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A POSTERIORI ERROR ESTIMATE FOR BOUNDARY CONTROL PROBLEMS GOVERNED BY THE PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS*

Wei Gong and Ningning Yan
LSEC, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
Eamil: gongwei@amss.ac.cn, ynn@amss.ac.cn

Abstract

In this paper, we discuss the a posteriori error estimate of the finite element approximation for the boundary control problems governed by the parabolic partial differential equations. Three different a posteriori error estimators are provided for the parabolic boundary control problems with the observations of the distributed state, the boundary state and the final state. It is proven that these estimators are reliable bounds of the finite element approximation errors, which can be used as the indicators of the mesh refinement in adaptive finite element methods.

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Key words: Boundary control problems, Finite element method, A posteriori error estimate, Parabolic partial differential equations.

1. Introduction

Finite element approximation plays a very important role in the numerical methods for optimal control problems. There have been extensive theoretical and numerical studies in this research direction. For instance, the error analysis for optimal control problems governed by linear elliptic equations has been established in [12,13], the error estimates for some important flow control problems are given in [14], the error estimates for Dirichlet boundary control governed by semilinear elliptic equations are provided in [6]. Some recent progress in this area has been summarized in [24].

In recent years, the adaptive finite element method has been investigated extensively. It has become one of the most popular methods in the scientific computation and numerical modelling. Adaptive finite element approximation ensures a higher density of nodes in a certain area of the given domain, where the solution is more difficult to approximate, indicated by a posteriori error estimators. Hence it is among the most important means to boost the accuracy and efficiency of finite element discretizations. We acknowledge the pioneering work due to Babuška and Rheinboldt [2]. Further references can be found in the monographs [1,3,28], and the references therein.

Earlier works on a posteriori error estimates are concentrated on the elliptic partial differential equations. Later, there are many works about the a posteriori error estimates for parabolic problems. We mention the work of Eriksson and Johnson [10,11], which is based on the analysis of linear dual problems of the corresponding error equations. The derived a posteriori error estimates depend on the H^2 regularity assumption on the underlying elliptic operator. In [25],

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Picasso derived a posteriori error estimator in the classical L^2-norm in time and H^1-norm in space based on the energy method, and a lower bound for the local error is also derived for the associated a posteriori error indicator. Recently, Chen and Jia [7] obtained an efficient and reliable a posteriori error estimate for linear parabolic equations, which is also in the energy norm and based on a direct energy estimate argument.

In the decades, there appear many works concentrating on the adaptivity of various optimal control problems. For example, [4] studied the adaptive finite element method for the optimal control problems governed by PDE, while a posteriori error estimators for convex distributed optimal control problems governed by elliptic equations, parabolic equations, Stokes equations, integral equations and integro-differential equations are derived in [5,17,19,21–23], respectively, the a posteriori error estimates for the boundary control problems governed by elliptic equation are also discussed in [15,20].

The main objective of this paper is to establish the a posteriori error estimate of the finite element approximation for the boundary control problems governed by the parabolic partial differential equations. Three different a posteriori error estimators are provided for the parabolic boundary control problems with the observations of the distributed state, the boundary state and the final state. It is proven that these estimators are reliable bounds of the finite element approximation errors, and can be used as the indicators of the mesh refinement in adaptive finite element methods. Although we use some ideas and techniques, which have been applied in our earlier work for the parabolic distributed optimal control and the elliptic boundary control (see, e.g., [19,20,23]), in the a posteriori error estimate analysis of this paper, there are some obviously different difficulties which should be solved for the parabolic boundary control problems.

The paper is organized as follows: In section 2, we introduce the model problems and their weak formulations, provide their fully discrete finite element approximation schemes. Then we discuss the a posteriori error estimate of the finite element approximation for the parabolic boundary control problems in Section 3. We provide three different a posteriori error estimators for the parabolic boundary control problems with the observations of the distributed state, the boundary state and the final state in Subsections 3.1, 3.2 and 3.3, respectively.

2. Model Problems and Finite Element Approximations

In this section, we will introduce the boundary control problems governed by the parabolic partial differential equations with three kinds of different observations and their finite element approximations.

Let Ω be a bounded domain in $\mathbf{R}^n (n \leq 3)$ with a Lipschitz boundary $\partial\Omega$. In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,p,\Omega}$ and seminorm $|\cdot|_{m,p,\Omega}$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and set $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v|_{\partial\Omega=0}\}$. We denote by $L^s(0,T;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from $(0,T)$ into $W^{m,p}(\Omega)$ with norm

$$\|v\|_{L^s\left(0,T;W^{m,p}(\Omega)\right)} = \left(\int_0^T \|v\|_{m,p,\Omega}^s dt\right)^{\frac{1}{s}} \quad \text{for} \quad s \in [1,\infty)$$

and the standard modification for $s = \infty$. Similarly, one define the spaces $H^1\big(0,T;W^{m,p}(\Omega)\big)$ and $C^l\big(0,T;W^{m,p}(\Omega)\big)$. In addition c or C denotes a general positive constant independent of the mesh size parameter h.

In the rest of the paper, we shall take the state space $W = L^2(0,T;V)$ with $V = H^1(\Omega)$, the control space $X = L^2(0,T;U)$ with $U = L^2(\partial\Omega)$, the observation space Y will be specified later depending on the different observations. The model problem we will study in this paper is the following control problems governed by parabolic equation:

$$\min_{u\in K}\{g^*(y)+j^*(u)\}, \tag{2.1a}$$

$$\frac{\partial y}{\partial t} - \mathrm{div}(A\nabla y) + a_0 y = f, \quad x\in\Omega,\ \ t\in(0,T], \tag{2.1b}$$

$$(A\nabla y)\cdot n = Bu + z_b, \quad x\in\partial\Omega,\ \ t\in[0,T], \tag{2.1c}$$

$$y(x,0) = y_0(x), \quad x\in\Omega, \tag{2.1d}$$

where control appears in the Neumann boundary conditions, g^* and j^* are strictly convex functionals which are continuously differentiable on the observation space Y and control space X. We further assume that $j^*(u)\to+\infty$ as $\|u\|_X\to+\infty$ and that the functional $g^*(\cdot)$ is bounded below. In (2.1), the given functions $f\in L^2\big(0,T;L^2(\Omega)\big)$, $y_0\in H^1(\Omega)$, $z_b\in L^2(\partial\Omega)$, $a_0\in L^\infty(\Omega)$ with $a_0\ge c$, and $A(\cdot)=\big(a_{i,j}(\cdot)\big)_{n\times n}\in\big(L^\infty(\Omega)\big)^{n\times n}$ such that there is a constant $c>0$ satisfying

$$X^tAX\ge c\|X\|^2_{\mathbf{R}^n},\quad \forall X\in\mathbf{R}^n.$$

Moreover, let the constraint set K be a closed convex set in the control space X, B be a linear continuous operator from X to $L^2\big(0,T;L^2(\partial\Omega)\big)$.

2.1. Boundary control problem with observation of the distributed state

Firstly, let us consider the boundary control problem (2.1) with observation of the distributed state, i.e., the observation space $Y=L^2\big(0,T;L^2(\Omega)\big)$. Set

$$g^*(y)=\int_0^T\int_\Omega g(y),\quad j^*(u)=\int_0^T\int_{\partial\Omega} j(u),$$

where $g(\cdot)$ and $j(\cdot)$ are strictly convex continuously differentiable functions such that the assumptions on $g^*(\cdot)$ and $j^*(\cdot)$ are satisfied. An example for $g^*(\cdot)$ and $j^*(\cdot)$ is

$$g^*(y)=\frac12\int_0^T\int_\Omega (y-y_s)^2,\quad j^*(u)=\frac{\alpha}{2}\int_0^T\int_{\partial\Omega}u^2,$$

where $y_s\in L^2(\Omega)$ is a given function, α is a positive number. Let

$$\begin{aligned}
a(y,w)&=\int_\Omega (A\nabla y)\cdot\nabla w + a_0 yw\ \ \forall y,w\in H^1(\Omega),\\
(f_1,f_2)&=\int_\Omega f_1f_2\ \ \forall f_1,f_2\in L^2(\Omega),\\
(v,w)_U&=\int_{\partial\Omega} vw\ \ \forall v,w\in L^2(\partial\Omega).
\end{aligned}$$

Then a weak formula of the parabolic boundary control problem (2.1) reads:

$$\min_{u\in K}\big\{g^*(y)+j^*(u)\big\}, \tag{2.2}$$

subject to

$$\Big(\frac{\partial y}{\partial t}, w\Big) + a(y, w) = (f, w) + (Bu + z_b, w)_U, \ \ \forall w \in V, \ t \in (0, T], \tag{2.3a}$$

$$y(x, 0) = y_0(x). \tag{2.3b}$$

It is well known (see, e.g., [18]) that the control problem (2.2)-(2.3) has a unique solution (y, u), and that a pair (y, u) is the solution of (2.2)-(2.3) if and only if there is a co-state $p \in W$ such that the triplet (y, p, u) satisfies the following optimality conditions:

$$\begin{aligned} &\left(\frac{\partial y}{\partial t}, w\right) + a(y, w) = (f, w) + (Bu + z_b, w)_U, \ \ \forall w \in V, \ t \in (0, T], \\ &\qquad y(x, 0) = y_0(x), \end{aligned} \tag{2.4}$$

$$\begin{aligned} &-\left(\frac{\partial p}{\partial t}, q\right) + a(q, p) = (g'(y), q), \ \ \forall q \in V, \ t \in [0, T), \\ &\qquad p(x, T) = 0, \end{aligned} \tag{2.5}$$

$$\int_0^T (j'(u) + B^* p, v - u)_U dt \geq 0, \ \ \forall v \in K, \tag{2.6}$$

where B^* is the adjoint operator of B.

Let us consider the finite element approximation of the control problem (2.2)-(2.3). Here we consider only n-simplex elements, as they are among the most widely used ones. Also we consider only conforming Lagrange elements.

Let Ω^h be a polygonal approximation to Ω with a boundary $\partial\Omega^h$. Let T^h be a partitioning of Ω^h into disjoint regular n-simplices τ, so that $\bar{\Omega}^h = \bigcup_{\tau \in T^h} \bar{\tau}$. For simplicity, we assume that Ω is a polygon or polyhedron such that $\Omega^h = \Omega$. Associated with T^h is a finite dimensional subspace V^h of $C(\bar{\Omega}^h)$, such that $\chi|_\tau$ are polynomials of m-order $(m \geq 1)$ for $\forall \chi \in V^h$ and $\tau \in T^h$. It is easy to see that $V^h \subset V$.

Let T_U^h be a partitioning of $\partial\Omega^h$ into disjoint regular $(n-1)$-simplices s, so that $\partial\Omega^h = \bigcup_{s \in T_U^h} \bar{s}$. Associated with T_U^h is another finite dimensional subspace U^h of $L^2(\partial\Omega^h)$, such that $\chi|_s$ are polynomials of m-order $(m \geq 0)$ for $\forall \chi \in U^h$ and $s \in T_U^h$. Here there is no requirement for the continuity. Let $X^h = L^2(0, T; U^h)$, $K^h = X^h \cap K$. Let h_τ and h_s denote the maximum diameter of the element τ in T^h and s in T_U^h, respectively.

Then the semi-discrete finite element approximation of (2.2)-(2.3) is as follows:

$$\min_{u_h \in K^h} \big\{ g^*(y_h) + j^*(u_h) \big\}, \tag{2.7}$$

subject to

$$\begin{aligned} &\Big(\frac{\partial y_h}{\partial t}, w_h\Big) + a(y_h, w_h) = (f, w_h) + (Bu_h + z_b, w_h)_U, \ \ \forall w_h \in V^h, \ \ t \in (0, T], \\ &y_h(x, 0) = y_0^h(x), \quad x \in \Omega, \end{aligned} \tag{2.8}$$

where $y_h \in H^1(0, T; V^h)$, and $y_0^h \in V^h$ is an approximation of y_0.

It follows that the control problem (2.7)-(2.8) has a unique solution (y_h, u_h) and that a pair (y_h, u_h) is the solution of the problem (2.7)-(2.8) if and only if there is a co-state p_h such that the triplet (y_h, p_h, u_h) satisfies the following optimality conditions:

$$\begin{aligned}
&\Big(\frac{\partial y_h}{\partial t}, w_h\Big) + a(y_h, w_h) = (f, w_h) + (Bu_h + z_b, w_h)_U, \quad \forall w_h \in V^h,\\
&\qquad y_h(x,0) = y_0^h(x), \quad x \in \Omega, \qquad (2.9)\\
&-\Big(\frac{\partial p_h}{\partial t}, q_h\Big) + a(q_h, p_h) = \big(g'(y_h), q_h\big), \quad \forall q_h \in V^h,\\
&\qquad p_h(x,T) = 0, \quad x \in \Omega, \qquad (2.10)\\
&\int_0^T \big(j'(u_h) + B^* p_h, v_h - u_h\big)_U dt \ge 0, \quad \forall v_h \in K^h. \qquad (2.11)
\end{aligned}$$

We next consider the fully discrete approximation for above semidiscrete problem by using the backward Euler scheme in time.

Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, $k_i = t_i - t_{i-1}$, $i = 1, 2, \cdots, N$, $k = \max_{i\in[1,N]}\{k_i\}$. For $i = 1, 2, \cdots, N$, construct the finite element spaces $V_i^h \in H^1(\Omega)$ (similar to V^h) with the mesh T_i^h. Similarly, construct the finite element spaces $U_i^h \in L^2(\partial\Omega)$ (similar to U^h) with the mesh $(T_U^h)_i$. Let $h_{\tau^i}(h_{s^i})$ denote the maximum diameter of the element $\tau^i(s^i)$ in $T_i^h((T_U^h)_i)$. Define mesh functions $\tau(\cdot)$, $s(\cdot)$ and mesh size functions $h_\tau(\cdot)$, $h_s(\cdot)$ such that

$$\tau(t)|_{t\in(t_{i-1},t_i]} = \tau^i, \quad s(t)|_{t\in(t_{i-1},t_i]} = s^i, \quad h_\tau(t)|_{t\in(t_{i-1},t_i]} = h_{\tau^i}, \quad h_s(t)|_{t\in(t_{i-1},t_i]} = h_{s^i}.$$

For ease of exposition, we shall denote $\tau(t), s(t), h_\tau(t)$ and $h_s(t)$ by τ, s, h_τ and h_s, respectively. Let $K_i^h = U_i^h \cap K$. Then the fully discrete approximation scheme of (2.7)-(2.8) is to find $(y_h^i, u_h^i) \in V_i^h \times K_i^h$, $i = 1, 2, \cdots, N$, such that

$$\min_{u_i^h \in K_i^h} \Big\{ \sum_{i=1}^N k_i \int_\Omega \big(g(y_h^i) + h(u_h^i)\big)\Big\}, \qquad (2.12)$$

$$\begin{aligned}
&\Big(\frac{y_h^i - y_h^{i-1}}{k_i}, w_h\Big) + a(y_h^i, w_h) = \big(f(x,t_i), w_h\big) + (Bu_h^i + z_b, w_h)_U,\\
&\qquad \forall w_h \in V_i^h, \quad i = 1, \cdots, N, \quad y_h^0(x) = y_0^h(x), \ x \in \Omega. \qquad (2.13)
\end{aligned}$$

It follows that the above control problem has a unique solution (Y_h^i, U_h^i), $i = 1, 2, \cdots, N$, and that a pair $(Y_h^i, U_h^i) \in V_i^h \times K_i^h$, $i = 1, 2, \cdots, N$, is the solution of (2.12)-(2.13) if and only if there is a co-state $P_h^{i-1} \in V_i^h$, $i = 1, 2, \cdots, N$, such that the triplet $(Y_h^i, P_h^{i-1}, U_h^i) \in V_i^h \times V_i^h \times K_i^h$, $i = 1, 2, \cdots, N$, satisfies the following optimality conditions:

$$\begin{aligned}
&\Big(\frac{Y_h^i - Y_h^{i-1}}{k_i}, w_h\Big) + a(Y_h^i, w_h) = \big(f(x,t_i), w_h\big) + (BU_h^i + z_b, w_h)_U,\\
&\qquad \forall w_h \in V_i^h, \ i = 1, \cdots, N, \quad Y_h^0(x) = y_0^h(x), \ x \in \Omega, \qquad (2.14)\\
&\Big(\frac{P_h^{i-1} - P_h^i}{k_i}, q_h\Big) + a(q_h, P_h^{i-1}) = \big(g'(Y_h^i), q_h\big),\\
&\qquad \forall q_h \in V_i^h, \ i = N, \cdots, 1, \quad P_h^N(x) = 0, \ x \in \Omega, \qquad (2.15)\\
&\big(j'(U_h^i) + B^* P_h^{i-1}, v_h - U_h^i\big)_U \ge 0, \quad \forall v_h \in K_i^h, i = 1, 2, \cdots, N. \qquad (2.16)
\end{aligned}$$

For $i = 1, 2, \cdots, N$, let

$$\begin{aligned}
&Y_h|_{(t_{i-1},t_i]} = \big((t - t_{i-1})Y_h^i + (t_i - t)Y_h^{i-1}\big)/k_i,\\
&P_h|_{(t_{i-1},t_i]} = \big((t - t_{i-1})P_h^i + (t_i - t)P_h^{i-1}\big)/k_i, \quad U_h|_{(t_{i-1},t_i]} = U_h^i.
\end{aligned}$$

For any function $w \in C\big(0,T;L^2(\Omega)\big)$, let $\hat{w}(x,t)|_{(t_{i-1},t_i]} = w(x,t_i)$, $\tilde{w}(x,t)|_{(t_{i-1},t_i]} = w(x,t_{i-1})$. Then the optimality conditions (2.14)-(2.16) can be restated as

$$\begin{aligned}
&\Big(\frac{\partial Y_h}{\partial t}, w_h\Big) + a(\hat{Y}_h, w_h) = (\hat{f}, w_h) + (BU_h + z_b, w_h)_U, \ \ \forall w_h \in V_i^h, \ \ 1 \le i \le N,\\
&\qquad Y_h(x,0) = y_0^h(x), \ \ x \in \Omega, && (2.17)\\
&-\Big(\frac{\partial P_h}{\partial t}, q_h\Big) + a(q_h, \tilde{P}_h) = \big(g'(\hat{Y}_h), q_h\big), \ \ \forall q_h \in V_i^h, \ \ N \le i \le 1,\\
&\qquad P_h(x,T) = 0, \ \ x \in \Omega, && (2.18)\\
&\big(j'(U_h) + B^*\tilde{P}_h, v_h - U_h\big)_U \ge 0, \ \ \forall v_h \in K_i^h, \ \ 1 \le i \le N. && (2.19)
\end{aligned}$$

2.2. Boundary control problem with observation of the boundary state

Next, let us consider another case: the boundary control problem (2.1) with observation of the boundary state, i.e., the observation space $Y = L^2\big(0,T;L^2(\partial\Omega)\big)$. Set

$$g^*(y) = \int_0^T \int_{\partial\Omega} g(y), \quad j^*(u) = \int_0^T \int_{\partial\Omega} j(u),$$

where again $g(\cdot)$ and $j(\cdot)$ are strictly convex continuously differentiable functions such that the assumptions on $g^*(\cdot)$ and $j^*(\cdot)$ are satisfied. Again, the example for $g^*(\cdot)$ can be

$$g^*(y) = \frac{1}{2}\int_0^T \int_{\partial\Omega} (y - y_s)^2,$$

where y_s is a given function, and $j^*(\cdot)$ is defined in the example in the last subsection. In this case, the optimality condition of the control problem (2.1) is

$$\begin{aligned}
&\Big(\frac{\partial y}{\partial t}, w\Big) + a(y,w) = (f,w) + (Bu + z_b, w)_U, \ \ \forall w \in V, \ t \in (0,T),\\
&\qquad y(x,0) = y_0(x), && (2.20)\\
&-\Big(\frac{\partial p}{\partial t}, q\Big) + a(q,p) = (g'(y), q)_U, \ \ \forall q \in V, \ t \in [0,T],\\
&\qquad p(x,T) = 0, && (2.21)\\
&\int_0^T (j'(u) + B^*p, v - u)_U dt \ge 0, \ \ \forall v \in K. && (2.22)
\end{aligned}$$

Comparing the problem with the observation of the distributed state, it can be found that only costate equation is changed. More precisely, the Neumann boundary of the costate equation is changed from zero to $g'(y)$, and the source term is changed from $g'(y)$ to zero.

Using the finite element space introduced in the last subsection, we have the fully discrete finite element approximation of the control problem similar to (2.12)-(2.13), which is equivalent to the following fully discrete optimality condition:

$$\begin{aligned}
&\Big(\frac{\partial Y_h}{\partial t}, w_h\Big) + a(\hat{Y}_h, w_h) = (\hat{f}, w_h) + (BU_h + z_b, w_h)_U, \ \ \forall w_h \in V_i^h, \ \ 1 \le i \le N,\\
&\qquad Y_h(x,0) = y_0^h(x), \ x \in \Omega, && (2.23)\\
&-\Big(\frac{\partial P_h}{\partial t}, q_h\Big) + a(q_h, \tilde{P}_h) = \big(g'(\hat{Y}_h), q_h\big)_U, \ \ \forall q_h \in V_i^h, \ \ N \le i \le 1,\\
&\qquad P_h(x,T) = 0, \ x \in \Omega, && (2.24)\\
&\big(j'(U_h) + B^*\tilde{P}_h, v_h - U_h\big)_U \ge 0, \ \ \forall v_h \in K_i^h, \ \ 1 \le i \le N. && (2.25)
\end{aligned}$$

2.3. Boundary control problem with observation of the final state

We now consider an important and more practical case: the boundary control problem (2.1) with observation of the final state, i.e., the observation space $Y = L^2(\Omega)$ for the final state $y(T)$. Set

$$g^*(y) = \int_\Omega g\big(y(T)\big), \quad j^*(u) = \int_0^T \int_{\partial\Omega} j(u),$$

where $g(\cdot)$ and $j(\cdot)$ are strictly convex continuously differentiable functions such that the assumptions on $g^*(\cdot)$ and $j^*(\cdot)$ are satisfied. An example for $g^*(\cdot)$ is

$$g^*(y) = \frac{1}{2}\int_\Omega \big(y(x,T) - y_s(x)\big)^2 dx,$$

where $y_s(\cdot)$ is a given function, and $j^*(\cdot)$ is defined in the example in Subsection 2.1. In this case, the costate equation is

$$\begin{aligned}
&-\frac{\partial p}{\partial t} - \mathrm{div}(A^*\nabla p) + a_0 p = 0, \quad x \in \Omega, \ \ t \in [0,T],\\
&(A^*\nabla p)\cdot n = 0, \quad x \in \partial\Omega, \ \ t \in [0,T],\\
&p(x,T) = g'\big(y(x,T)\big), \quad x \in \Omega.
\end{aligned}$$

Then we have the following optimality conditions:

$$\begin{aligned}
&\Big(\frac{\partial y}{\partial t}, w\Big) + a(y,w) = (f,w) + (Bu + z_b, w)_U, \ \ \forall w \in V, \ \ t \in (0,T),\\
&\qquad y(x,0) = y_0(x), && (2.26)\\
&-\Big(\frac{\partial p}{\partial t}, q\Big) + a(q,p) = 0, \ \ \forall q \in V, \ \ t \in [0,T],\\
&\qquad p(x,T) = g'\big(y(x,T)\big), && (2.27)\\
&\big(j'(u) + B^*p, v-u\big)_U \geq 0, \ \ \forall v \in K. && (2.28)
\end{aligned}$$

Using the finite element space introduced in the Subsection 2.1, we have the fully discrete finite element approximation of the control problem similar to (2.12)-(2.13), which is equivalent to the following fully discrete optimality condition:

$$\begin{aligned}
&\Big(\frac{\partial Y_h}{\partial t}, w_h\Big) + a(\hat{Y}_h, w_h) = (\hat{f}, w_h) + (BU_h + z_b, w_h)_U, \ \ \forall w_h \in V_i^h, \ \ 1 \leq i \leq N,\\
&\qquad Y_h(x,0) = y_0^h(x), \quad x \in \Omega, && (2.29)\\
&-\Big(\frac{\partial P_h}{\partial t}, q_h\Big) + a(q_h, \tilde{P}_h) = 0, \ \ \forall q_h \in V_i^h, \ \ N \leq i \leq 1,\\
&\qquad P_h(x,T) = g'\big(Y_h(x,T)\big)_h, \quad x \in \Omega, && (2.30)\\
&\big(j'(U_h) + B^*\tilde{P}_h, v_h - U_h\big)_U \geq 0, \ \ \forall v_h \in K_i^h, \ \ 1 \leq i \leq N, && (2.31)
\end{aligned}$$

where $g'(Y_h(x,T))_h \in V^h$ is an approximation of $g'\big(Y_h(x,T)\big)$. For instance when

$$g(y) = \frac{1}{2}\big(y(x,T) - y_s(x)\big)^2,$$

we have that

$$g'\big(Y_h(x,T)\big)_h = Y_h(x,T) - y_s^h(x),$$

where $y_s^h \in V^h$ is an approximation of y_s.

3. A Posteriori Error Estimates for Boundary Control Problem

In this section, we will derive a posteriori error estimates for the boundary control problems governed by the parabolic equations. We shall assume the following convexity conditions:

$$\big(j'(u) - j'(v), u - v\big)_U \geq c\|u - v\|_U^2, \ \ \forall u, v \in U, \tag{3.1}$$

$$\big(g'(v) - g'(w), v - w\big)_Y \geq 0, \ \ \forall v, w \in Y, \tag{3.2}$$

where Y is the observation space which has been defined in Subsections 2.1-2.3.

In this paper, we assume that the constraint on the control is an obstacle type such that

$$K = \big\{v \in X: \ v \geq 0, \ \text{a.e. in } \partial\Omega \times (0, T]\big\}. \tag{3.3}$$

To derive the sharp a posteriori error estimates, we divide the boundary $\partial\Omega$ into three parts:

$$\begin{aligned}
\partial\Omega^- &= \big\{x \in \partial\Omega: \ (B^*\tilde{P}_h)(x) + j'(0) \leq 0\big\},\\
\partial\Omega^0 &= \big\{x \in \partial\Omega: \ (B^*\tilde{P}_h)(x) + j'(0) > 0, \ U_h(x) = 0\big\},\\
\partial\Omega^+ &= \big\{x \in \partial\Omega: \ (B^*\tilde{P}_h)(x) + j'(0) > 0, \ U_h(x) > 0\big\}.
\end{aligned}$$

It is easy to see that the partition of above three subsets is dependent on t. For all t, the three subsets are not intersected each other, and

$$\partial\Omega = \partial\Omega^- \cup \partial\Omega^0 \cup \partial\Omega^+.$$

Moreover, we introduce the well known error estimates for the Clément type interpolation (see [9] and [26] for more details), which will be used in a posteriori error estimate analysis in this section.

Lemma 3.1. *Let $\hat{\pi}$ be the Clément type interpolation operator defined in* [9]. *Then for any $v \in H^1(\Omega)$ and all element τ,*

$$\|v - \hat{\pi}v\|_{L^2(\tau)} + h_\tau\|\nabla(v - \hat{\pi}v)\|_{L^2(\tau)} \leq \sum_{\bar{\tau}'\cap\bar{\tau}\neq\emptyset} Ch_\tau|\nabla v|_{L^2(\tau')}, \tag{3.4}$$

$$\|v - \hat{\pi}v\|_{L^2(e)} \leq \sum_{e\subset\bar{\tau}'} Ch_e^{1/2}|\nabla v|_{L^2(\tau')}, \tag{3.5}$$

where e is the edge or face of the element.

3.1. Boundary control problem with observation of the distributed state

In this subsection, we will consider the a posteriori error estimate for the boundary control problem with observation of the distributed state. Firstly, let us derive the a posteriori error estimate for the control u.

Lemma 3.2. *Let (y, p, u) and (Y_h, P_h, U_h) be the solutions of* (2.4)-(2.6) *and* (2.17)-(2.19), *respectively. Assume that $j(\cdot)$, $g(\cdot)$ satisfy the convex assumptions* (3.1)-(3.2), *and $j'(\cdot)$ and $g'(\cdot)$ are locally Lipschitz continuous. Then we have that*

$$\|u - U_h\|^2_{L^2(0,T;L^2(\partial\Omega))} \leq C\eta_1^2 + C\|\tilde{P}_h - p(U_h)\|^2_{L^2(0,T;L^2(\partial\Omega))}, \tag{3.6}$$

where

$$\eta_1^2 = \|j'(U_h) + B^*\tilde{P}_h\|^2_{L^2(0,T;L^2(\partial\Omega^-\cup\partial\Omega^+))},$$

and $p(U_h)$ *is the solution of the following system:*

$$\begin{aligned}
&\Big(\frac{\partial y(U_h)}{\partial t}, w\Big) + a\big(y(U_h), w\big) = (f, w) + (BU_h + z_b, w)_U, \quad \forall w \in V, \\
&\qquad y(U_h)(x, 0) = y_0. \qquad (3.7) \\
&-\Big(\frac{\partial p(U_h)}{\partial t}, q\Big) + a\big(q, p(U_h)\big) = (g'(y(U_h)), q), \quad \forall q \in V, \\
&\qquad p(U_h)(x, T) = 0. \qquad (3.8)
\end{aligned}$$

Proof. From the uniform convexity of j, we have

$$\begin{aligned}
&c\|u - U_h\|^2_{L^2(0,T;L^2(\partial\Omega))} \\
\le& \int_0^T \big(j'(u) - j'(U_h), u - U_h\big)_U dt \\
=& \int_0^T \big(j'(u) + B^*p, u - U_h\big)_U dt + \int_0^T \big(j'(U_h) + B^*\tilde{P}_h, U_h - u\big)_U dt \\
&+ \int_0^T \big(B^*(\tilde{P}_h - p(U_h)), u - U_h\big)_U dt + \int_0^T \big(B^*(p(U_h) - p), u - U_h\big)_U dt \\
=:& \sum_{i=1}^4 I_i. \qquad (3.9)
\end{aligned}$$

We first estimate I_1. Note that $U_h \in K^h \subset K$. It follows from (2.6) that

$$I_1 = \int_0^T \big(j'(u) + B^*p, u - U_h\big)_U dt \le 0. \qquad (3.10)$$

Next we estimate I_2. Note that

$$\begin{aligned}
I_2 =& \int_0^T \big(j'(U_h) + B^*\tilde{P}_h, U_h - u\big)_U dt \\
=& \int_0^T \int_{\partial\Omega^- \cup \partial\Omega^+} \big(j'(U_h) + B^*\tilde{P}_h\big)(U_h - u) \\
&+ \int_0^T \int_{\partial\Omega^0} \big(j'(U_h) + B^*\tilde{P}_h\big)(U_h - u), \qquad (3.11)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \int_{\partial\Omega^- \cup \partial\Omega^+} \big(j'(U_h) + B^*\tilde{P}_h\big)(U_h - u) \\
\le& C(\delta)\|j'(U_h) + B^*\tilde{P}_h\|^2_{L^2\left(0,T;L^2(\partial\Omega^- \cup \partial\Omega^+)\right)} \\
&+ \delta\|u - U_h\|^2_{L^2\left(0,T;L^2(\partial\Omega^- \cup \partial\Omega^+)\right)} \\
\le& C(\delta)\eta_1^2 + \delta\|u - U_h\|^2_{L^2\left(0,T;L^2(\partial\Omega)\right)}, \qquad (3.12)
\end{aligned}$$

where δ is an arbitrary small positive number. Furthermore, we have that

$$j'(U_h) + B^*\tilde{P}_h \ge j'(0) + B^*\tilde{P}_h > 0, \quad U_h - u = 0 - u \le 0 \text{ on } \partial\Omega^0.$$

It yields that

$$\int_0^T \int_{\partial\Omega^0} \big(j'(U_h) + B^* \tilde{P}_h\big)(U_h - u) \leq 0. \tag{3.13}$$

Then (3.11)-(3.13) imply that

$$I_2 \leq C(\delta)\eta_1^2 + \delta \|u - U_h\|^2_{L^2(0,T;L^2(\partial\Omega))}. \tag{3.14}$$

Moreover, it is clear that

$$\begin{aligned}
I_3 &= \int_0^T \big(B^*(\tilde{P}_h - p(U_h)), u - U_h\big)_U dt \\
&\leq C \int_0^T \|B^*\big(\tilde{P}_h - p(U_h)\big)\|_{L^2(\partial\Omega)} \|u - U_h\|_{L^2(\partial\Omega)} dt \\
&\leq C(\delta) \int_0^T \|B^*\big(\tilde{P}_h - p(U_h)\big)\|^2_{L^2(\partial\Omega)} dt + \delta \int_0^T \|u - U_h\|^2_{L^2(\partial\Omega)} dt \\
&\leq C(\delta) \|\tilde{P}_h - p(U_h)\|^2_{L^2\left(0,T;L^2(\partial\Omega)\right)} + \delta \|u - U_h\|^2_{L^2\left(0,T;L^2(\partial\Omega)\right)}.
\end{aligned} \tag{3.15}$$

Now we turn to I_4. Note that

$$y(x,0) = y(U_h)(x,0) = y_0(x) \quad p(x,T) = y(U_h)(x,T) = 0.$$

It follows from (2.4)-(2.5) and (3.7)-(3.8) that

$$\begin{aligned}
I_4 &= \int_0^T \big(B^*(p(U_h) - p), u - U_h\big)_U dt \\
&= \int_0^T \big(p(U_h) - p, B(u - U_h)\big)_U dt \\
&= \int_0^T \left(\Big(\frac{\partial\big(y - y(U_h)\big)}{\partial t}, p(U_h) - p\Big) + a\big(y - y(U_h), p(U_h) - p\big) \right) dt \\
&= \int_0^T \left(-\Big(\frac{\partial\big(p(U_h) - p\big)}{\partial t}, y - y(U_h)\Big) + a\big(y - y(U_h), p(U_h) - p\big) \right) dt \\
&= \int_0^T \big(g'(y(U_h)) - g'(y), y - y(U_h)\big) dt \leq 0.
\end{aligned} \tag{3.16}$$

Thus, we obtain from (3.9)-(3.10) and (3.14)-(3.16) that

$$\|u - U_h\|^2_{L^2\left(0,T;L^2(\partial\Omega)\right)} \leq C\eta_1^2 + C\|\tilde{P}_h - p(U_h)\|^2_{L^2\left(0,T;L^2(\partial\Omega)\right)},$$

which proves (3.6). □

Next we estimate the error $\|\tilde{P}_h - p(U_h)\|_{L^2(0,T;L^2(\partial\Omega))}$. Let $|||\varphi||| = (a(\varphi,\varphi))^{\frac{1}{2}}$. It is easy to see that the norms $|||\cdot|||$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent.

Lemma 3.3. *Let* (Y_h, P_h, U_h) *be the solutions of* (2.17)-(2.19), *and let* $(y(U_h), p(U_h))$ *be the solution of* (3.7)-(3.8). *Assume that* $g'(\cdot)$ *is locally Lipschitz continuous. Then,*

$$\|\tilde{P}_h - p(U_h)\|^2_{L^2\left(0,T;H^1(\Omega)\right)} + \|y(U_h) - \hat{Y}_h\|^2_{L^2\left(0,T;H^1(\Omega)\right)} \leq C\sum_{i=2}^{9} \eta_i^2, \tag{3.17}$$

where

$$\eta_2^2=\int_0^T\sum_\tau h_\tau^2\|\frac{\partial P_h}{\partial t}+div(A^*\nabla\tilde P_h)-a_0\tilde P_h+g'(\hat Y_h)\|^2_{L^2(\tau)},$$
$$\eta_3^2=\int_0^T\sum_{e\cap\partial\Omega=\emptyset}h_e\|[A^*\nabla\tilde P_h\cdot n]\|^2_{L^2(e)}+\int_0^T\sum_{e\subset\partial\Omega}h_e\|A^*\nabla\tilde P_h\cdot n\|^2_{L^2(e)},$$
$$\eta_4^2=\int_0^T|||P_h-\tilde P_h|||^2,\quad \eta_5^2=\int_0^T\sum_\tau h_\tau^2\|-\frac{\partial Y_h}{\partial t}+div(A\nabla\hat Y_h)-a_0\hat Y_h+\hat f\|^2_{L^2(\tau)},$$
$$\eta_6^2=\int_0^T\sum_{e\cap\partial\Omega=\emptyset}h_e\|[A\nabla\hat Y_h\cdot n]\|^2_{L^2(e)}+\int_0^T\sum_{e\subset\partial\Omega}h_e\|A\nabla\hat Y_h\cdot n-BU_h-z_b\|^2_{L^2(e)},$$
$$\eta_7^2=\int_0^T|||Y_h-\hat Y_h|||^2,\quad \eta_8^2=\|Y_h(x,0)-y_0(x)\|^2_{L^2(\Omega)},\quad \eta_9^2=\|f-\hat f\|^2_{L^2(0,T;L^2(\Omega))},$$

where h_e is the size of face e, $[A\nabla\hat Y_h\cdot n]_e$ and $[A^*\nabla\tilde P_h\cdot n]_e$ are the $A-$normal and A^*-normal derivative jumps over the interelement face e respectively, defined by

$$[A\nabla\hat Y_h\cdot n]_e=\big(A\nabla\hat Y_h|_{\tau_e^1}-A\nabla\hat Y_h|_{\tau_e^2}\big)\cdot n,$$
$$[A^*\nabla\tilde P_h\cdot n]_e=\big(A^*\nabla\tilde P_h|_{\tau_e^1}-A^*\nabla\tilde P_h|_{\tau_e^2}\big)\cdot n,$$

where n is the unit normal vector on $e=\tau_e^1\cap\tau_e^2$ outwards τ_e^1.

Proof. It follows from (2.18) that for all $\varphi\in H^1(\Omega)$ and $q\in V_i^h$, $t\in(t_{i-1},t_i]$,

$$\begin{aligned}&-\Big(\frac{\partial P_h}{\partial t},\varphi\Big)+a(\varphi,\tilde P_h)\\&=-\Big(g'(\hat Y_h)+\frac{\partial P_h}{\partial t},\varphi-q\Big)+a(\varphi-q,\tilde P_h)+\big(g'(\hat Y_h),\varphi\big).\end{aligned}\tag{3.18}$$

Then from (3.8) and (3.18) we have

$$\begin{aligned}&\left(\frac{\partial\big(P_h-p(U_h)\big)}{\partial t},\varphi\right)+a(\varphi,p(U_h)-\tilde P_h)\\&=\Big(g'(\hat Y_h)+\frac{\partial P_h}{\partial t},\varphi-q\Big)-a(\varphi-q,\tilde P_h)+\big(g'(y(U_h))-g'(\hat Y_h),\varphi\big).\end{aligned}\tag{3.19}$$

Set

$$\varphi=p(U_h)-P_h,\ q=\hat\pi(p(U_h)-P_h)$$

in (3.19), where $\hat\pi:\ H^1(\Omega)\to V_i^h$ is the Clément type interpolation operator defined in Lemma 3.1. Note that

$$\begin{aligned}a(\varphi-q,\tilde P_h)&=\int_\Omega\nabla(\varphi-q)A^*\nabla\tilde P_h+a_0\tilde P_h(\varphi-q)\\&=\int_\Omega\big(a_0\tilde P_h-\mathrm{div}(A^*\nabla\tilde P_h)\big)(\varphi-q)+\sum_\tau\int_{\partial\tau}(A^*\nabla\tilde P_h)\cdot n(\varphi-q)ds\\&=\int_\Omega\big(a_0\tilde P_h-\mathrm{div}(A^*\nabla\tilde P_h)\big)(\varphi-q)+\sum_{e\cap\partial\Omega=\emptyset}\int_e[A^*\nabla\tilde P_h\cdot n](\varphi-q)ds\\&\quad+\sum_{e\subset\partial\Omega}\int_e(A^*\nabla\tilde P_h\cdot n)(\varphi-q)ds\end{aligned}$$

and

$$
\begin{aligned}
& a\big(p(U_h)-P_h, p(U_h)-\tilde{P}_h\big) \\
=& \frac{1}{2}|||\tilde{P}_h - p(U_h)|||^2 + \frac{1}{2}|||P_h - p(U_h)|||^2 - \frac{1}{2}|||\tilde{P}_h - P_h|||^2 .
\end{aligned}
$$

We have that

$$
\begin{aligned}
& -\frac{1}{2}\frac{d}{dt}\|P_h - p(U_h)\|^2_{L^2(\Omega)} + \frac{1}{2}|||\tilde{P}_h - p(U_h)|||^2 + \frac{1}{2}|||P_h - p(U_h)|||^2 \\
=& \frac{1}{2}|||\tilde{P}_h - P_h|||^2 \\
& + \int_\Omega \left(g'(\hat{Y}_h) + \frac{\partial P_h}{\partial t} + \operatorname{div}(A^*\nabla \tilde{P}_h) - a_0\tilde{P}_h\right)\big(p(U_h) - P_h - \hat{\pi}(p(U_h)-P_h)\big) \\
& - \sum_{e\cap\partial\Omega=\emptyset}\int_e [A^*\nabla\tilde{P}_h\cdot n]\big(p(U_h)-P_h-\hat{\pi}(p(U_h)-P_h)\big)ds \\
& - \sum_{e\subset\partial\Omega}\int_e (A^*\nabla\tilde{P}_h\cdot n)\big(p(U_h)-P_h-\hat{\pi}(p(U_h)-P_h)\big)ds \\
& + \big(g'(y(U_h)) - g'(\hat{Y}_h), p(U_h)-P_h\big).
\end{aligned} \tag{3.20}
$$

By integrating (3.20) in time from t_{i-1} to t_i and using (3.4) and (3.5), we have

$$
\begin{aligned}
& \frac{1}{2}\|\big(P_h - p(U_h)\big)(t_{i-1})\|^2_{L^2(\Omega)} + \frac{1}{2}\int_{t_{i-1}}^{t_i}\big(|||\tilde{P}_h - p(U_h)|||^2 + |||P_h - p(U_h)|||^2\big)dt \\
\le& \frac{1}{2}\|\big(P_h - p(U_h)\big)(t_i)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_{t_{i-1}}^{t_i} |||\tilde{P}_h - P_h|||^2 dt \\
& + C\int_{t_{i-1}}^{t_i}\Big(\sum_\tau h_\tau^2\|g'(\hat{Y}_h) + \frac{\partial P_h}{\partial t} + \operatorname{div}(A^*\nabla\tilde{P}_h) - a_0\tilde{P}_h\|^2_{L^2(\tau)}\Big)dt \\
& + C\int_{t_{i-1}}^{t_i}\Big(\sum_{e\cap\partial\Omega=\emptyset} h_e\|[A^*\nabla\tilde{P}_h\cdot n]\|^2_{L^2(e)} + \sum_{e\subset\partial\Omega} h_e\|A^*\nabla\tilde{P}_h\cdot n\|^2_{L^2(e)}\Big)dt \\
& + C\int_{t_{i-1}}^{t_i}\|g'\big(y(U_h)\big) - g'(\hat{Y}_h)\|^2_{L^2(\Omega)}dt + \frac{1}{2}\int_{t_{i-1}}^{t_i}|||p(U_h) - P_h|||^2 dt.
\end{aligned}
$$

Summing up from 1 to N and noting that $(P_h - p(U_h))(x,T) = 0$, we have

$$
\begin{aligned}
& \frac{1}{2}\|(P_h - p(U_h))(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^T (|||\tilde{P}_h - p(U_h)|||^2 + |||P_h - p(U_h)|||^2)dt \\
\le& \frac{1}{2}\int_0^T\Big(|||\tilde{P}_h - P_h|||^2 + C\sum_\tau h_\tau^2\|g'(\hat{Y}_h) + \frac{\partial P_h}{\partial t} + \operatorname{div}(A^*\nabla\tilde{P}_h) - a_0\tilde{P}_h\|^2_{L^2(\tau)}\Big)dt \\
& + C\int_0^T\Big(\sum_{e\cap\partial\Omega=\emptyset} h_e\|[A^*\nabla\tilde{P}_h\cdot n]\|^2_{L^2(e)} + \sum_{e\subset\partial\Omega} h_e\|A^*\nabla\tilde{P}_h\cdot n\|^2_{L^2(e)}\Big)dt \\
& + C\|y(U_h) - \hat{Y}_h\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2}\int_0^T |||P_h - p(U_h)|||^2 dt \\
\le& C\sum_{i=2}^4 \eta_i^2 + C\|y(U_h) - \hat{Y}_h\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2}\int_0^T |||P_h - p(U_h)|||^2 dt,
\end{aligned}
$$

which implies that

$$\|\tilde{P}_h - p(U_h)\|^2_{L^2\left(0,T:H^1(\Omega)\right)} \le C\sum_{i=2}^{4}\eta_i^2 + C\|y(U_h) - \hat{Y}_h\|^2_{L^2\left(0,T;L^2(\Omega)\right)}. \tag{3.21}$$

Similarly, from (2.17) we have that for any $\phi \in H^1(\Omega)$ and $w \in V_i^h$, $t \in (t_{i-1}, t_i]$

$$\begin{aligned}
&\Big(\frac{\partial Y_h}{\partial t}, \phi\Big) + a(\hat{Y}_h, \phi)\\
&= -\Big(\hat{f} - \frac{\partial Y_h}{\partial t}, \phi - w\Big) + a(\hat{Y}_h, \phi - w) + (BU_h + z_b, w)_U + (\hat{f}, \phi).
\end{aligned} \tag{3.22}$$

Then from (3.7) and (3.22) we have

$$\begin{aligned}
&\Big(\frac{\partial (y(U_h) - Y_h)}{\partial t}, \phi\Big) + a(y(U_h) - \hat{Y}_h, \phi)\\
&= \Big(\hat{f} - \frac{\partial Y_h}{\partial t}, \phi - w\Big) - a(\hat{Y}_h, \phi - w) + (BU_h + z_b, \phi - w)_U + (f - \hat{f}, \phi).
\end{aligned} \tag{3.23}$$

Set $\phi = y(U_h) - Y_h$ and $w = \hat{\pi}\big(y(U_h) - Y_h\big)$ in (3.23), where $\hat{\pi} : H^1(\Omega) \to V_i^h$ is the Clément type interpolation operator defined in Lemma 3.1. Note that

$$\begin{aligned}
a(\hat{Y}_h, \phi - w) &= \int_\Omega A\nabla\hat{Y}_h\nabla(\phi - w) + a_0\hat{Y}_h(\phi - w)\\
&= \int_\Omega \big(a_0\hat{Y}_h - \operatorname{div}(A\nabla\hat{Y}_h)\big)(\phi - w) + \sum_\tau\int_{\partial\tau}(A\nabla\hat{Y}_h)\cdot n(\phi - w)ds\\
&= \int_\Omega \big(a_0\hat{Y}_h - \operatorname{div}(A\nabla\hat{Y}_h)\big)(\phi - w) + \sum_{e\cap\partial\Omega=\emptyset}\int_e[A\nabla\hat{Y}_h\cdot n](\phi - w)ds\\
&\quad + \sum_{e\subset\partial\Omega}\int_e(A\nabla\hat{Y}_h\cdot n)(\phi - w)ds
\end{aligned}$$

and

$$\begin{aligned}
&a\big(y(U_h) - \hat{Y}_h, y(U_h) - Y_h\big)\\
&= \frac{1}{2}|||y(U_h) - \hat{Y}_h|||^2 + \frac{1}{2}|||Y_h - y(U_h)|||^2 - \frac{1}{2}|||\hat{Y}_h - Y_h|||^2.
\end{aligned}$$

We have that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Y_h - y(U_h)\|^2_{L^2(\Omega)} + \frac{1}{2}|||\hat{Y}_h - y(U_h)|||^2 + \frac{1}{2}|||Y_h - y(U_h)|||^2\\
&= \frac{1}{2}|||\hat{Y}_h - Y_h|||^2\\
&\quad + \int_\Omega\Big(\hat{f} - \frac{\partial Y_h}{\partial t} + \operatorname{div}(A\nabla\hat{Y}_h) - a_0\hat{Y}_h\Big)\big(y(U_h) - Y_h - \hat{\pi}(y(U_h) - Y_h)\big)\\
&\quad - \sum_{e\cap\partial\Omega=\emptyset}\int_e[A\nabla\hat{Y}_h\cdot n]\big(y(U_h) - Y_h - \hat{\pi}(y(U_h) - Y_h)\big)ds\\
&\quad - \sum_{e\subset\partial\Omega}\int_e(A\nabla\hat{Y}_h\cdot n - BU_h - z_b)\big(y(U_h) - Y_h - \hat{\pi}(y(U_h) - Y_h)\big)ds\\
&\quad + \big(f - \hat{f}, y(U_h) - Y_h\big).
\end{aligned} \tag{3.24}$$

By integrating (3.24) in time from t_{i-1} to t_i and summing up from 1 to N we have

$$\begin{aligned}
&\frac{1}{2}\|\big(Y_h-y(U_h)\big)(T)\|^2_{L^2(\Omega)}+\frac{1}{2}\int_0^T(|||\hat{Y}_h-y(U_h)|||^2+|||Y_h-y(U_h)|||^2)dt\\
\leq&\frac{1}{2}\|Y_h(\cdot,0)-y_0(\cdot)\|^2_{L^2(\Omega)}+C\int_0^T\Big(|||\hat{Y}_h-Y_h|||^2\\
&+\sum_\tau h_\tau^2\|\hat{f}-\frac{\partial Y_h}{\partial t}+\mathrm{div}(A\nabla\hat{Y}_h)-a_0\hat{Y}_h\|^2_{L^2(\tau)}\\
&+\sum_{e\cap\partial\Omega=\emptyset}h_e\|[A\nabla\hat{Y}_h\cdot n]\|^2_{L^2(e)}+\sum_{e\subset\partial\Omega}h_e\|(A\nabla\hat{Y}_h\cdot n-BU_h-z_b)\|^2_{L^2(e)}\Big)dt\\
&+\frac{1}{2}\int_0^T|||Y_h-y(U_h)|||^2dt+C\int_0^T\|f-\hat{f}\|^2_{L^2(\Omega)}dt,\\
\leq&C\sum_{i=5}^9\eta_i^2+\frac{1}{2}\int_0^T|||Y_h-y(U_h)|||^2dt,
\end{aligned}$$

which implies that

$$\|\big(Y_h-y(U_h)\big)(T)\|^2_{L^2(\Omega)}+\|\hat{Y}_h-y(U_h)\|^2_{L^2\big(0,T;H^1(\Omega)\big)}\leq C\sum_{i=5}^9\eta_i^2. \tag{3.25}$$

Note that

$$\|y(U_h)-\hat{Y}_h\|_{L^2\big(0,T;L^2(\Omega)\big)}\leq C\|y(U_h)-\hat{Y}_h\|_{L^2\big(0,T;H^1(\Omega)\big)}.$$

Then (3.17) follows from (3.21) and (3.25). □

Using Lemmas 3.2 and 3.3, we can derive following a posteriori error estimate:

Theorem 3.4. *Let* (y,p,u) *and* (Y_h,P_h,U_h) *be the solutions of* (2.4)-(2.6) *and* (2.17)-(2.19), *respectively. Assume that all the conditions in Lemma* 3.2 *and* 3.3 *are valid. Then*

$$\begin{aligned}
&\|u-U_h\|^2_{L^2\big(0,T;L^2(\partial\Omega)\big)}+\|y-Y_h\|^2_{L^2\big(0,T;H^1(\Omega)\big)}+\|p-P_h\|^2_{L^2\big(0,T;H^1(\Omega)\big)}\\
\leq&C\sum_{i=1}^9\eta_i^2,
\end{aligned}\tag{3.26}$$

where η_1 *is defined in Lemma* 3.2, $\eta_i, 2\leq i\leq 9$, *are defined in Lemma* 3.3.

Proof. It follows from Lemmas 3.2 and 3.3 that

$$\begin{aligned}
&\|u-U_h\|^2_{L^2(0,T;L^2(\partial\Omega))}\\
\leq&C\eta_1^2+C\|\tilde{P}_h-p(U_h)\|^2_{L^2(0,T;L^2(\partial\Omega))}\\
\leq&C\eta_1^2+C\|\tilde{P}_h-p(U_h)\|^2_{L^2(0,T;H^1(\Omega))}\leq C\sum_{i=1}^9\eta_i^2.
\end{aligned}\tag{3.27}$$

Note that

$$\begin{aligned}
&\|y-Y_h\|_{L^2(0,T;H^1(\Omega))}\\
\leq&\|y-y(U_h)\|_{L^2(0,T;H^1(\Omega))}+\|y(U_h)-\hat{Y}_h\|_{L^2(0,T;H^1(\Omega))}\\
&+\|\hat{Y}_h-Y_h\|_{L^2(0,T;H^1(\Omega))},
\end{aligned}\tag{3.28}$$

$$
\begin{aligned}
&\|p-P_h\|_{L^2(0,T;H^1(\Omega))}\\
\le&\|p-p(U_h)\|_{L^2(0,T;H^1(\Omega))}+\|p(U_h)-\tilde{P}_h\|_{L^2(0,T;H^1(\Omega))}\\
&+\|\tilde{P}_h-P_h\|_{L^2(0,T;H^1(\Omega))},
\end{aligned}\tag{3.29}
$$

and

$$
\|y-y(U_h)\|_{L^2(0,T;H^1(\Omega))}\le C\|u-U_h\|_{L^2(0,T;L^2(\partial\Omega))},\tag{3.30}
$$

$$
\begin{aligned}
\|p-p(U_h)\|_{L^2(0,T;H^1(\Omega))}&\le C\|y-y(U_h)\|_{L^2(0,T;H^1(\Omega))}\\
&\le C\|u-U_h\|_{L^2(0,T;L^2(\partial\Omega))}.
\end{aligned}\tag{3.31}
$$

Then, it follows from Lemma 3.3 and (3.27)-(3.31) that

$$
\begin{aligned}
&\|y-Y_h\|^2_{L^2\left(0,T;H^1(\Omega)\right)}+\|p-P_h\|^2_{L^2\left(0,T;H^1(\Omega)\right)}\\
\le&C\|u-U_h\|^2_{L^2\left(0,T;L^2(\partial\Omega)\right)}+C\|y(U_h)-\hat{Y}_h\|^2_{L^2\left(0,T;H^1(\Omega)\right)}\\
&+C\|p(U_h)-\tilde{P}_h\|^2_{L^2\left(0,T;H^1(\Omega)\right)}+C\eta_7^2+C\eta_4^2\\
\le&C\sum_{i=1}^{9}\eta_i^2.
\end{aligned}\tag{3.32}
$$

Therefore, (3.26) follows from (3.27) and (3.32). □

3.2. Boundary control problem with observation of the boundary state

Note that for all $v\in L^2(0,T;H^1(\Omega))$,

$$
\|v\|_{L^2(0,T;L^2(\partial\Omega))}\le C\|v\|_{L^2(0,T;H^1(\Omega))}.
$$

Then we can derive a posteriori error estimates for the boundary control problem with observation of the boundary state similar to the last subsection. Because the proof is similar, we just state the results in the following, and omit the proof.

Lemma 3.5. *Let (y,p,u) and (Y_h,P_h,U_h) be the solutions of* (2.20)-(2.22) *and* (2.23)-(2.25), *respectively. Assume that $j(\cdot)$, $g(\cdot)$ satisfy the convex assumptions* (3.1)-(3.2), *and $j'(\cdot)$ and $g'(\cdot)$ are locally Lipschitz continuous. Then we have that*

$$
\|u-U_h\|^2_{L^2(0,T;L^2(\partial\Omega))}\le C\tilde{\eta}_1^2+C\|\tilde{P}_h-p(U_h)\|^2_{L^2(0,T;L^2(\partial\Omega))},\tag{3.33}
$$

where $\tilde{\eta}_1$ is defined in Lemma 3.2, *and $p(U_h)$ is the solution of the following system:*

$$
\begin{aligned}
&\Big(\frac{\partial y(U_h)}{\partial t},w\Big)+a\big(y(U_h),w\big)=(f,w)+(BU_h+z_b,w)_U,\quad\forall w\in V,\\
&\quad y(U_h)(x,0)=y_0,
\end{aligned}\tag{3.34}
$$

$$
\begin{aligned}
&-\Big(\frac{\partial p(U_h)}{\partial t},q\Big)+a\big(q,p(U_h)\big)=\big(g'(y(U_h)),q\big)_U,\quad\forall q\in V,\\
&\quad p(U_h)(x,T)=0.
\end{aligned}\tag{3.35}
$$

Lemma 3.6. *Let* (Y_h, P_h, U_h) *be the solutions of* (2.23)-(2.25) *and let* $(y(U_h), p(U_h))$ *be the solution of* (3.34)-(3.35). *If* $g'(\cdot)$ *is locally Lipschitz continuous, then,*

$$\|\tilde{P}_h - p(U_h)\|^2_{L^2(0,T;H^1(\Omega))} + \|y(U_h) - \hat{Y}_h\|^2_{L^2(0,T;H^1(\Omega))} \leq C\sum_{i=2}^{9} \tilde{\eta}_i^2, \tag{3.36}$$

where $\tilde{\eta}_i = \eta_i$, η_i *is defined in Lemma* 3.3, $4 \leq i \leq 9$, *and*

$$\begin{aligned}\tilde{\eta}_2^2 &= \int_0^T \sum_\tau h_\tau^2 \|\frac{\partial P_h}{\partial t} + div(A^*\nabla\tilde{P}_h) - a_0\tilde{P}_h\|^2_{L^2(\tau)},\\ \tilde{\eta}_3^2 &= \int_0^T \sum_{e\cap\partial\Omega=\emptyset} h_e \|[A^*\nabla\tilde{P}_h\cdot n]\|^2_{L^2(e)}\\ &\quad + \int_0^T \sum_{e\subset\partial\Omega} h_e\|A^*\nabla\tilde{P}_h\cdot n - g'(\hat{Y}_h)\|^2_{L^2(e)},\end{aligned}$$

where h_e *and* $[A^*\nabla\tilde{P}_h\cdot n]_e$ *are defined in Lemma* 3.3.

Theorem 3.7. *Let* (y, p, u) *and* (Y_h, P_h, U_h) *be the solutions of* (2.20)-(2.22) *and* (2.23)-(2.25), *respectively. Assume that all the conditions in Lemmas* 3.5 *and* 3.6 *are valid. Then*

$$\|u - U_h\|^2_{L^2(0,T;L^2(\partial\Omega))} + \|y - Y_h\|^2_{L^2(0,T;H^1(\Omega))} + \|p - P_h\|^2_{L^2(0,T;H^1(\Omega))} \leq C\sum_{i=1}^{9}\tilde{\eta}_i^2, \tag{3.37}$$

where $\tilde{\eta}_1$ *is defined in Lemma* 3.5, $\tilde{\eta}_i, 2 \leq i \leq 9$, *are defined in Lemma* 3.6.

3.3. Boundary control problem with observation of final state

Last we will derive a posteriori error estimates for the boundary control problems with observation of the final state.

Lemma 3.8. *Let* (y, p, u) *and* (Y_h, P_h, U_h) *be the solutions of* (2.26)-(2.28) *and* (2.29)-(2.31), *respectively. Assume that* $j(\cdot)$, $g(\cdot)$ *satisfy the convex assumptions* (3.1)-(3.2), *and* $j'(\cdot)$ *and* $g'(\cdot)$ *are locally Lipschitz continuous. Then we have that*

$$\|u - U_h\|^2_{L^2(0,T;L^2(\partial\Omega))} \leq C\hat{\eta}_1^2 + C\|\tilde{P}_h - p(U_h)\|^2_{L^2(0,T;L^2(\partial\Omega))}, \tag{3.38}$$

where $\hat{\eta}_1 = \eta_1$ *is defined in Lemma* 3.2 *and* $p(U_h)$ *is the solution of the following system:*

$$\begin{aligned}&\Big(\frac{\partial y(U_h)}{\partial t}, w\Big) + a\big(y(U_h), w\big) = (f, w) + (BU_h + z_b, w)_U, \quad \forall w \in V,\\ &\quad y(U_h)(x, 0) = y_0,\end{aligned} \tag{3.39}$$

$$\begin{aligned}&-\Big(\frac{\partial p(U_h)}{\partial t}, q\Big) + a\big(q, p(U_h)\big) = 0, \quad \forall q \in V,\\ &\quad p(U_h)(x, T) = g'\big(y(U_h)(x, T)\big).\end{aligned} \tag{3.40}$$

Proof. Similar to Lemma 3.2, it can be proved that

$$\begin{aligned}
&c\|u-U_h\|^2_{L^2(0,T;L^2(\partial\Omega))}\\
\le&\int_0^T \big(j'(u)-j'(U_h),u-U_h\big)_U dt\\
=&\int_0^T \big(j'(u)+B^*p,u-U_h\big)_U dt+\int_0^T (j'(U_h)+B^*\tilde{P}_h,U_h-u)_U dt\\
&+\int_0^T \big(B^*(\tilde{P}_h-p(U_h)),u-U_h\big)_U dt+\int_0^T \big(B^*(p(U_h)-p),u-U_h\big)_U dt\\
=:&\sum_{i=1}^4 I_i.
\end{aligned}\tag{3.41}$$

The estimates for I_1, I_2, I_3 are the same as those in Lemma 3.2. As to I_4, using (2.26)-(2.27), (3.39)-(3.40) and setting $w=p(U_h)-p$, $q=y-y(U_h)$, we have

$$\begin{aligned}
I_4=&\int_0^T \big(B^*(p(U_h)-p),u-U_h\big)_U dt=\int_0^T \big(p(U_h)-p,B(u-U_h)\big)_U dt\\
=&\int_0^T \Big(\Big(\frac{\partial\big(y-y(U_h)\big)}{\partial t},p(U_h)-p\Big)+a\big(y-y(U_h),p(U_h)-p\big)\Big)dt\\
=&\int_0^T \Big(-\Big(\frac{\partial\big(p(U_h)-p\big)}{\partial t},y-y(U_h)\Big)+a\big(y-y(U_h),p(U_h)-p\big)\Big)dt\\
&+\big(y-y(U_h),p(U_h)-p\big)(T)-\big(y-y(U_h),p(U_h)-p\big)(0)\\
=&\Big(y(T)-y(U_h)(T),g'\big(y(U_h)\big)(T)-g'(y)(T)\Big)\le 0.
\end{aligned}\tag{3.42}$$

Thus, we obtain from (3.10), (3.14), (3.15), (3.41) and (3.42) that

$$\|u-U_h\|^2_{L^2(0,T;L^2(\partial\Omega))}\le C\hat{\eta}_1^2+C\|\tilde{P}_h-p(U_h)\|^2_{L^2(0,T;L^2(\partial\Omega))}.$$

This proves (3.38). □

Below we estimate the error $\|\tilde{P}_h-p(U_h)\|_{L^2(0,T;L^2(\partial\Omega))}$.

Lemma 3.9. *Let* (Y_h,P_h,U_h) *be the solution of* (2.29)-(2.31) *and let* $(y(U_h),p(U_h))$ *be the solution of* (3.39)-(3.40). *Assume that* $g'(\cdot)$ *is locally Lipschitz continuous. Then,*

$$\|\tilde{P}_h-p(U_h)\|^2_{L^2(0,T;H^1(\Omega))}+\|y(U_h)-\hat{Y}_h\|^2_{L^2(0,T;H^1(\Omega))}\le C\sum_{i=2}^{10}\hat{\eta}_i^2,\tag{3.43}$$

where $\hat{\eta}_2=\tilde{\eta}_2$ *is defined in Lemma* 3.6, $\hat{\eta}_i=\eta_i$, $i=3,\cdots,9$, *are defined in Lemma* 3.3, *and*

$$\hat{\eta}_{10}^2=\|(g'(Y_h)_h-g'(Y_h))(T)\|^2_{L^2(\Omega)}.$$

Proof. From (2.30) we know that, $\forall\varphi\in H^1(\Omega)$ and $q\in V_i^h$, $t\in(t_{i-1},t_i]$,

$$-\Big(\frac{\partial P_h}{\partial t},\varphi\Big)+a(\varphi,\tilde{P}_h)=-\Big(\frac{\partial P_h}{\partial t},\varphi-q\Big)+a(\varphi-q,\tilde{P}_h).\tag{3.44}$$

Then it follows from (3.40) and (3.44) that

$$\begin{aligned}&\Big(\frac{\partial\big(P_h-p(U_h)\big)}{\partial t},\varphi\Big)+a\big(\varphi,p(U_h)-\tilde P_h\big)\\=&\Big(\frac{\partial P_h}{\partial t},\varphi-q\Big)-a(\varphi-q,\tilde P_h).\end{aligned}\tag{3.45}$$

Let $\varphi=p(U_h)-P_h,\ q=\hat\pi(p(U_h)-P_h)$ in (3.45), where $\hat\pi:\ H^1(\Omega)\to V_i^h$ be the Clément type interpolation operator defined in Lemma 3.1. Similar to Lemma 3.3, we have that

$$\begin{aligned}&-\frac12\frac{d}{dt}\|P_h-p(U_h)\|^2_{L^2(\Omega)}+\frac12|||\tilde P_h-p(U_h)|||^2+\frac12|||P_h-p(U_h)|||^2\\=&\frac12|||\tilde P_h-P_h|||^2\\&+\int_\Omega\Big(\frac{\partial P_h}{\partial t}+\mathrm{div}(A^*\nabla\tilde P_h)-a_0\tilde P_h\Big)\big(p(U_h)-P_h-\hat\pi\big(p(U_h)-P_h\big)\big)\\&+\sum_{e\cap\partial\Omega=\emptyset}\int_e[A^*\nabla\tilde P_h\cdot n]\big(p(U_h)-P_h-\hat\pi\big(p(U_h)-P_h\big)\big)ds\\&+\sum_{e\subset\partial\Omega}\int_e(A^*\nabla\tilde P_h\cdot n)\big(p(U_h)-P_h-\hat\pi\big(p(U_h)-P_h\big)\big)ds,\end{aligned}$$

and

$$\begin{aligned}&\frac12\|\big(P_h-p(U_h)\big)(t_{i-1})\|^2_{L^2(\Omega)}+\frac12\int_{t_{i-1}}^{t_i}\big(|||\tilde P_h-p(U_h)|||^2+|||P_h-p(U_h)|||^2\big)dt\\\le&\frac12\|\big(P_h-p(U_h)\big)(t_i)\|^2_{L^2(\Omega)}+\frac12\int_{t_{i-1}}^{t_i}|||\tilde P_h-P_h|||^2dt\\&+C\int_{t_{i-1}}^{t_i}\Bigg(\sum_\tau h_\tau^2\|\frac{\partial P_h}{\partial t}+\mathrm{div}(A^*\nabla\tilde P_h)-a_0\tilde P_h\|^2_{L^2(\tau)}\\&+\sum_{e\cap\partial\Omega=\emptyset}h_e\|[A^*\nabla\tilde P_h\cdot n]\|^2_{L^2(e)}+\sum_{e\subset\partial\Omega}h_e\|A^*\nabla\tilde P_h\cdot n\|^2_{L^2(e)}\Bigg)dt\\&+\frac12\int_{t_{i-1}}^{t_i}|||p(U_h)-P_h|||^2dt.\end{aligned}$$

Summing up from 1 to N, we have that

$$\begin{aligned}&\frac12\|\big(P_h-p(U_h)\big)(0)\|^2_{L^2(\Omega)}+\frac12\int_0^T\big(|||\tilde P_h-p(U_h)|||^2+|||P_h-p(U_h)|||^2\big)dt\\\le&\frac12\|\big(P_h-p(U_h)\big)(T)\|^2_{L^2(\Omega)}+\frac12\int_0^T\Bigg(|||\tilde P_h-P_h|||^2+|||p(U_h)-P_h|||^2\\&+C\sum_\tau h_\tau^2\|\frac{\partial P_h}{\partial t}+\mathrm{div}(A^*\nabla\tilde P_h)-a_0\tilde P_h\|^2_{L^2(\tau)}\\&+C\sum_{e\cap\partial\Omega=\emptyset}h_e\|[A^*\nabla\tilde P_h\cdot n]\|^2_{L^2(e)}+C\sum_{e\subset\partial\Omega}h_e\|A^*\nabla\tilde P_h\cdot n\|^2_{L^2(e)}\Bigg)dt\\\le&C\sum_{i=2}^4\hat\eta_i^2+C\|\big(g'(Y_h)_h-g'(y(U_h))\big)(T)\|^2_{L^2(\Omega)}+\frac12\int_0^T|||p(U_h)-P_h|||^2.\end{aligned}$$

Note that

$$\begin{aligned}
&\|\big(g'(Y_h)_h - g'(y(U_h))\big)(T)\|^2_{L^2(\Omega)}\\
\le& C\|\big(g'(Y_h)_h - g'(Y_h)\big)(T)\|^2_{L^2(\Omega)}\\
&+ C\|\big(g'(Y_h) - g'(y(U_h))\big)(T)\|^2_{L^2(\Omega)}\\
\le& C\hat{\eta}^2_{10} + C\|\big(Y_h - y(U_h)\big)(T)\|^2_{L^2(\Omega)}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\|\tilde{P}_h - p(U_h)\|^2_{L^2(0,T:H^1(\Omega))}\\
\le& C\sum_{i=2}^{4}\hat{\eta}^2_i + C\hat{\eta}^2_{10} + C\|\big(Y_h - y(U_h)\big)(T)\|^2_{L^2(\Omega)}. \qquad (3.46)
\end{aligned}$$

Again similar to Lemma 3.3, it can be proved that (see (3.25))

$$\|\big(Y_h - y(U_h)\big)(T)\|^2_{L^2(\Omega)} + \|\hat{Y}_h - y(U_h)\|^2_{L^2(0,T;H^1(\Omega))} \le C\sum_{i=5}^{9}\hat{\eta}^2_i. \qquad (3.47)$$

Then (3.43) follows from (3.46) and (3.47). □

Theorem 3.10. *Let* (y,p,u) *and* (Y_h,P_h,U_h) *be the solutions of* (2.26)-(2.28) *and* (2.29)-(2.31), *respectively. Assume that all the conditions in Lemmas* 3.8 *and* 3.9 *are valid. Then*

$$\begin{aligned}
&\|u - U_h\|^2_{L^2(0,T;L^2(\partial\Omega))} + \|y - Y_h\|^2_{L^2(0,T;H^1(\Omega))} + \|p - P_h\|^2_{L^2(0,T;H^1(\Omega))}\\
\le& C\sum_{i=1}^{10}\hat{\eta}^2_i, \qquad (3.48)
\end{aligned}$$

where $\hat{\eta}_1$ *is defined in Lemma* 3.8, $\hat{\eta}_i$, $2 \le i \le 10$, *are defined in Lemma* 3.9.

Proof. Similar to the proof of Theorem 3.4, it follows from Lemmas 3.8 and 3.9 that

$$\begin{aligned}
&\|u - U_h\|^2_{L^2(0,T;L^2(\partial\Omega))}\\
\le& C\hat{\eta}^2_1 + C\|\tilde{P}_h - p(U_h)\|^2_{L^2(0,T;L^2(\partial\Omega))}\\
\le& C\hat{\eta}^2_1 + C\|\tilde{P}_h - p(U_h)\|^2_{L^2(0,T;H^1(\Omega))} \le C\sum_{i=1}^{10}\hat{\eta}^2_i. \qquad (3.49)
\end{aligned}$$

Note that

$$\begin{aligned}
&\|\big(y - y(U_h)\big)(T)\|_{L^2(\Omega)} + \|y - y(U_h)\|_{L^2(0,T;H^1(\Omega))}\\
\le& C\|u - U_h\|_{L^2(0,T;L^2(\partial\Omega))}, \qquad (3.50)\\
&\|p - p(U_h)\|_{L^2(0,T;H^1(\Omega))} \le C\|\big(y - y(U_h)\big)(T)\|_{L^2(\Omega)}\\
\le& C\|u - U_h\|_{L^2(0,T;L^2(\partial\Omega))}. \qquad (3.51)
\end{aligned}$$

Then, it follows from (3.28)-(3.29), (3.49)-(3.51) and Lemma 3.9 that

$$\begin{aligned}
&\|y - Y_h\|^2_{L^2(0,T;H^1(\Omega))} + \|p - P_h\|^2_{L^2(0,T;H^1(\Omega))}\\
\le& C\|u - U_h\|^2_{L^2(0,T;L^2(\partial\Omega))} + C\|y(U_h) - \hat{Y}_h\|^2_{L^2(0,T;H^1(\Omega))}\\
&+C\|p(U_h) - \tilde{P}_h\|^2_{L^2(0,T;H^1(\Omega))} + C\hat{\eta}^2_7 + C\hat{\eta}^2_4 \le C\sum_{i=1}^{10}\hat{\eta}^2_i. \qquad (3.52)
\end{aligned}$$

Therefore, (3.48) follows from (3.49) and (3.52). □

4. Discussions

In this paper, we investigate the a posteriori error estimate of the finite element approximation for the boundary control problems governed by the parabolic partial differential equations. We provide three different a posteriori error estimators for the parabolic boundary control problems with the observation of the distributed state, the boundary state and the final state.

There are many important issues still to be addressed in this area. For example, the discontinuous Galerkin method and the high order approximation scheme in time variable should be investigated. The optimal control problems governed by evolutionary advection-diffusion equations are also very important. Especially, many computational issues have to be addressed, e.g., adaptive refinement strategy should be investigated the optimal control problems governed by evolution equations.

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