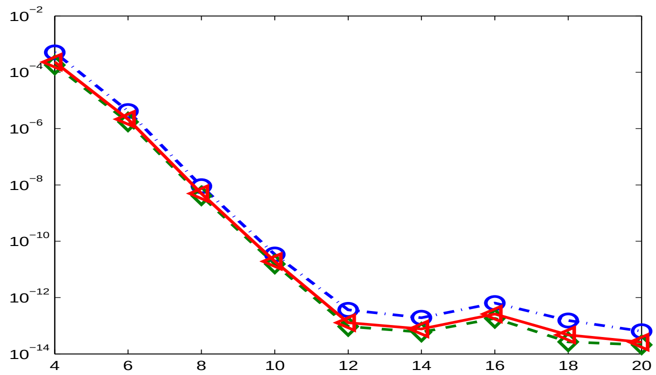

Fig. 5.1. Example 5.1: $L^\infty(\circ), L^2(\triangleleft)$ and $L^1(\diamond)$ errors obtained by using (2.7) for $q = .99$.

such that the exact solution is given by

$$y(t) = \sin(t/q).$$

The point-wise error (for different norms) between the numerical solution and the exact solution for $q = 0.01, q = 0.5, q = .99$ and for different values of N, is shown in Fig. 5.1 and Tables 5.1 and 5.2, respectively.

Example 5.2. Choose $a(t) = t$, $b(t) = e^{-q^2 t}$, $K_0(t-s) = e^{q(t-s)}$, $K_1(t-s) = e^{q(t-s)}$, and

$$g(t) = qe^{qt}\cos(t/q) - \frac{1}{q}e^{qt}\sin(t/q) - te^{qt}\cos(t/q) - \cos(t) - \frac{1}{q}e^{qt}\sin(t/q) - e^{qt}q\sin(t).$$

For these data the exact solution is

$$y(t) = e^{qt}\cos(t/q).$$

The point-wise error (in different norms) between the numerical solution and the exact solution for $q = 0.01, q = 0.5, q = .99$ and for different values of N, is shown in Figs. 5.2 and 5.3 and Table 5.3, respectively.

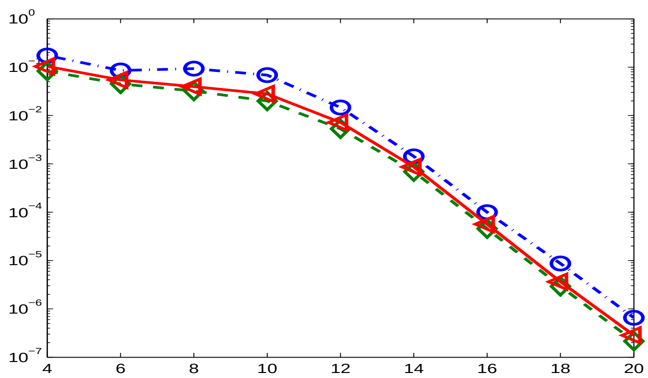

Fig. 5.2. Example 5.2: $L^\infty(\circ), L^2(\triangleleft)$ and $L^1(\diamond)$ errors against N obtained by using (2.7) for $q = 0.01$.

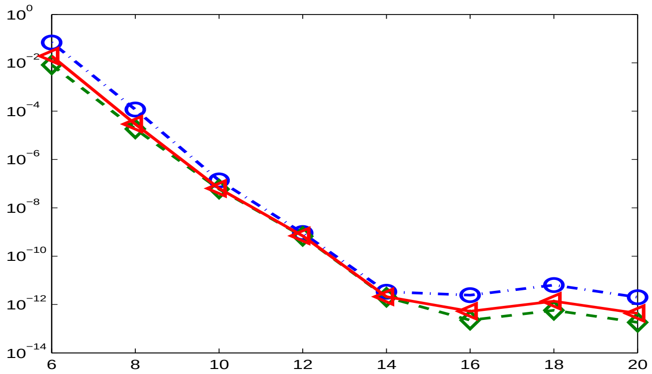

Fig. 5.3. Example 5.2: $L^\infty(\circ), L^2(\triangleleft)$ and $L^1(\diamond)$ errors against N obtained by using (2.7) for $q = .99$.

Table 5.3: Example 5.2: The point-wise error for $q = 0.5$ using (2.7).

N	$L^\infty(q = 0.5)$	N	$L^1(q = 0.5)$ error	N	$L^2(q = 0.5)$ error
6	1.286e+00	6	1.658e-01	6	3.671e-01
12	1.645e-07	12	1.987e-08	12	4.154e-08
14	3.390e-09	14	4.749e-10	14	8.150e-10
16	1.941e-11	16	3.669e-12	16	5.068e-12
18	9.530e-13	18	1.083e-13	18	2.163e-13
20	6.750e-14	20	7.605e-15	20	1.601e-14

6. Concluding Remarks

We have shown that the spectral method yields an efficient and very accurate numerical method for the approximation of solutions to integro-differential equation with proportional delay. The method has spectral accuracy, which means that a very accurate solution can be obtained using relatively few collocation points. The method is readily extended to pantograph-type integro-differential equations with nonlinear vanishing delays.

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