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# UNIFIED ANALYSIS OF TIME DOMAIN MIXED FINITE ELEMENT METHODS FOR MAXWELL'S EQUATIONS IN DISPERSIVE MEDIA*

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**Abstract**

In this paper, we consider the time dependent Maxwell's equations when dispersive media are involved. The Crank-Nicolson mixed finite element methods are developed for three most popular dispersive medium models: the isotropic cold plasma, the one-pole Debye medium and the two-pole Lorentz medium. Optimal error estimates are proved for all three models solved by the Raviart-Thomas-Nédélec spaces. Extensions to multiple pole dispersive media are presented also.

*Mathematics subject classification:* 65N30, 35L15, 78-08.
*Key words:* Maxwell's equations, Dispersive media, Mixed finite element method.

## 1. Introduction

The dispersive medium is characterized by a frequency-dependent susceptibility or permittivity, so that monochromatic waves of different frequencies travel in the medium at different velocities and undergo different attenuations. The most common dispersive media include biological tissue, ionosphere, water, soil, snow, ice, plasma, optical fibers and radar absorbing materials. Hence the study of wave or pulse propagation in dispersive media is important in many applications.

Starting early 1990's, considerable attention has been devoted to numerical modeling of wave propagation in dispersive media. Approaches such as the recursive convolution method and auxiliary differential equation method have been developed under the framework of the finite-difference time-domain (FDTD) method, details and early references can be found in books [19, Ch.8] and [29, Ch.9]. However, due to its complexity, the time-domain finite element method (TDFEM) for the dispersive media has not explored until 2001 by Jiao and Jin [18]. Their TDFEM is based on the second-order vector wave equation. Recently, the time-domain discontinuous Galerkin method has been investigated by Lu *et al.* [24] by solving the first-order Maxwell's equations directly. The one dimensional TDFEM was studied for Debye and Lorentz dispersive media by Bank *et al.* [4] recently.

Since 1980's, there has been a growing interest in finite element analysis of Maxwell's equations (e.g. [3, 5–9, 12–15, 17, 23, 27, 28, 32]). However, almost all studies are restricted to the simple medium case. Very recently, we initiated the error analysis of TDFEM for dispersive

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media [20–22]. In [22], we discussed the superconvergence results for some semi-discrete schemes developed for dispersive medium models. While in [20], we analyzed the backward Euler mixed finite element methods (FEMs) for three most popular dispersive medium models. In [21], we studied the backward Euler scheme for the vector wave equation resulting from the isotropic non-magnetized cold plasma model. In all our previous work, the FEMs are all built on the integro-differential equations. In this paper, we propose some Crank-Nicolson mixed FEMs directly on the governing equations without introducing integral terms. It turns out that this algorithm is simpler and the error analysis can be beautifully carried through by skillful manipulations. Here we provide a unified optimal error analysis for all three popular dispersive medium models.

We conclude the section with an outline of the remainder of the paper. In next section, we consider the single pole Debye medium solved by the Crank-Nicolson mixed method using the lowest Raviart-Thomas-Nédélec (RTN) space. Optimal error estimates are proved under proper regularity assumptions. Then we extend the results to the multiple pole Debye medium. In Section 3, we generalize the numerical scheme and error analysis to both the two-pole and multiple pole Lorentz media. Section 4 is devoted to the isotropic cold plasma model. Similar numerical scheme and results are presented. Finally, we conclude the paper in Section 5.

In this paper, $C$ (sometimes with sub-index) denotes a generic constant, which is independent of the finite element mesh size $h$ and time step size $\tau$. We also use some common notation

$$H^{\alpha}(\mathrm{curl};\Omega)=\left\{\boldsymbol{v}\in(H^{\alpha}(\Omega))^3;\quad \nabla\times\boldsymbol{v}\in(H^{\alpha}(\Omega))^3\right\},$$
$$H_0(\mathrm{curl};\Omega)=\left\{\boldsymbol{v}\in H(\mathrm{curl};\Omega);\quad \mathbf{n}\times\boldsymbol{v}=0\ \text{ on }\ \partial\Omega\right\},$$

where $\alpha\geq 0$ is a real number, and $\Omega$ is a bounded and convex Lipschitz polyhedral domain in $\mathcal{R}^3$ with connected boundary $\partial\Omega$ and unit outward normal $\mathbf{n}$. When $\alpha=0$, we simply denote $H^0(\mathrm{curl};\Omega)=H(\mathrm{curl};\Omega)$. Let $(H^{\alpha}(\Omega))^3$ be the standard Sobolev space equipped with the norm $\|\cdot\|_{\alpha}$ and semi-norm $|\cdot|_{\alpha}$. In particular, $\|\cdot\|_0$ will mean the $(L^2(\Omega))^3$-norm. Also $H(\mathrm{curl};\Omega)$ and $H^{\alpha}(\mathrm{curl};\Omega)$ are equipped with the norm

$$\|\boldsymbol{v}\|_{0,\mathrm{curl}}=\left(\|\boldsymbol{v}\|_0^2+\|\mathrm{curl}\ \boldsymbol{v}\|_0^2\right)^{1/2},$$
$$\|\boldsymbol{v}\|_{\alpha,\mathrm{curl}}=\left(\|\boldsymbol{v}\|_{\alpha}^2+\|\mathrm{curl}\ \boldsymbol{v}\|_{\alpha}^2\right)^{1/2}.$$

Finally, we denote $C^m(0,T;X)$ the space of $m$ times continuously differentiable functions from $[0,T]$ into the Hilbert space $X$.

## 2. Debye Medium

For the single pole Debye medium model, we have the governing equations [20,30]

$$\epsilon_0\epsilon_\infty\frac{\partial\boldsymbol{E}}{\partial t}=\nabla\times\boldsymbol{H}-\frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{t_0}\boldsymbol{E}+\frac{1}{t_0}\boldsymbol{P}, \tag{2.1}$$

$$\mu_0\frac{\partial\boldsymbol{H}}{\partial t}=-\nabla\times\boldsymbol{E}, \tag{2.2}$$

$$\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\frac{\partial\boldsymbol{P}}{\partial t}+\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0 t_0}\boldsymbol{P}=\frac{1}{t_0}\boldsymbol{E}, \tag{2.3}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field, $\epsilon_0$ is the permittivity of free space, $\mu_0$ is the permeability of free space, $\mathbf{P}$ is the polarization vector, $\epsilon_\infty$ is the permittivity at infinite frequency, $\epsilon_s$ ($> \epsilon_\infty$) is the permittivity at zero frequency, $t_0$ is the relaxation time.

We assume a perfect conducting boundary condition

$$\mathbf{n} \times \boldsymbol{E} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{2.4}$$

and the initial conditions

$$\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0(\boldsymbol{x}), \quad \boldsymbol{P}(\boldsymbol{x},0) = \boldsymbol{P}_0(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega, \tag{2.5}$$

where $\boldsymbol{E}_0, \boldsymbol{H}_0$ and $\boldsymbol{P}_0$ are given functions.

To obtain a finite element scheme, we multiply (2.1)-(2.3) by test functions and integrate over $\Omega$. Then using the boundary condition (2.4) and integration by parts for the curl term in (2.1) with the identity

$$\int_\Omega \nabla \times \boldsymbol{H} \cdot \phi = \int_\Omega \boldsymbol{H} \cdot \nabla \times \phi - \int_{\partial\Omega} \boldsymbol{H} \cdot \boldsymbol{n} \times \phi,$$

we obtain the weak formulation for (2.1)-(2.3): Find $\boldsymbol{E} \in C(0,T; H_0(\text{curl};\Omega)) \cap C^1(0,T; (L_2(\Omega))^3)$, $\boldsymbol{H} \in C^1(0,T;(L_2(\Omega))^3)$ and $\boldsymbol{P} \in C^1(0,T;(L_2(\Omega))^3)$ such that

$$\epsilon_0\epsilon_\infty(\frac{\partial \boldsymbol{E}}{\partial t}, \phi) - (\boldsymbol{H}, \nabla \times \phi) + \frac{(\epsilon_s - \epsilon_\infty)\epsilon_0}{t_0}(\boldsymbol{E}, \phi) - \frac{1}{t_0}(\boldsymbol{P}, \phi) = 0, \quad \forall\, \phi \in H_0(\text{curl};\Omega), \tag{2.6}$$

$$\mu_0(\frac{\partial \boldsymbol{H}}{\partial t}, \psi) + (\nabla \times \boldsymbol{E}, \psi) = 0, \qquad \forall\, \psi \in (L_2(\Omega))^3, \tag{2.7}$$

$$\frac{1}{(\epsilon_s - \epsilon_\infty)\epsilon_0}(\frac{\partial \boldsymbol{P}}{\partial t}, \tilde{\phi}) + \frac{1}{(\epsilon_s - \epsilon_\infty)\epsilon_0 t_0}(\boldsymbol{P}, \tilde{\phi}) - \frac{1}{t_0}(\boldsymbol{E}, \tilde{\phi}) = 0, \qquad \forall\, \tilde{\phi} \in (L_2(\Omega))^3. \tag{2.8}$$

**Lemma 2.1.** *Let* $(\boldsymbol{E}(t), \boldsymbol{H}(t), \boldsymbol{P}(t))$ *be the solution of* (2.6)-(2.8). *Then for* $0 \le t \le T$, *we have*

$$\begin{aligned} &\epsilon_0\epsilon_\infty||\boldsymbol{E}(t)||_0^2 + \mu_0||\boldsymbol{H}(t)||_0^2 + \frac{1}{(\epsilon_s - \epsilon_\infty)\epsilon_0}||\boldsymbol{P}(t)||_0^2 \\ \le&\epsilon_0\epsilon_\infty||\boldsymbol{E}_0||_0^2 + \mu_0||\boldsymbol{H}_0||_0^2 + \frac{1}{(\epsilon_s - \epsilon_\infty)\epsilon_0}||\boldsymbol{P}_0||_0^2. \end{aligned}$$

*Proof.* Choosing $\phi = \boldsymbol{E}$ in (2.6), $\psi = \boldsymbol{H}$ in (2.7), $\tilde{\phi} = \boldsymbol{P}$ in (2.8), and adding the resultants together, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[\epsilon_0\epsilon_\infty||\boldsymbol{E}(t)||_0^2 + \mu_0||\boldsymbol{H}(t)||_0^2 + \frac{1}{(\epsilon_s - \epsilon_\infty)\epsilon_0}||\boldsymbol{P}(t)||_0^2] \\ &+\frac{1}{(\epsilon_s - \epsilon_\infty)\epsilon_0 t_0}||\boldsymbol{P}(t)||_0^2 + \frac{(\epsilon_s - \epsilon_\infty)\epsilon_0}{t_0}||\boldsymbol{E}(t)||_0^2 - \frac{2}{t_0}(\boldsymbol{P}, \boldsymbol{E}) = 0, \end{aligned}$$

which, along with the identity

$$\begin{aligned} &\frac{1}{(\epsilon_s - \epsilon_\infty)\epsilon_0 t_0}||\boldsymbol{P}(t)||_0^2 + \frac{(\epsilon_s - \epsilon_\infty)\epsilon_0}{t_0}||\boldsymbol{E}(t)||_0^2 - \frac{2}{t_0}(\boldsymbol{P}, \boldsymbol{E}) \\ =&\left\| \frac{1}{\sqrt{(\epsilon_s - \epsilon_\infty)\epsilon_0 t_0}}\boldsymbol{P}(t) - \sqrt{\frac{(\epsilon_s - \epsilon_\infty)\epsilon_0}{t_0}}\boldsymbol{E}(t) \right\|_0^2, \end{aligned} \tag{2.9}$$

completes the proof. □

**Remark 2.1.** The Debye model (2.1)-(2.3) can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} \epsilon_0\epsilon_\infty \boldsymbol{E} \\ \mu_0 \boldsymbol{H} \\ \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\boldsymbol{P} \end{pmatrix} = \begin{pmatrix} 0 & \nabla\times & 0 \\ -\nabla\times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \\ \boldsymbol{P} \end{pmatrix} - \mathcal{A}\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \\ \boldsymbol{P} \end{pmatrix}, \tag{2.10}$$

where the matrix

$$\mathcal{A} = \begin{pmatrix} \frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{t_0} & 0 & -\frac{1}{t_0} \\ 0 & 0 & 0 \\ -\frac{1}{t_0} & 0 & \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0 t_0} \end{pmatrix}$$

is symmetric positive semi-definite. The stability can be obtained by multiplying (2.10) by a row vector $(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{P})$.

To design our mixed finite element method, we partition $\Omega$ by a family of regular tetrahedral meshes $T^h$ with maximum mesh size $h$. Considering the usual low regularity of Maxwell's equations [2,10], we only employ the lowest order Raviart-Thomas-Nédélec's mixed spaces [26]:

$$\mathbf{V}_h = \left\{ \boldsymbol{v}_h \in H(\text{div};\Omega) : \ \boldsymbol{v}_h|_K = \boldsymbol{c}_K + d_K\boldsymbol{x} \quad \forall\, K \in T^h \right\}, \tag{2.11}$$

$$\mathbf{U}_h = \left\{ \boldsymbol{u}_h \in H(\text{curl};\Omega) : \ \boldsymbol{u}_h|_K = \boldsymbol{a}_K + \boldsymbol{b}_K \times \boldsymbol{x} \quad \forall\, K \in T^h \right\}, \tag{2.12}$$

$$\mathbf{U}_h^0 = \left\{ \boldsymbol{u}_h \in \mathbf{U}_h, \boldsymbol{n} \times \boldsymbol{u}_h = 0 \ \text{ on } \partial\Omega \right\}, \tag{2.13}$$

where $\boldsymbol{a}_K, \boldsymbol{b}_K, \boldsymbol{c}_K$ are constant vectors in $R^3$, and $d_K$ is a real constant.

For any $\boldsymbol{u} \in H^\alpha(curl;\Omega), \frac{1}{2} < \alpha \le 1$, it is well known [26] that its interpolant $\Pi_h\boldsymbol{u} \in \mathbf{U}_h$ can be defined on each tetrahedron $K \in T^h$ by the degrees of freedom $\int_e \boldsymbol{u}\cdot\tau$ on each edge $e$ of $K$, where $\tau$ is the unit vector along the edge $e$. Furthermore, we have (see [8] and [25, (5.42)]):

$$||\boldsymbol{u} - \Pi_h\boldsymbol{u}||_0 + ||\nabla\times(\boldsymbol{u} - \Pi_h\boldsymbol{u})||_0 \le Ch^\alpha ||\boldsymbol{u}||_{\alpha,\text{curl}} \quad \forall \boldsymbol{u} \in H^\alpha(curl;\Omega). \tag{2.14}$$

Denoting by $Q_h\boldsymbol{v} \in \mathbf{V}_h$ the standard $(L^2(\Omega))^3$-projection, we have

$$||\boldsymbol{v} - Q_h\boldsymbol{v}||_0 \le Ch^\alpha ||\boldsymbol{v}||_\alpha \quad \forall \boldsymbol{v} \in H^\alpha(\Omega). \tag{2.15}$$

To construct a fully discrete scheme, we divide the time interval $(0,T)$ into $M$ uniform subintervals using points $0 = t^0 < t^1 < \cdots < t^M = T$, where $t^k = k\tau$, and denote subinterval $I^k = [t^{k-1}, t^k]$. Moreover, we define $\boldsymbol{u}^k = \boldsymbol{u}(\cdot, k\tau)$ for $0 \le k \le M$, and introduce the following finite difference operators:

$$\delta_\tau \boldsymbol{u}^k = \frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\tau}, \quad \overline{\boldsymbol{u}}^k = \frac{1}{2}(\boldsymbol{u}^k + \boldsymbol{u}^{k-1}), \quad 1 \le k \le M.$$

Now we can formulate the Crank-Nicolson mixed finite element scheme for (2.6)-(2.8) as follows: for $k = 1, \cdots, M$, find $(\boldsymbol{E}_h^k, \boldsymbol{H}_h^k, \boldsymbol{P}_h^k) \in \mathbf{U}_h^0 \times \mathbf{V}_h \times \mathbf{U}_h$ such that

$$\epsilon_0\epsilon_\infty(\delta_\tau \boldsymbol{E}_h^k, \phi) - (\overline{\boldsymbol{H}}_h^k, \nabla\times\phi) + \frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{t_0}(\overline{\boldsymbol{E}}_h^k, \phi) - \frac{1}{t_0}(\overline{\boldsymbol{P}}_h^k, \phi) = 0, \quad \forall\, \phi \in \mathbf{U}_h^0, \tag{2.16}$$

$$\mu_0(\delta_\tau \boldsymbol{H}_h^k, \psi) + (\nabla\times\overline{\boldsymbol{E}}_h^k, \psi) = 0, \quad \forall\, \psi \in \mathbf{V}_h, \tag{2.17}$$

$$\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}(\delta_\tau \boldsymbol{P}_h^k, \tilde{\phi}) + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0 t_0}(\overline{\boldsymbol{P}}_h^k, \tilde{\phi}) - \frac{1}{t_0}(\overline{\boldsymbol{E}}_h^k, \tilde{\phi}) = 0, \quad \forall\, \tilde{\phi} \in \mathbf{U}_h, \tag{2.18}$$

subject to the initial conditions

$$\boldsymbol{E}_h^0 = \Pi_h \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}_h^0 = Q_h \boldsymbol{H}_0(\boldsymbol{x}), \quad \boldsymbol{P}_h^0 = \Pi_h \boldsymbol{P}_0(\boldsymbol{x}). \tag{2.19}$$

**Lemma 2.2.** *Let* $(\boldsymbol{E}_h^k, \boldsymbol{H}_h^k, \boldsymbol{P}_h^k)$ *be the solution of* (2.16)-(2.18). *Then we have*

$$\begin{aligned}&\epsilon_0\epsilon_\infty||\boldsymbol{E}_h^k||_0^2 + \mu_0||\boldsymbol{H}_h^k||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}||\boldsymbol{P}_h^k||_0^2\\ \le&\epsilon_0\epsilon_\infty||\boldsymbol{E}_h^0||_0^2 + \mu_0||\boldsymbol{H}_h^0||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}||\boldsymbol{P}_h^0||_0^2.\end{aligned}$$

*Proof.* Choosing $\phi = \overline{\boldsymbol{E}}_h^k$ in (2.16), $\psi = \overline{\boldsymbol{H}}_h^k$ in (2.17), $\tilde{\phi} = \overline{\boldsymbol{P}}_h^k$ in (2.18), then adding the resultants together, we obtain

$$\begin{aligned}&\frac{1}{2\tau}\left[\epsilon_0\epsilon_\infty\left(||\boldsymbol{E}_h^k||_0^2 - ||\boldsymbol{E}_h^{k-1}||_0^2\right) + \mu_0\left(||\boldsymbol{H}_h^k||_0^2 - ||\boldsymbol{H}_h^{k-1}||_0^2\right) + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\left(||\boldsymbol{P}_h^k||_0^2 - ||\boldsymbol{P}_h^{k-1}||_0^2\right)\right]\\ &+\frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{t_0}||\overline{\boldsymbol{E}}_h^k||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0 t_0}||\overline{\boldsymbol{P}}_h^k||_0^2 - \frac{2}{t_0}(\overline{\boldsymbol{P}}_h^k, \overline{\boldsymbol{E}}_h^k) = 0,\end{aligned}$$

which, along with the inequality (2.9), concludes the proof. □

Notice that $\boldsymbol{E}_h^k$ and $\boldsymbol{P}_h^k$ are chosen from the same finite element space, hence (2.18) is equivalent to

$$\frac{\boldsymbol{P}_h^k - \boldsymbol{P}_h^{k-1}}{\tau} + \frac{1}{2t_0}(\boldsymbol{P}_h^k + \boldsymbol{P}_h^{k-1}) - \frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{2t_0}(\boldsymbol{E}_h^k + \boldsymbol{E}_h^{k-1}) = 0,$$

or

$$\boldsymbol{P}_h^k = \left[\left(\frac{1}{\tau} - \frac{1}{2t_0}\right)\boldsymbol{P}_h^{k-1} + \frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{2t_0}(\boldsymbol{E}_h^k + \boldsymbol{E}_h^{k-1})\right]\left(\frac{1}{\tau} + \frac{1}{2t_0}\right)^{-1}. \tag{2.20}$$

Substituting (2.20) into (2.16), we can rewrite (2.16) as follows:

$$\begin{aligned}&\left(\frac{\epsilon_0\epsilon_\infty}{\tau} + \frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{2t_0+\tau}\right)(\boldsymbol{E}_h^k, \phi) - \frac{1}{2}(\boldsymbol{H}_h^k, \nabla\times\phi)\\ =&\left(\frac{\epsilon_0\epsilon_\infty}{\tau} - \frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{2t_0+\tau}\right)(\boldsymbol{E}_h^{k-1}, \phi)\\ &+\frac{2}{2t_0+\tau}(\boldsymbol{P}_h^{k-1}, \phi) + \frac{1}{2}(\boldsymbol{H}_h^{k-1}, \nabla\times\phi), \quad \forall\, \phi\in\mathbf{U}_h^0.\end{aligned} \tag{2.21}$$

On the other hand, we can rewrite (2.17) as follows:

$$\frac{\mu_0}{\tau}(\boldsymbol{H}_h^k, \psi) + \frac{1}{2}(\nabla\times\boldsymbol{E}_h^k, \psi) = \frac{\mu_0}{\tau}(\boldsymbol{H}_h^{k-1}, \psi) - \frac{1}{2}(\nabla\times\boldsymbol{E}_h^{k-1}, \psi), \quad \forall\, \psi\in\mathbf{V}_h. \tag{2.22}$$

Hence, the Crank-Nicolson mixed finite element scheme (2.16)-(2.18) can be realized in practice as follows: at each time step, we first solve a system of (2.21)-(2.22) for $\boldsymbol{E}_h^k$ and $\boldsymbol{H}_h^k$, then update $\boldsymbol{P}_h^k$ by (2.20).

Finally, notice that the coefficient matrix for the system of (2.21)-(2.22) can be written as

$$R \equiv \begin{pmatrix} A & -B \\ B' & D \end{pmatrix},$$

where matrices

$$A = \left(\frac{\epsilon_0\epsilon_\infty}{\tau} + \frac{(\epsilon_s - \epsilon_\infty)\epsilon_0}{2t_0+\tau}\right)(\mathbf{U}_h, \mathbf{U}_h),$$
$$B = \frac{1}{2}(\mathbf{V}_h, \nabla\times\mathbf{U}_h), \quad D = \frac{\mu_0}{\tau}(\mathbf{V}_h, \mathbf{V}_h),$$

and $B'$ denotes the transpose of matrix $B$. Hence, the determinant of $R$ can be obtained as

$$\det(R) = \det(A)\det(D + B'A^{-1}B),$$

which is guaranteed to be non-zero. Hence the system of (2.21)-(2.22) is guaranteed to have a unique solution $(\boldsymbol{E}_h^k, \boldsymbol{H}_h^k)$ at each time step.

**Lemma 2.3.** *(i)* ([8,21]) *For any* $\boldsymbol{u} \in H^1(0,T;(L_2(\Omega))^3)$, *we have*

$$\|\delta_\tau \boldsymbol{u}^k\|_0^2 = \left\|\frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\tau}\right\|_0^2 \le \frac{1}{\tau}\int_{t^{k-1}}^{t^k} \|\boldsymbol{u}_t(t)\|_0^2 dt.$$

*(ii) For any* $\boldsymbol{u} \in H^2(0,T;(L_2(\Omega))^3)$, *we have*

$$\left\|\overline{\boldsymbol{u}}^k - \frac{1}{\tau}\int_{t^{k-1}}^{t^k} \boldsymbol{u}(t)dt\right\|_0^2 \le \frac{1}{4}\tau^3\int_{t^{k-1}}^{t^k} \|\boldsymbol{u}_{tt}(t)\|_0^2 dt.$$

*Proof.* The proof for (ii) can be obtained easily as follows:

$$\begin{aligned}
&\left|\frac{1}{2}(\boldsymbol{u}^k + \boldsymbol{u}^{k-1}) - \frac{1}{\tau}\int_{t^{k-1}}^{t^k} \boldsymbol{u}(t)dt\right|^2 \\
=&\left|\frac{1}{2\tau}\int_{t^{k-1}}^{t^k} (t-t^{k-1})(t^k-t)\boldsymbol{u}_{tt}(t)dt\right|^2 \\
\le&\frac{1}{4\tau^2}\left(\int_{t^{k-1}}^{t^k}(t-t^{k-1})^2(t^k-t)^2dt\right)\left(\int_{t^{k-1}}^{t^k}|\boldsymbol{u}_{tt}(t)|^2dt\right) \\
\le&\frac{\tau^3}{4}\int_{t^{k-1}}^{t^k}|\boldsymbol{u}_{tt}(t)|^2dt.
\end{aligned}$$

This completes the proof of the lemma. □

**Theorem 2.1.** *Let* $(\boldsymbol{E}^n, \boldsymbol{H}^n, \boldsymbol{P}^n)$ *and* $(\boldsymbol{E}_h^n, \boldsymbol{H}_h^n, \boldsymbol{P}_h^n)$ *be the solutions of* (2.1)-(2.3) *and* (2.16)-(2.18) *at time* $t^n = n\tau$, *respectively. Furthermore, assume that*

$$\begin{aligned}
&\boldsymbol{E}_t, \boldsymbol{P}_t \in L^2(0,T;H^\alpha(curl;\Omega)), \\
&\boldsymbol{E}_{tt}, \nabla\times\boldsymbol{E}_{tt}, \boldsymbol{P}_{tt}, \nabla\times\boldsymbol{H}_{tt} \in L^2(0,T;(L^2(\Omega))^3), \\
&\boldsymbol{E}, \boldsymbol{P} \in L^\infty(0,T;H^\alpha(curl;\Omega)), \boldsymbol{H} \in L^\infty(0,T;(H^\alpha(\Omega))^3).
\end{aligned}$$

*Then there is a constant $C = C(T, \epsilon_0, \mu_0, \epsilon_s, \epsilon_\infty, t_0, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{P})$, independent of both the time step $\tau$ and the mesh size $h$, such that*

$$\max_{1\le n\le M}\left(||\boldsymbol{E}^n - \boldsymbol{E}_h^n||_0 + ||\boldsymbol{H}^n - \boldsymbol{H}_h^n||_0 + ||\boldsymbol{P}^n - \boldsymbol{P}_h^n||_0\right) \le C(h^\alpha + \tau^2).$$

*Proof.* Denote

$$\xi_h^k = \Pi_h\boldsymbol{E}^k - \boldsymbol{E}_h^k, \eta_h^k = Q_h\boldsymbol{H}^k - \boldsymbol{H}_h^k, \tilde{\xi}_h^k = \Pi_h\boldsymbol{P}^k - \boldsymbol{P}_h^k.$$

Integrating (2.6)-(2.8) in time over $I^k = [t^{k-1}, t^k]$, dividing by $\tau$, then subtracting the resultants from (2.16)-(2.18), respectively, we obtain the error equations

$$\begin{aligned}
(i)\quad & \epsilon_0\epsilon_\infty(\delta_\tau\xi_h^k, \phi) - (\overline{\eta}_h^k, \nabla\times\phi) + \frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{t_0}(\overline{\xi}_h^k, \phi) - \frac{1}{t_0}(\overline{\tilde{\xi}}_h^k, \phi)\\
& = \epsilon_0\epsilon_\infty(\delta_\tau(\Pi_h\boldsymbol{E}^k - \boldsymbol{E}^k), \phi) - \left(Q_h\overline{\boldsymbol{H}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{H}(s)ds, \nabla\times\phi\right)\\
& \quad + \frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{t_0}\left(\Pi_h\overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{E}(s)ds, \phi\right) - \frac{1}{t_0}\left(\Pi_h\overline{\boldsymbol{P}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{P}(s)ds, \phi\right);\\
(ii)\quad & \mu_0(\delta_\tau\eta_h^k, \psi) + \left(\nabla\times\overline{\xi}_h^k, \psi\right)\\
& = \mu_0(\delta_\tau(Q_h\boldsymbol{H}^k - \boldsymbol{H}^k), \psi) + \left(\nabla\times(\Pi_h\overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{E}(s)ds), \psi\right);\\
(iii)\quad & \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}(\delta_\tau\tilde{\xi}_h^k, \tilde{\phi}) + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0 t_0}\left(\overline{\tilde{\xi}}_h^k, \tilde{\phi}\right) - \frac{1}{t_0}(\overline{\xi}_h^k, \tilde{\phi})\\
& = \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\left(\delta_\tau(\Pi_h\boldsymbol{P}^k - \boldsymbol{P}^k), \tilde{\phi}\right)\\
& \quad + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0 t_0}\left(\Pi_h\overline{\boldsymbol{P}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{P}(s)ds, \tilde{\phi}\right) - \frac{1}{t_0}\left(\Pi_h\overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{E}(s)ds, \tilde{\phi}\right).
\end{aligned}$$

Choosing $\phi = 2\tau\overline{\xi}_h^k, \psi = 2\tau\overline{\eta}_h^k, \tilde{\phi} = 2\tau\overline{\tilde{\xi}}_h^k$ in the above respective error equations, adding the resultants together, using the projection property of $Q_h$ and the fact $\nabla\times\mathbf{U}_h \subset \mathbf{V}_h$, we obtain

$$\begin{aligned}
& \epsilon_0\epsilon_\infty\left(||\xi_h^k||_0^2 - ||\xi_h^{k-1}||_0^2\right) + \mu_0\left(||\eta_h^k||_0^2 - ||\eta_h^{k-1}||_0^2\right)\\
& + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\left(||\tilde{\xi}_h^k||_0^2 - ||\tilde{\xi}_h^{k-1}||_0^2\right)\\
& + 2\tau\left[\frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{t_0}||\overline{\xi}_h^k||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0 t_0}||\overline{\tilde{\xi}}_h^k||_0^2 - \frac{2}{t_0}(\overline{\tilde{\xi}}_h^k, \overline{\xi}_h^k)\right]\\
= & 2\tau\epsilon_0\epsilon_\infty\left(\delta_\tau(\Pi_h\boldsymbol{E}^k - \boldsymbol{E}^k), \overline{\xi}_h^k\right) - 2\tau\left(\overline{\boldsymbol{H}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{H}(s)ds, \nabla\times\overline{\xi}_h^k\right)\\
& + 2\tau\frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{t_0}\left(\Pi_h\overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{E}(s)ds, \overline{\xi}_h^k\right) - \frac{2\tau}{t_0}\left(\Pi_h\overline{\boldsymbol{P}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{P}(s)ds, \overline{\xi}_h^k\right)\\
& + 2\tau\left(\nabla\times(\Pi_h\overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{E}(s)ds), \overline{\eta}_h^k\right) + \frac{2\tau}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\left(\delta_\tau(\Pi_h\boldsymbol{P}^k - \boldsymbol{P}^k), \overline{\tilde{\xi}}_h^k\right)\\
& + \frac{2\tau}{(\epsilon_s-\epsilon_\infty)\epsilon_0 t_0}\left(\Pi_h\overline{\boldsymbol{P}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{P}(s)ds, \overline{\tilde{\xi}}_h^k\right) - \frac{2\tau}{t_0}\left(\Pi_h\overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{E}(s)ds, \overline{\tilde{\xi}}_h^k\right)\\
= & \sum_{i=1}^{8}(Err)_i. \qquad (2.23)
\end{aligned}$$

In the rest of the proof, we will frequently use the basic arithmetic-geometric mean inequality

$$|ab| \leq \delta a^2 + \frac{1}{4\delta} b^2, \tag{2.24}$$

and the inequality

$$|\overline{u}^k|^2 = \frac{1}{4}|u^k + u^{k-1}|^2 \leq \frac{1}{2}\Big(|u^k|^2 + |u^{k-1}|^2\Big). \tag{2.25}$$

Using inequalities (2.24) and (2.25), Lemma 2.3, and estimate (2.14), we obtain

$$\begin{aligned}
(Err)_1 \leq & 2\tau\epsilon_0\epsilon_\infty[\delta_1||\overline{\xi}_h^k||_0^2 + \frac{1}{4\delta_1}||\delta_\tau(\Pi_h \boldsymbol{E}^k - \boldsymbol{E}^k)||_0^2] \\
\leq & \epsilon_0\epsilon_\infty\tau\delta_1\Big(||\xi_h^k||_0^2 + ||\xi_h^{k-1}||_0^2\Big) + \frac{\epsilon_0\epsilon_\infty}{2\delta_1}\int_{I^k}\left\|\frac{\partial}{\partial t}(\Pi_h\boldsymbol{E} - \boldsymbol{E})(t)\right\|_0^2 dt \\
\leq & \epsilon_0\epsilon_\infty\tau\delta_1\Big(||\xi_h^k||_0^2 + ||\xi_h^{k-1}||_0^2\Big) + \frac{\epsilon_0\epsilon_\infty}{2\delta_1}\cdot Ch^{2\alpha}\int_{I^k}||\boldsymbol{E}_t||_{\alpha,\mathrm{curl}}^2 dt.
\end{aligned}$$

Similarly, using integration by parts, the boundary condition (2.4), and Lemma 2.3, we have

$$\begin{aligned}
(Err)_2 \leq & 2\tau\left[\epsilon_0\epsilon_\infty\delta_2||\overline{\xi}_h^k||_0^2 + \frac{1}{\epsilon_0\epsilon_\infty\cdot 4\delta_2}||\nabla\times(\overline{\boldsymbol{H}}_h^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{H}(s)ds)||_0^2\right] \\
\leq & \epsilon_0\epsilon_\infty\tau\delta_2\Big(||\xi_h^k||_0^2 + ||\xi_h^{k-1}||_0^2\Big) + \frac{\tau^4}{\epsilon_0\epsilon_\infty\cdot 8\delta_2}\int_{I^k}||\nabla\times\boldsymbol{H}_{tt}(s)||_0^2 ds.
\end{aligned}$$

By similar arguments, we have

$$\begin{aligned}
(Err)_3 \leq & 2\tau(\epsilon_s - \epsilon_\infty)\epsilon_0\left[\delta_3||\overline{\xi}_h^k||_0^2 + \frac{1}{t_0^2\cdot 4\delta_3}\left\|\Pi_h\overline{\boldsymbol{E}}^k - \overline{\boldsymbol{E}}^k + \overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{E}(s)ds\right\|_0^2\right] \\
\leq & (\epsilon_s - \epsilon_\infty)\epsilon_0\cdot\tau\delta_3\Big(||\xi_h^k||_0^2 + ||\xi_h^{k-1}||_0^2\Big) \\
& + \frac{(\epsilon_s - \epsilon_\infty)\epsilon_0\tau}{t_0^2\delta_3}\Big(||\Pi_h\overline{\boldsymbol{E}}^k - \overline{\boldsymbol{E}}^k||_0^2 + ||\overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k}\boldsymbol{E}(s)ds||_0^2\Big) \\
\leq & (\epsilon_s - \epsilon_\infty)\epsilon_0\cdot\tau\delta_3\Big(||\xi_h^k||_0^2 + ||\xi_h^{k-1}||_0^2\Big) \\
& + \frac{(\epsilon_s - \epsilon_\infty)\epsilon_0\tau}{t_0^2\delta_3}\left[Ch^{2\alpha}(||\boldsymbol{E}^k||_{\alpha,\mathrm{curl}}^2 + ||\boldsymbol{E}^{k-1}||_{\alpha,\mathrm{curl}}^2) + \frac{\tau^3}{4}\int_{I^k}||\boldsymbol{E}_{tt}(s)||_0^2 ds\right].
\end{aligned}$$

Using the fact that $\epsilon_s$ is usually a multiple of $\epsilon_\infty$ [29,30], i.e., $\epsilon_s = a\epsilon_\infty$ for some constant $a > 1$, and the inequality

$$||\boldsymbol{E}^k||_{\alpha,\mathrm{curl}}^2 + ||\boldsymbol{E}^{k-1}||_{\alpha,\mathrm{curl}}^2 \leq 2||\boldsymbol{E}||_{L^\infty(0,T;H^\alpha(\mathrm{curl};\Omega))}^2,$$

we can simplify $(Err)_3$ further as follows

$$\begin{aligned}
(Err)_3 \leq & (a-1)\epsilon_\infty\epsilon_0\cdot\tau\delta_3\Big(||\xi_h^k||_0^2 + ||\xi_h^{k-1}||_0^2\Big) \\
& + \frac{(\epsilon_s - \epsilon_\infty)\epsilon_0\tau}{t_0^2\delta_3}\left[Ch^{2\alpha}||\boldsymbol{E}||_{L^\infty(0,T;H^\alpha(\mathrm{curl};\Omega))}^2 + \frac{\tau^3}{4}\int_{I^k}||\boldsymbol{E}_{tt}(s)||_0^2 ds\right].
\end{aligned}$$

With analogous calculations, we can obtain the rest estimates.

$$
\begin{aligned}
(Err)_4 \le& \epsilon_0\epsilon_\infty \cdot 2\tau\delta_4 ||\overline{\xi}_h^k||_0^2 + \frac{2\tau}{\epsilon_0\epsilon_\infty t_0^2 \cdot 4\delta_4} \left\| \Pi_h \overline{\boldsymbol{P}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{P}(s)ds \right\|_0^2 \\
\le& \epsilon_0\epsilon_\infty \tau\delta_4 \left( ||\xi_h^k||_0^2 + ||\xi_h^{k-1}||_0^2 \right) \\
&+ \frac{\tau}{\epsilon_0\epsilon_\infty t_0^2 \delta_4} \left[ Ch^{2\alpha} ||\boldsymbol{P}||^2_{L^\infty(0,T;H^\alpha(\text{curl};\Omega))} + \frac{\tau^3}{4}\int_{I^k} ||\boldsymbol{P}_{tt}(s)||_0^2 ds \right]; \\
(Err)_5 \le& \mu_0 \cdot 2\tau\delta_5 \left\| \overline{\eta}_h^k ||_0^2 + \frac{2\tau}{\mu_0 \cdot 4\delta_5} \right\| \nabla \times (\Pi_h \overline{\boldsymbol{E}}^k - \overline{\boldsymbol{E}}^k + \overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{E}(s)ds)||_0^2 \\
\le& \mu_0\tau\delta_5 \left( ||\eta_h^k||_0^2 + ||\eta_h^{k-1}||_0^2 \right) \\
&+ \frac{\tau}{\mu_0 \cdot 4\delta_5} \left[ Ch^{2\alpha} ||\boldsymbol{E}||^2_{L^\infty(0,T;H^\alpha(\text{curl};\Omega))} + \frac{\tau^3}{4}\int_{I^k} ||(\nabla\times \boldsymbol{E}_{tt})(s)||_0^2 ds \right]; \\
(Err)_6 \le& 2\tau\epsilon^* \left[ \delta_6 ||\overline{\tilde{\xi}}_h^k||_0^2 + \frac{1}{4\delta_6} ||\delta_\tau(\Pi_h \boldsymbol{P}^k - \boldsymbol{P}^k)||_0^2 \right] \\
\le& \epsilon^*\tau\delta_6 \left( ||\tilde{\xi}_h^k||_0^2 + ||\tilde{\xi}_h^{k-1}||_0^2 \right) + \frac{\epsilon^*}{2\delta_6}\int_{I^k} \left\| \frac{\partial}{\partial t}(\Pi_h \boldsymbol{P} - \boldsymbol{P}) \right\|_0^2 dt; \\
\le& \epsilon^*\tau\delta_6 \left( ||\tilde{\xi}_h^k||_0^2 + ||\tilde{\xi}_h^{k-1}||_0^2 \right) + \frac{\epsilon^*}{2\delta_6} \cdot Ch^{2\alpha}\int_{I^k} ||\boldsymbol{P}_t||^2_{\alpha,\text{curl}} dt; \\
(Err)_7 \le& 2\tau\epsilon^* \left[ \delta_7 \left\| \overline{\tilde{\xi}}_h ||_0^2 + \frac{1}{4t_0^2\delta_7} ||\Pi_h \overline{\boldsymbol{P}}^k - \overline{\boldsymbol{P}}^k + \overline{\boldsymbol{P}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{P}(s)ds \right\|_0^2 \right] \\
\le& \epsilon^*\tau\delta_7 \left( ||\tilde{\xi}_h^k||_0^2 + ||\tilde{\xi}_h^{k-1}||_0^2 \right) \\
&+ \frac{\tau\epsilon^*}{t_0^2\delta_7} \left[ Ch^{2\alpha} ||\boldsymbol{P}||^2_{L^\infty(0,T;H^\alpha(\text{curl};\Omega))} + \frac{\tau^3}{4}\int_{I^k} ||\boldsymbol{P}_{tt}(s)||_0^2 ds \right]; \\
(Err)_8 \le& 2\tau\epsilon^*\delta_8 ||\overline{\tilde{\xi}}_h^k||_0^2 + \frac{(\epsilon_s - \epsilon_\infty)\epsilon_0 \cdot 2\tau}{t_0^2 \cdot 4\delta_8} \left\| \Pi_h \overline{\boldsymbol{E}}^k - \overline{\boldsymbol{E}}^k + \overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{E}(s)d \right\|_0^2 \\
\le& \epsilon^*\tau\delta_8 \left( ||\tilde{\xi}_h^k||_0^2 + ||\tilde{\xi}_h^{k-1}||_0^2 \right) \\
&+ \frac{\tau}{t_0^2\delta_8\epsilon^*} \left[ Ch^{2\alpha} ||\boldsymbol{E}||^2_{L^\infty(0,T;H^\alpha(\text{curl};\Omega))} + \frac{\tau^3}{4}\int_{I^k} ||\boldsymbol{E}_{tt}(s)||_0^2 ds \right],
\end{aligned}
$$

where

$$\epsilon^* = \frac{1}{(\epsilon_s - \epsilon_\infty)\epsilon_0}.$$

Substituting the estimates $(Err)_i$ into (2.23), dropping the bracket terms on the left hand side of (2.23) due to the same argument as (2.9), then summing up from $k = 1$ to $n$ ($n \le M-1$), and using the bound $n\tau \le T$ and the facts $\xi_h^0 = \eta_h^0 = \tilde{\xi}_h^0 = 0$, we obtain

$$
\begin{aligned}
&\epsilon_0\epsilon_\infty ||\xi_h^n||_0^2 + \mu_0 ||\eta_h^n||_0^2 + \frac{1}{(\epsilon_s - \epsilon_\infty)\epsilon_0} ||\tilde{\xi}_h^n||_0^2 \\
\le& 2\epsilon_0\epsilon_\infty\tau \left( \delta_1 + \delta_2 + (a-1)\delta_3 + \delta_4 \right) \sum_{k=1}^n ||\xi_h^k||_0^2 + 2\mu_0\tau \cdot \delta_5 \sum_{k=1}^n ||\eta_h^k||_0^2 \\
&+ \frac{2\tau}{(\epsilon_s - \epsilon_\infty)\epsilon_0} \left( \delta_6 + \delta_7 + \delta_8 \right) \sum_{k=1}^n ||\tilde{\xi}_h^k||_0^2 + \frac{Ch^{2\alpha}}{\delta_1}\int_0^T ||\boldsymbol{E}_t||^2_{\alpha,\text{curl}} ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C\tau^4}{\delta_2}\int_0^T ||\nabla\times \boldsymbol{H}_{tt}(s)||_0^2 ds + Ch^{2\alpha}\left(\frac{1}{\delta_3}+\frac{1}{\delta_5}+\frac{1}{\delta_8}\right)||\boldsymbol{E}||^2_{L^\infty(0,T;H^\alpha(\mathrm{curl};\Omega))}\\
&+C\tau^4\left(\frac{1}{\delta_3}+\frac{1}{\delta_8}\right)\int_0^T ||\boldsymbol{E}_{tt}(s)||_0^2 ds + Ch^{2\alpha}\left(\frac{1}{\delta_4}+\frac{1}{\delta_7}\right)||\boldsymbol{P}||^2_{L^\infty(0,T;H^\alpha(\mathrm{curl};\Omega))}\\
&+C\tau^4\left(\frac{1}{\delta_4}+\frac{1}{\delta_7}\right)\int_0^T ||\boldsymbol{P}_{tt}(s)||_0^2 ds\\
&+\frac{C\tau^4}{\delta_5}\int_0^T ||\nabla\times \boldsymbol{E}_{tt}(s)||_0^2 ds + \frac{Ch^{2\alpha}}{\delta_6}\int_0^T ||\boldsymbol{P}_t||^2_{\alpha,\mathrm{curl}} ds,
\end{aligned}
$$

where in the above we absorbed the explicit dependence of those physical parameters into the generic positive constant $C$.

Choosing those $\delta_i$ small enough so that the $n$th terms in the summation can be controlled by the left hand side terms, we have

$$
\begin{aligned}
&\epsilon_0\epsilon_\infty||\xi_h^n||_0^2 + \mu_0||\eta_h^n||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}||\tilde{\xi}_h^n||_0^2\\
\le & C\tau\sum_{k=1}^{n-1}\left(||\xi_h^k||_0^2 + ||\eta_h^k||_0^2 + ||\tilde{\xi}_h^k||_0^2\right) + C(h^{2\alpha}+\tau^4),
\end{aligned}
$$

which, along with the discrete Gronwall inequality [11, p.153] and the triangle inequality, concludes the proof. □

**Remark 2.2.** In practical applications, multiple pole Debye model is often used. For example, Hurt's five pole model is used for muscle [30]. The $N$ ($N \ge 1$) pole Debye medium can be described by the system of equations [31]:

$$
\begin{aligned}
&\epsilon_0\epsilon_\infty\frac{\partial \boldsymbol{E}}{\partial t} = \nabla\times\boldsymbol{H} - \sum_{k=1}^N \frac{(\epsilon_{sk}-\epsilon_\infty)\epsilon_0}{\tau_k}\boldsymbol{E} + \sum_{k=1}^N \frac{1}{\tau_k}\boldsymbol{P}_k,\\
&\mu_0\frac{\partial \boldsymbol{H}}{\partial t} = -\nabla\times\boldsymbol{E},\\
&\frac{1}{(\epsilon_{sk}-\epsilon_\infty)\epsilon_0}\frac{\partial \boldsymbol{P}_k}{\partial t} + \frac{1}{(\epsilon_{sk}-\epsilon_\infty)\epsilon_0\tau_k}\boldsymbol{P}_k = \frac{1}{\tau_k}\boldsymbol{E}, \qquad k=1,\cdots,N,
\end{aligned}
$$

where $\tau_k$ is the $k$th relaxation time, and $\epsilon_{sk}$ is the zero-frequency permittivity of the $k$th relaxation.

Extension of the Crank-Nicolson scheme (2.16)-(2.18) to the $N$ pole Debye model is straightforward: replacing the last two terms of (2.16) as a summation from $k=1$ to $N$, and replacing (2.18) by $N$ similar equations. The new scheme can be realized as follows: first solve a similar system of (2.21)-(2.22) for $(\boldsymbol{E}_h^k, \boldsymbol{H}_h^k)$, then update all $\boldsymbol{P}_h^k$ ($k=1,\cdots,N$) in parallel. Similar results as Lemmas 2.1-2.2 and Theorem 2.1 can be proved by following the same arguments used for one pole model.

**Remark 2.3.** If the solution under consideration has enough regularity, then high-order tetrahedral and cubic Raviart-Thomas-Nédélec spaces [25, 26] can be used and similar high-order accuracy can be proved by following the same technique used above.

## 3. Lorentz Medium

We now turn to the Lorentzian two pole model, which are described by the following equations [20, 30]:

$$\epsilon_0\epsilon_\infty\frac{\partial \boldsymbol{E}}{\partial t} = \nabla\times\boldsymbol{H} - \boldsymbol{J}, \tag{3.1}$$

$$\mu_0\frac{\partial \boldsymbol{H}}{\partial t} = -\nabla\times\boldsymbol{E}, \tag{3.2}$$

$$\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}\frac{\partial \boldsymbol{J}}{\partial t} + \frac{\nu}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}\boldsymbol{J} = \boldsymbol{E} - \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\boldsymbol{P}, \tag{3.3}$$

$$\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\frac{\partial \boldsymbol{P}}{\partial t} = \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\boldsymbol{J}, \tag{3.4}$$

with the perfect conducting boundary condition (2.4) and the initial conditions

$$\boldsymbol{E}(\boldsymbol{x},0)=\boldsymbol{E}_0(\boldsymbol{x}),\quad \boldsymbol{H}(\boldsymbol{x},0)=\boldsymbol{H}_0(\boldsymbol{x}),\quad \boldsymbol{J}(\boldsymbol{x},0)=\boldsymbol{J}_0(\boldsymbol{x}),\quad \boldsymbol{P}(\boldsymbol{x},0)=\boldsymbol{P}_0(\boldsymbol{x})\quad \boldsymbol{x}\in\Omega, \tag{3.5}$$

where $\boldsymbol{E}_0, \boldsymbol{H}_0, \boldsymbol{J}_0$ and $\boldsymbol{P}_0$ are given functions. Here in addition to the notation defined earlier, $\omega_1$ is the resonant frequency, $\nu$ is the damping coefficient, $\boldsymbol{P}$ is the polarization vector, and $\boldsymbol{J}$ is the polarization current.

Similar to the Debye medium, we can obtain the weak formulation for (3.1)-(3.4): Find $\boldsymbol{E}\in C(0,T;H_0(\mathrm{curl};\Omega))\cap C^1(0,T;(L_2(\Omega))^3), \boldsymbol{H},\boldsymbol{J},\boldsymbol{P}\in C^1(0,T;(L_2(\Omega))^3)$ such that

$$\epsilon_0\epsilon_\infty\left(\frac{\partial \boldsymbol{E}}{\partial t},\phi\right) - (\boldsymbol{H},\nabla\times\phi) + (\boldsymbol{J},\phi) = 0, \qquad \forall\ \phi\in H_0(\mathrm{curl};\Omega), \tag{3.6}$$

$$\mu_0\left(\frac{\partial \boldsymbol{H}}{\partial t},\psi\right) + (\nabla\times\boldsymbol{E},\psi) = 0, \qquad \forall\ \psi\in (L_2(\Omega))^3, \tag{3.7}$$

$$\begin{aligned}&\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}\left(\frac{\partial \boldsymbol{J}}{\partial t},\phi_1\right) + \frac{\nu}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}(\boldsymbol{J},\phi_1)\\ &\quad - (\boldsymbol{E},\phi_1) + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}(\boldsymbol{P},\phi_1) = 0, \qquad \forall\ \phi_1\in (L_2(\Omega))^3,\end{aligned} \tag{3.8}$$

$$\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}(\frac{\partial \boldsymbol{P}}{\partial t},\phi_2) - \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}(\boldsymbol{J},\phi_2) = 0, \quad \forall\ \phi_2\in (L_2(\Omega))^3. \tag{3.9}$$

**Lemma 3.1.** *Let $(\boldsymbol{E}(t),\boldsymbol{H}(t),\boldsymbol{J}(t),\boldsymbol{P}(t))$ be the solution of (3.6)-(3.9). Then for $0\le t\le T$,*

$$\begin{aligned}&\epsilon_0\epsilon_\infty||\boldsymbol{E}(t)||_0^2 + \mu_0||\boldsymbol{H}(t)||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}||\boldsymbol{J}(t)||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}||\boldsymbol{P}(t)||_0^2\\ \le&\epsilon_0\epsilon_\infty||\boldsymbol{E}_0||_0^2 + \mu_0||\boldsymbol{H}_0||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}||\boldsymbol{J}_0||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}||\boldsymbol{P}_0||_0^2.\end{aligned}$$

*Proof.* Let $\phi=\boldsymbol{E}, \psi=\boldsymbol{H}, \phi_1=\boldsymbol{J}$ and $\phi_2=\boldsymbol{P}$ in (3.6)-(3.9), respectively, then add the resultants together, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\epsilon_0\epsilon_\infty||\boldsymbol{E}(t)||_0^2 + \mu_0||\boldsymbol{H}(t)||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}||\boldsymbol{J}(t)||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}||\boldsymbol{P}(t)||_0^2\right)\\ &+\frac{\nu}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}||\boldsymbol{J}(t)||_0^2 = 0,\end{aligned}$$

which concludes the proof by using the fact $\nu\ge 0$ and integrating with respect to $t$. $\square$

**Remark 3.1.** The Lorentz model (3.1)-(3.4) can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} \epsilon_0\epsilon_\infty \boldsymbol{E} \\ \mu_0 \boldsymbol{H} \\ \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}\boldsymbol{J} \\ \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\boldsymbol{P} \end{pmatrix} = \begin{pmatrix} 0 & \nabla\times & 0 & 0 \\ -\nabla\times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \\ \boldsymbol{J} \\ \boldsymbol{P} \end{pmatrix} + \mathcal{B}\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \\ \boldsymbol{J} \\ \boldsymbol{P} \end{pmatrix}, \tag{3.10}$$

where the matrix

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{\nu}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2} & -\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0} \\ 0 & 0 & \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0} & 0 \end{pmatrix}$$

is skew-symmetric. The stability can be obtained by multiplying (3.10) by a row vector $(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J}, \boldsymbol{P})$.

Similar to the Debye medium case, we can formulate the Crank-Nicolson mixed finite element scheme for (3.6)-(3.9) as follows: for $k = 1, \cdots, M$, find $(\boldsymbol{E}_h^k, \boldsymbol{H}_h^k, \boldsymbol{J}_h^k, \boldsymbol{P}_h^k) \in \mathbf{U}_h^0 \times \mathbf{V}_h \times \mathbf{U}_h \times \mathbf{U}_h$ such that

$$\epsilon_0\epsilon_\infty(\delta_\tau \boldsymbol{E}_h^k, \phi) - (\overline{\boldsymbol{H}}_h^k, \nabla\times\phi) + (\overline{\boldsymbol{J}}_h^k, \phi) = 0, \qquad \forall\ \phi \in \mathbf{U}_h^0, \tag{3.11}$$

$$\mu_0(\delta_\tau \boldsymbol{H}_h^k, \psi) + (\nabla\times\overline{\boldsymbol{E}}_h^k, \psi) = 0, \qquad \forall\ \psi \in \mathbf{V}_h, \tag{3.12}$$

$$\begin{aligned}&\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}(\delta_\tau \boldsymbol{J}_h^k, \phi_1) + \frac{\nu}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}(\overline{\boldsymbol{J}}_h^k, \phi_1) \\ &\quad - (\overline{\boldsymbol{E}}_h^k, \phi_1) + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}(\overline{\boldsymbol{P}}_h^k, \phi_1) = 0, \qquad \forall\ \phi_1 \in \mathbf{U}_h,\end{aligned} \tag{3.13}$$

$$\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}(\delta_\tau \boldsymbol{P}_h^k, \phi_2) - \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}(\overline{\boldsymbol{J}}_h^k, \phi_2) = 0, \quad \forall\ \phi_2 \in \mathbf{U}_h, \tag{3.14}$$

for $0 < t \le T$, subject to the initial conditions

$$\boldsymbol{E}_h^0 = \Pi_h \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}_h^0 = Q_h \boldsymbol{H}_0(\boldsymbol{x}), \quad \boldsymbol{J}_h^0 = \Pi_h \boldsymbol{J}_0(\boldsymbol{x}), \quad \boldsymbol{P}_h^0 = \Pi_h \boldsymbol{P}_0(\boldsymbol{x}). \tag{3.15}$$

Choosing $\phi = \overline{\boldsymbol{E}}_h^k, \psi = \overline{\boldsymbol{H}}_h^k, \phi_1 = \overline{\boldsymbol{J}}_h^k$ and $\phi_2 = \overline{\boldsymbol{P}}_h^k$ in (3.11)-(3.14), respectively, then adding the resultants together, we obtain

$$\begin{aligned}&\frac{1}{2\tau}\Bigg[\epsilon_0\epsilon_\infty\Big(||\boldsymbol{E}_h^k||_0^2 - ||\boldsymbol{E}_h^{k-1}||_0^2\Big) + \mu_0\Big(||\boldsymbol{H}_h^k||_0^2 - ||\boldsymbol{H}_h^{k-1}||_0^2\Big) + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}\Big(||\boldsymbol{J}_h^k||_0^2 - ||\boldsymbol{J}_h^{k-1}||_0^2\Big) \\ &+ \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}(||\boldsymbol{P}_h^k||_0^2 - ||\boldsymbol{P}_h^{k-1}||_0^2)\Bigg] + \frac{\nu}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}||\overline{\boldsymbol{J}}_h^k||_0^2 = 0,\end{aligned}$$

which easily leads to the following unconditional stability:

**Lemma 3.2.** *We have*

$$\begin{aligned}&\epsilon_0\epsilon_\infty||\boldsymbol{E}_h^k||_0^2 + \mu_0||\boldsymbol{H}_h^k||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}||\boldsymbol{J}_h^k||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}||\boldsymbol{P}_h^k||_0^2 \\ \le&\epsilon_0\epsilon_\infty||\boldsymbol{E}_h^0||_0^2 + \mu_0||\boldsymbol{H}_h^0||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}||\boldsymbol{J}_h^0||_0^2 + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}||\boldsymbol{P}_h^0||_0^2.\end{aligned}$$

Note that (3.13) is equivalent to

$$\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}\cdot\frac{\boldsymbol{J}_h^k-\boldsymbol{J}_h^{k-1}}{\tau}+\frac{\nu}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}\cdot\frac{\boldsymbol{J}_h^k+\boldsymbol{J}_h^{k-1}}{2}-\overline{\boldsymbol{E}}_h^k$$
$$+\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\cdot\frac{\boldsymbol{P}_h^k+\boldsymbol{P}_h^{k-1}}{2}=0, \tag{3.16}$$

and (3.14) can be rewritten as

$$\boldsymbol{P}_h^k=\boldsymbol{P}_h^{k-1}+\frac{\tau}{2}(\boldsymbol{J}_h^k+\boldsymbol{J}_h^{k-1}). \tag{3.17}$$

Solving for $\boldsymbol{P}_h^k$ from (3.17) and then substituting it into (3.16), we obtain

$$\beta\boldsymbol{J}_h^k+\left(-\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2\tau}+\frac{\nu}{2(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}+\frac{\tau}{4(\epsilon_s-\epsilon_\infty)\epsilon_0}\right)\boldsymbol{J}_h^{k-1}$$
$$-\overline{\boldsymbol{E}}_h^k+\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\boldsymbol{P}_h^{k-1}=0, \tag{3.18}$$

where we denote

$$\beta=\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2\tau}+\frac{\nu}{2(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}+\frac{\tau}{4(\epsilon_s-\epsilon_\infty)\epsilon_0}.$$

Then substituting (3.18) into (3.11), we can rewrite (3.11) as follows:

$$\begin{aligned}&\left(\frac{\epsilon_0\epsilon_\infty}{\tau}+\frac{1}{4\beta}\right)(\boldsymbol{E}_h^k,\phi)-\frac{1}{2}(\boldsymbol{H}_h^k,\nabla\times\phi)\\=&\left(\frac{\epsilon_0\epsilon_\infty}{\tau}-\frac{1}{4\beta}\right)(\boldsymbol{E}_h^{k-1},\phi)+\frac{1}{2}(\boldsymbol{H}_h^{k-1},\nabla\times\phi)\\&-\frac{1}{\beta(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2\tau}(\boldsymbol{J}_h^{k-1},\phi)+\frac{1}{2\beta(\epsilon_s-\epsilon_\infty)\epsilon_0}(\boldsymbol{P}_h^{k-1},\phi),\quad\forall\,\phi\in\mathbf{U}_h^0.\end{aligned} \tag{3.19}$$

On the other hand, we can rewrite (3.12) as

$$\frac{\mu_0}{\tau}(\boldsymbol{H}_h^k,\psi)+\frac{1}{2}(\nabla\times\boldsymbol{E}_h^k,\psi)=\frac{\mu_0}{\tau}(\boldsymbol{H}_h^{k-1},\psi)-\frac{1}{2}(\nabla\times\boldsymbol{E}_h^{k-1},\psi),\quad\forall\,\psi\in\mathbf{V}_h. \tag{3.20}$$

Hence, the Crank-Nicolson mixed finite element scheme (3.11)-(3.14) can be programmed as follows: at each time step, we first solve a system of (3.19)-(3.20) for $\boldsymbol{E}_h^k$ and $\boldsymbol{H}_h^k$, then update $\boldsymbol{J}_h^k$ by (3.18), and finally update $\boldsymbol{P}_h^k$ by (3.17). By the same technique used for Debye medium, the coefficient matrix of the system (3.19)-(3.20) can be proved to be non-singular.

**Theorem 3.1.** *Let* $(\boldsymbol{E}^n,\boldsymbol{H}^n,\boldsymbol{J}^n,\boldsymbol{P}^n)$ *and* $(\boldsymbol{E}_h^n,\boldsymbol{H}_h^n,\boldsymbol{J}_h^n,\boldsymbol{P}_h^n)$ *be the solutions of* (3.1)-(3.4) *and* (3.11)-(3.14) *at time* $t^n=n\tau$, *respectively. Then there is a constant* $C=C(T,\epsilon_0,\mu_0,\epsilon_s,\epsilon_\infty,\omega_1,\nu,\boldsymbol{E},\boldsymbol{H},\boldsymbol{J},\boldsymbol{P})$, *independent of both the time step* $\tau$ *and the mesh size* $h$, *such that*

$$\begin{aligned}&\max_{1\le n\le M}\Big(||\boldsymbol{E}^n-\boldsymbol{E}_h^n||_0+||\boldsymbol{H}^n-\boldsymbol{H}_h^n||_0+||\boldsymbol{J}^n-\boldsymbol{J}_h^n||_0+||\boldsymbol{P}^n-\boldsymbol{P}_h^n||_0\Big)\\\le&C(h^\alpha+\tau^2).\end{aligned}$$

*Proof.* Denote

$$\xi_h^k = \Pi_h \boldsymbol{E}^k - \boldsymbol{E}_h^k, \quad \eta_h^k = Q_h \boldsymbol{H}^k - \boldsymbol{H}_h^k, \quad \xi_{1h}^k = \Pi_h \boldsymbol{J}^k - \boldsymbol{J}_h^k, \quad \xi_{2h}^k = \Pi_h \boldsymbol{P}^k - \boldsymbol{P}_h^k.$$

Integrating (3.6)-(3.9) in time over $I^k$, dividing by $\tau$, then subtracting the resultants from (3.11)-(3.14), respectively, we obtain the error equations

$$\begin{aligned}
(i) \qquad & \epsilon_0\epsilon_\infty(\delta_\tau \xi_h^k, \phi) - (\overline{\eta}_h^k, \nabla\times\phi) + (\overline{\xi}_{1h}^k, \phi) \\
&= \epsilon_0\epsilon_\infty(\delta_\tau(\Pi_h \boldsymbol{E}^k - \boldsymbol{E}^k), \phi) \\
&\quad - \left(Q_h \overline{\boldsymbol{H}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{H}(s)ds, \nabla\times\phi\right) + \left(\Pi_h \overline{\boldsymbol{J}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{J}(s)ds, \phi\right), \\
(ii) \qquad & \mu_0(\delta_\tau \eta_h^k, \psi) + (\nabla\times\overline{\xi}_h^k, \psi) \\
&= \mu_0(\delta_\tau(Q_h \boldsymbol{H}^k - \boldsymbol{H}^k), \psi) + \left(\nabla\times(\Pi_h \overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{E}(s)ds), \psi\right), \\
(iii) \qquad & \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}(\delta_\tau \xi_{1h}^k, \phi_1) + \frac{\nu}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}(\overline{\xi}_{1h}^k, \phi_1) - (\overline{\xi}_h^k, \phi_1) + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}(\overline{\xi}_{2h}^k, \phi_1) \\
&= \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}\left(\delta_\tau(\Pi_h \boldsymbol{J}^k - \boldsymbol{J}^k), \phi_1\right) + \frac{\nu}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}\left(\Pi_h \overline{\boldsymbol{J}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{J}(s)ds, \phi_1\right) \\
&\quad - \left(\Pi_h \overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{E}(s)ds, \phi_1\right) + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\left(\Pi_h \overline{\boldsymbol{P}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{P}(s)ds, \phi_1\right), \\
(iv) \qquad & \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}(\delta_\tau \xi_{2h}^k, \phi_2) - \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}(\overline{\xi}_{1h}^k, \phi_2) \\
&= \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\left(\delta_\tau(\Pi_h \boldsymbol{P}^k - \boldsymbol{P}^k), \phi_2\right) - \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\left(\Pi_h \overline{\boldsymbol{J}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{J}(s)ds, \phi_2\right).
\end{aligned}$$

Choosing $\phi = 2\tau\overline{\xi}_h^k, \psi = 2\tau\overline{\eta}_h^k, \phi_1 = 2\tau\overline{\xi}_{1h}^k, \phi_2 = 2\tau\overline{\xi}_{2h}^k$ in the above equations, respectively, then adding the resultants together, we obtain

$$\begin{aligned}
&\epsilon_0\epsilon_\infty\left(||\xi_h^k||_0^2 - ||\xi_h^{k-1}||_0^2\right) + \mu_0\left(||\eta_h^k||_0^2 - ||\eta_h^{k-1}||_0^2\right) + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}\left(||\xi_{1h}^k||_0^2 - ||\xi_{1h}^{k-1}||_0^2\right) \\
&\quad + \frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\left(||\xi_{2h}^k||_0^2 - ||\xi_{2h}^{k-1}||_0^2\right) + \frac{2\tau\nu}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}||\overline{\xi}_{1h}^k||_0^2 \\
&= 2\tau\epsilon_0\epsilon_\infty(\delta_\tau(\Pi_h \boldsymbol{E}^k - \boldsymbol{E}^k), \overline{\xi}_h^k) - 2\tau\left(\overline{\boldsymbol{H}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{H}(s)ds, \nabla\times\overline{\xi}_h^k\right) \\
&\quad + 2\tau\left(\Pi_h \overline{\boldsymbol{J}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{J}(s)ds, \overline{\xi}_h^k\right) + 2\tau\left(\nabla\times(\Pi_h \overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{E}(s)ds), \overline{\eta}_h^k\right) \\
&\quad + \frac{2\tau}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}(\delta_\tau(\Pi_h \boldsymbol{J}^k - \boldsymbol{J}^k), \overline{\xi}_{1h}^k) + \frac{2\tau\nu}{(\epsilon_s-\epsilon_\infty)\epsilon_0\omega_1^2}\left(\Pi_h \overline{\boldsymbol{J}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{J}(s)ds, \overline{\xi}_{1h}^k\right) \\
&\quad - 2\tau(\Pi_h \overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{E}(s)ds, \overline{\xi}_{1h}^k) + \frac{2\tau}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\left(\Pi_h \overline{\boldsymbol{P}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{P}(s)ds, \overline{\xi}_{1h}^k\right) \\
&\quad + \frac{2\tau}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\left(\delta_\tau(\Pi_h \boldsymbol{P}^k - \boldsymbol{P}^k), \overline{\xi}_{2h}^k\right) - \frac{2\tau}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\left(\Pi_h \overline{\boldsymbol{J}}^k - \frac{1}{\tau}\int_{I^k} \boldsymbol{J}(s)ds, \overline{\xi}_{2h}^k\right).
\end{aligned}$$

The rest proof follows exactly the same way as we did for the Debye medium case. ☐

**Remark 3.2.** Similar to the Debye medium, a Lorentz medium having $N$ $(N \geq 1)$ pole pairs is often used and can be described by the following system of equations [31]:

$$
\begin{aligned}
&\epsilon_0\epsilon_\infty \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{H} - \sum_{k=1}^{N} \boldsymbol{J}_k,\\
&\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = -\nabla \times \boldsymbol{E},\\
&\frac{1}{(\epsilon_{s1}-\epsilon_\infty)\epsilon_0\omega_1^2}\frac{\partial \boldsymbol{J}_1}{\partial t} + \frac{\nu_1}{(\epsilon_{s1}-\epsilon_\infty)\epsilon_0\omega_1^2}\boldsymbol{J}_1 = G_1\boldsymbol{E} - \frac{1}{(\epsilon_{s1}-\epsilon_\infty)\epsilon_0}\boldsymbol{P}_1,\\
&\frac{1}{(\epsilon_{s1}-\epsilon_\infty)\epsilon_0}\frac{\partial \boldsymbol{P}_1}{\partial t} = \frac{1}{(\epsilon_{s1}-\epsilon_\infty)\epsilon_0}\boldsymbol{J}_1,\\
&\cdots\cdots\\
&\frac{1}{(\epsilon_{sN}-\epsilon_\infty)\epsilon_0\omega_N^2}\frac{\partial \boldsymbol{J}_N}{\partial t} + \frac{\nu_N}{(\epsilon_{sN}-\epsilon_\infty)\epsilon_0\omega_N^2}\boldsymbol{J}_N = G_N\boldsymbol{E} - \frac{1}{(\epsilon_{sN}-\epsilon_\infty)\epsilon_0}\boldsymbol{P}_N,\\
&\frac{1}{(\epsilon_{sN}-\epsilon_\infty)\epsilon_0}\frac{\partial \boldsymbol{P}_N}{\partial t} = \frac{1}{(\epsilon_{sN}-\epsilon_\infty)\epsilon_0}\boldsymbol{J}_N.
\end{aligned}
$$

Here the constants $\nu_k \geq 0, G_k \geq 0, k = 1, \cdots, N$, and $\sum_{k=1}^{N} G_k = 1$.

Note that we rewrite and generalize the original equations of [31] into the above form in order to see clearly that similar results as Lemmas 3.1-3.2 and Theorem 3.1 can be extended directly to the multiple pole Lorentz model. For example, Lemma 3.1 should be extended to

$$
\begin{aligned}
&\epsilon_0\epsilon_\infty||\boldsymbol{E}(t)||_0^2 + \mu_0||\boldsymbol{H}(t)||_0^2 + \sum_{k=1}^{N}\left(\frac{1}{(\epsilon_{sk}-\epsilon_\infty)\epsilon_0\omega_k^2}||\boldsymbol{J}_k(t)||_0^2 + \frac{1}{(\epsilon_{sk}-\epsilon_\infty)\epsilon_0}||\boldsymbol{P}_k(t)||_0^2\right)\\
\leq &\epsilon_0\epsilon_\infty||\boldsymbol{E}_0||_0^2 + \mu_0||\boldsymbol{H}_0||_0^2 + \sum_{k=1}^{N}\left(\frac{1}{(\epsilon_{sk}-\epsilon_\infty)\epsilon_0\omega_k^2}||\boldsymbol{J}_{k0}||_0^2 + \frac{1}{(\epsilon_{sk}-\epsilon_\infty)\epsilon_0}||\boldsymbol{P}_{k0}||_0^2\right).
\end{aligned}
$$

## 4. Isotropic Cold Plasma

The governing equations that describe electromagnetic wave propagation in isotropic non-magnetized cold electron plasma are [20, 30]:

$$\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{H} - \boldsymbol{J}, \tag{4.1}$$

$$\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = -\nabla \times \boldsymbol{E}, \tag{4.2}$$

$$\frac{1}{\epsilon_0\omega_p^2}\frac{\partial \boldsymbol{J}}{\partial t} + \frac{\nu}{\epsilon_0\omega_p^2}\boldsymbol{J} = \boldsymbol{E}, \tag{4.3}$$

with the perfect conducting boundary condition (2.4) and the initial conditions

$$\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0(\boldsymbol{x}), \quad \boldsymbol{J}(\boldsymbol{x},0) = \boldsymbol{J}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{4.4}$$

where $\boldsymbol{E}_0, \boldsymbol{H}_0$ and $\boldsymbol{J}_0$ are given functions. Here in addition to the notation defined earlier, $\omega_p$ is the plasma frequency, $\boldsymbol{J}$ is the polarization current, and $\nu$ is the electron-neutral collision frequency. Note that $\nu = 0$ reduces to the collisionless case.

Similar to the Debye medium, we can obtain the weak formulation for (4.1)-(4.3): Find $\boldsymbol{E} \in C(0,T;H_0(\text{curl};\Omega)) \cap C^1(0,T;(L_2(\Omega))^3), \boldsymbol{H}, \boldsymbol{J} \in C^1(0,T;(L_2(\Omega))^3)$ such that

$$\epsilon_0 \left(\frac{\partial \boldsymbol{E}}{\partial t}, \phi\right) - (\boldsymbol{H}, \nabla \times \phi) + (\boldsymbol{J}, \phi) = 0, \qquad \forall\, \phi \in H_0(\text{curl};\Omega), \tag{4.5}$$

$$\mu_0 \left(\frac{\partial \boldsymbol{H}}{\partial t}, \psi\right) + (\nabla \times \boldsymbol{E}, \psi) = 0, \qquad \forall\, \psi \in (L_2(\Omega))^3, \tag{4.6}$$

$$\frac{1}{\epsilon_0 \omega_p^2} \left(\frac{\partial \boldsymbol{J}}{\partial t}, \phi_1\right) + \frac{\nu}{\epsilon_0 \omega_p^2}(\boldsymbol{J}, \phi_1) - (\boldsymbol{E}, \phi_1) = 0, \quad \forall\, \phi_1 \in (L_2(\Omega))^3. \tag{4.7}$$

**Lemma 4.1.** *Let $(\boldsymbol{E}(t), \boldsymbol{H}(t), \boldsymbol{J}(t))$ be the solution of (4.5)-(4.7). Then for $0 \le t \le T$,*

$$\epsilon_0 ||\boldsymbol{E}(t)||_0^2 + \mu_0 ||\boldsymbol{H}(t)||_0^2 + \frac{1}{\epsilon_0 \omega_p^2} ||\boldsymbol{J}(t)||_0^2 \le \epsilon_0 ||\boldsymbol{E}_0||_0^2 + \mu_0 ||\boldsymbol{H}_0||_0^2 + \frac{1}{\epsilon_0 \omega_1^2} ||\boldsymbol{J}_0||_0^2.$$

*Proof.* Let $\phi = \boldsymbol{E}, \psi = \boldsymbol{H}, \phi_1 = \boldsymbol{J}$ and $\phi_2 = \boldsymbol{P}$ in (4.5)-(4.7), respectively, we can easily obtain

$$\frac{1}{2}\frac{d}{dt}\left(\epsilon_0 ||\boldsymbol{E}(t)||_0^2 + \mu_0 ||\boldsymbol{H}(t)||_0^2 + \frac{1}{\epsilon_0 \omega_p^2} ||\boldsymbol{J}(t)||_0^2\right) + \frac{\nu}{\epsilon_0 \omega_p^2} ||\boldsymbol{J}(t)||_0^2 = 0,$$

which concludes the proof by using the fact $\nu \ge 0$ and integrating with respect to $t$. □

**Remark 4.1.** The plasma model (4.1)-(4.3) can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} \epsilon_0 \boldsymbol{E} \\ \mu_0 \boldsymbol{H} \\ \frac{1}{\epsilon_0 \omega_p^2} \boldsymbol{J} \end{pmatrix} = \begin{pmatrix} 0 & \nabla\times & 0 \\ -\nabla\times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \\ \boldsymbol{J} \end{pmatrix} + \mathcal{B} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \\ \boldsymbol{J} \end{pmatrix}, \tag{4.8}$$

where the matrix

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -\frac{\nu}{\epsilon_0 \omega_p^2} \end{pmatrix}$$

is skew-symmetric. The stability can be obtained by multiplying (4.8) by a row vector $(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J})$.

Similar to the Debye medium case, we can formulate the Crank-Nicolson mixed finite element scheme for (4.5)-(4.7) as follows: for $k = 1, \cdots, M$, find $(\boldsymbol{E}_h^k, \boldsymbol{H}_h^k, \boldsymbol{J}_h^k) \in \mathbf{U}_h^0 \times \mathbf{V}_h \times \mathbf{U}_h$ such that

$$\epsilon_0 (\delta_\tau \boldsymbol{E}_h^k, \phi) - (\overline{\boldsymbol{H}}_h^k, \nabla \times \phi) + (\overline{\boldsymbol{J}}_h^k, \phi) = 0, \qquad \forall\, \phi \in \mathbf{U}_h^0, \tag{4.9}$$

$$\mu_0 (\delta_\tau \boldsymbol{H}_h^k, \psi) + (\nabla \times \overline{\boldsymbol{E}}_h^k, \psi) = 0, \qquad \forall\, \psi \in \mathbf{V}_h, \tag{4.10}$$

$$\frac{1}{\epsilon_0 \omega_p^2} (\delta_\tau \boldsymbol{J}_h^k, \phi_1) + \frac{\nu}{\epsilon_0 \omega_p^2} (\overline{\boldsymbol{J}}_h^k, \phi_1) - (\overline{\boldsymbol{E}}_h^k, \phi_1) = 0, \quad \forall\, \phi_1 \in \mathbf{U}_h, \tag{4.11}$$

for $0 < t \le T$, subject to the initial conditions

$$\boldsymbol{E}_h^0 = \Pi_h \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}_h^0 = Q_h \boldsymbol{H}_0(\boldsymbol{x}), \quad \boldsymbol{J}_h^0 = \Pi_h \boldsymbol{J}_0(\boldsymbol{x}). \tag{4.12}$$

Choosing $\phi = \overline{\boldsymbol{E}}_h^k, \psi = \overline{\boldsymbol{H}}_h^k$ and $\phi_1 = \overline{\boldsymbol{J}}_h^k$ in (4.9)-(4.11), respectively, and adding the resultants together, we obtain

$$\frac{1}{2\tau}\Bigg[\epsilon_0\left(||\boldsymbol{E}_h^k||_0^2 - ||\boldsymbol{E}_h^{k-1}||_0^2\right) + \mu_0\left(||\boldsymbol{H}_h^k||_0^2 - ||\boldsymbol{H}_h^{k-1}||_0^2\right) + \frac{1}{\epsilon_0 \omega_p^2}\left(||\boldsymbol{J}_h^k||_0^2 - ||\boldsymbol{J}_h^{k-1}||_0^2\right)\Bigg] + \frac{\nu}{\epsilon_0 \omega_p^2} ||\overline{\boldsymbol{J}}_h^k||_0^2 = 0,$$

which easily leads to the following unconditional stability:

**Lemma 4.2.** *Let* $\boldsymbol{E}_h^k$, $\boldsymbol{H}_h^k$ *and* $\boldsymbol{J}_h^k$ *be the solution of* (4.9)-(4.12). *Then*

$$\epsilon_0||\boldsymbol{E}_h^k||_0^2 + \mu_0||\boldsymbol{H}_h^k||_0^2 + \frac{1}{\epsilon_0\omega_p^2}||\boldsymbol{J}_h^k||_0^2 \le \epsilon_0||\boldsymbol{E}_h^0||_0^2 + \mu_0||\boldsymbol{H}_h^0||_0^2 + \frac{1}{\epsilon_0\omega_1^2}||\boldsymbol{J}_h^0||_0^2.$$

Notice that the scheme (4.9)-(4.11) for the plasma medium is reduced to the scheme (3.11)-(3.14) for the Lorentz medium (except differences in coefficients) by dropping the last term of (3.13) and the equation (3.14). Hence the previous proof for the Lorentz medium can be carried out directly to the plasma medium, in which case we have the following optimal error estimate:

**Theorem 4.1.** *Let* $(\boldsymbol{E}^n, \boldsymbol{H}^n, \boldsymbol{J}^n)$ *and* $(\boldsymbol{E}_h^n, \boldsymbol{H}_h^n, \boldsymbol{J}_h^n)$ *be the solutions of* (4.5)-(4.7) *and* (4.9)-(4.11) *at time* $t^n = n\tau$*, respectively. Then there is a constant* $C = C(T, \epsilon_0, \mu_0, \omega_p, \nu, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J})$*, independent of both the time step* $\tau$ *and the mesh size* $h$*, such that*

$$\max_{1\le n\le M}\left(||\boldsymbol{E}^n - \boldsymbol{E}_h^n||_0 + ||\boldsymbol{H}^n - \boldsymbol{H}_h^n||_0 + ||\boldsymbol{J}^n - \boldsymbol{J}_h^n||_0\right) \le C(h^\alpha + \tau^2).$$

## 5. Conclusions

In this paper we have studied mixed finite element methods for the time-dependent Maxwell's equations in dispersive media. We propose a general framework that allows us to obtain a unified analysis for all three most popular dispersive medium models. Our error analysis show that the proposed Crank-Nicolson scheme is optimally convergent in the $L_2$ norm for regular meshes. Since this is our first comprehensive study of the dispersive medium models (especially the multiple pole models), more advanced algorithms such as the mixed discontinuous Galerkin method and posteriori error estimators will be explored in the future.

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## References

[1] M. Ainsworth and J. Coyle, Hierarchic *hp*-edge element families for Maxwell's equations on hybrid quadrilateral/triangular meshes, *Comput. Method. Appl. M.*, **190** (2001), 6709-6733.

[2] C. Amrouche, C. Bernardi, M. Dauge and V. Girault, Vector potentials in three-dimensional non-smooth domains, *Math. Method. Appl. Sci.*, **21** (1998), 823-864.

[3] D.N. Arnold, R.S. Falk and R. Winther, Multigrid in $H$(div) and $H$(curl), *Numer. Math.*, **85** (2000), 197-217.

[4] H.T. Bank, V.A. Bokil and N.L. Gibson, Analysis of stability and dispersion in a finite element method for Debye and Lorentz dispersive media. Report, North Carolina State University, August 2006. 74 pp.

[5] G. Bao, Z. Chen and H. Wu, Adaptive finite-element method for diffraction gratings, *J. Opt. Soc. Am. A*, **22** (2005), 1106-1114.

[6] R. Beck, R. Hiptmair, R.H.W. Hoppe and B. Wohlmuth, Residual based a posteriori error estimators for eddy current computation, *M2AN Math. Model. Numer. Anal.*, **34** (2000), 159-182.

[7] D. Boffi, F. Kikuchi and J. Schöberl, Edge element computation of Maxwell's eigenvalues on general quadrilateral meshes, *Math. Mod. Meth. Appl. S.*, **16** (2006), 265-273.

[8] P. Ciarlet, Jr. and J. Zou, Fully discrete finite element approaches for time-dependent Maxwell's equations, *Numer. Math.*, **82** (1999), 193-219.

[9] B. Cockburn, F. Li and C.-W. Shu, Locally divergence-free discontinuous Galerkin methods for the Maxwell equations, *J. Comput. Phys.*, **194** (2004), 588-610.

[10] M. Costabel, M. Dauge ans S. Nicaise, Singularities of Maxwell interface problems, *M2AN Math. Model. Numer. Anal.*, **33** (1999), 627-649.

[11] G. Fairweather, Finite Element Galerkin Methods for Differential Equations, Marcel Dekker, New York, 1978.

[12] J. Gopalakrishnan and J.E. Pasciak and L. Demkowicz, Analysis of a multigrid algorithm for time harmonic Maxwell equations, *SIAM J. Numer. Anal.*, **42** (2004), 90-108.

[13] J.S. Hesthaven and T. Warburton, High-order nodal methods on unstructured grids. I. time-domain solution of Maxwell's equations, *J. Comput. Phys.*, **181** (2002), 186-221.

[14] R. Hiptmair, Finite elements in computational electromagnetism, *Acta Numerica*, **11** (2002), 237-339.

[15] R. Hiptmair and J. Xu, Nodal auxiliary space preconditioning in H(curl) and H(div) spaces, *SIAM J. Numer. Anal.*, **45** (2007), 2483-2509.

[16] P. Houston, I. Perugia and D. Schötzau, Energy norm a posteriori error estimation for mixed discontinuous Galerkin approximations of the Maxwell operator, *Comput. Method. Appl. M.*, **194** (2005), 499-510.

[17] Q. Hu, S. Shu and J. Zou, A mortar edge element method with nearly optimal convergence for three-dimensional Maxwell's equations, *Math. Comput.*, **77** (2008), 1333-1353.

[18] D. Jiao and J.-M. Jin, Time-domain finite-element modeling of dispersive media, *IEEE Microw. Wirel. Co.*, **11** (2001), 220-223.

[19] K. Kunz and R.J. Luebbers, The Finite-Difference Time-Domain Method for Electromagnetics, CRC Press, Boca Raton, FL, 1993.

[20] J. Li, Error analysis of fully discrete mixed finite element schemes for 3-D Maxwell's equations in dispersive media, *Comput. Method. Appl. M.*, **196** (2007), 3081-3094.

[21] J. Li and Y. Chen, Analysis of a time-domain finite element method for 3-D Maxwell's equations in dispersive media, *Comput. Method. Appl. M.*, **195** (2006), 4220-4229.

[22] Q. Lin and J. Li, Superconvergence analysis for Maxwell's equations in dispersive media, *Math. Comput.*, **77** (2008), 757-771.

[23] Q. Lin and N. Yan, Global superconvergence for Maxwell's equations, *Math. Comput.*, **69** (1999), 159-176.

[24] T. Lu, P. Zhang and W. Cai, Discontinuous Galerkin methods for dispersive and lossy Maxwell's equations and PML boundary conditions, *J. Comput. Phys.*, **200** (2004), 549-580.

[25] P. Monk, Finite Element Methods for Maxwell's Equations, Oxford University Press, 2003.

[26] J.-C. Nédélec, Mixed finite elements in $R^3$, *Numer. Math.*, **35** (1980), 315-341.

[27] A.A. Rodríguez, R. Hiptmair and A. Valli, Mixed finite element approxiamtion of eddy current problems, *IMA J. Numer. Anal.*, **24** (2004), 255-271.

[28] M. Slodicka and J. Busa Jr., Error estimates for the time discretization for nonlinear Maxwell's equations, *J. Comput. Math.*, **26** (2008), 677-688.

[29] A. Taflove and C. Hagness, Computational Electrodynamics: the Finite-Difference Time-Domain Method, 2nd ed., Artech House, Norwood, MA, 2000.

[30] J.L. Young, Propagation in linear dispersive media: finite difference time-domain methodologies, *IEEE T. Antenn. Propag.*, **43** (1995), 422-426.

[31] J.L. Young and R.O. Nelson, A summary and systematic analysis of FDTD algorithms for linearly dispersive media, *IEEE Antenn. Propag. M.*, **43** (2001), 61-77.

[32] L. Zhong, S. Shu, G. Wittum and J. Xu Optimal error estimates for Nedelec edge elements for time-harmonic Maxwell's equations, *J. Comput. Math.*, **27** (2009), 563-572.