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ESTIMATING ERROR BOUNDS FOR TERNARY SUBDIVISION CURVES/SURFACES *1)

Ghulam Mustafa
(*Department of Mathematics, Islamia University, Bahawalpur, Pakistan*
Department of Mathematics, University of Science and Technology of China, Hefei 230026, China
Email: mustafa_rakib@yahoo.com)

Jiansong Deng
(*Department of Mathematics, University of Science and Technology of China, Hefei 230026, China*
Email: dengjs@ustc.edu.cn)

Abstract

We estimate error bounds between ternary subdivision curves/surfaces and their control polygons after k-fold subdivision in terms of the maximal differences of the initial control point sequences and constants that depend on the subdivision mask. The bound is independent of the process of subdivision and can be evaluated without recursive subdivision. Our technique is independent of parametrization therefore it can be easily and efficiently implemented. This is useful and important for pre-computing the error bounds of subdivision curves/surfaces in advance in many engineering applications such as surface/surface intersection, mesh generation, NC machining, surface rendering and so on.

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1. Introduction

Subdivision is an important method for generating smooth curves and surfaces, see, e.g., [1, 2, 8]. Efficiency of subdivision algorithms, their flexibility and simplicity have found their way into wide applications in Computer Graphics and Computer Aided Geometric Design (CAGD). A widely used, efficient and intuitive way to specify, represent and reason about curved, surfaces, nonlinear geometry for design and modeling is the control polygon paradigm. For many applications, e.g., rendering, intersection testing or design, this raises the question just how well the control polygon approximates the exact curved and surface geometry. Several researchers give several answers to this question. Nairn et al. [7] show that the maximal distance between a Bézier segment and its control polygon is bounded in terms of the differences of the control point sequence and a constant that depends only on the degree of the polynomial. Lutterkort and Peters [6] derived a sharp bound on the distance between a spline and its B-spline control polygon. Their bound yields a piecewise linear envelope enclosing the spline and the control polygon. Recently, Karavelas et al. [5] derived sharp bounds for the distance between a planar parametric Bézier curve and parameterizations of its control polygon based on the Greville abscissae. In [1], Cheng gave an algorithm to estimate subdivision depths for rational curves and surfaces. The subdivision depth is not estimated for the given curve/surface

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directly. Their algorithm computes a subdivision depth for the polynomial curve/surface of which the given rational curve/surface is the image under the standard perspective projection. The existing methods for computing the bounds on the approximation of polynomials and splines by their control structures are all based on the parameterizations, so that it is very difficult for them to be generalized to the subdivision surfaces.

In this paper, we estimate error bounds for ternary subdivision curves/surfaces in terms of the maximal differences of the initial control point sequence and constants that depend on the subdivision mask. Our technique is independent of parameterizations and therefore it can be easily and efficiently implemented. The paper is organized as follows.

In Section 2 we prove the first main result of the paper about the estimation of error bounds between ternary subdivision curves and their control polygon after k-fold subdivision. Then as an application of our result we find error bounds for 3-point ternary approximating [3], 3-point ternary interpolatory [3] and 4-point ternary interpolatory [4] subdivision schemes. In Section 3 we generalize the main result of Section 2 to estimate the error bounds between subdivision surfaces and their control polygons. In Section 4, we summarize the results obtained and make some comments for future research directions.

2. The Error Bounds for Ternary Subdivision Curves

Let $p_i^k \in \mathbb{R}^N$, $i \in \mathbb{Z}$, denote a sequence of points in $\mathbb{R}^N$, $N \geqslant 2$, where k is a nonnegative integer. A ternary subdivision process [3] is defined by

$$p_{3i+s}^{k+1} = \sum_{j=0}^{m} a_{s,j} p_{i+j}^k, \quad s = 0, 1, 2, \tag{2.1}$$

where $m > 0$ and

$$\sum_{j=0}^{m} a_{s,j} = 1, \quad s = 0, 1, 2. \tag{2.2}$$

The coefficients $\{a_{s,j}\}_{j=0}^m$, $0 \le s \le 2$, are called subdivision mask. Given initial values $p_i^0 \in \mathbb{R}^N, i \in \mathbb{Z}$. Then in the limit $k \to \infty$, the process defines an infinite set of points in $\mathbb{R}^N$. The sequence of control points $\{p_i^k\}$ is related, in a natural way, with the diadic mesh points $t_i^k = i/3^k$, $i \in \mathbb{Z}$. The process (2.1) then defines a scheme whereby p_{3i}^{k+1} replaces the value p_i^k at the mesh point $t_{3i}^{k+1} = t_i^k$ and p_{3i+1}^{k+1} and p_{3i+2}^{k+1} are inserted at the new mesh points $t_{3i+1}^{k+1} = \frac{1}{3}(2t_i^k + t_{i+1}^k)$ and $t_{3i+2}^{k+1} = \frac{1}{3}(t_i^k + 2t_{i+1}^k)$ respectively.

We now establish our first main result for error bounds between subdivision curves and their control polygons.

Theorem 2.1. *Given initial control polygon $p_i^0 = p_i$, $i \in \mathbb{Z}$, and let the values p_i^k, $k \geqslant 0$ be defined recursively by subdivision process* (2.1) *together with* (2.2)*. Suppose P^k be the piecewise linear interpolant to the values p_i^k and P^∞ be the limit curve of the process* (2.1)*. If*

$$\delta = \max\left\{\sum_{j=0}^{m} |d_j|, \sum_{j=0}^{m} |e_j|, \sum_{j=0}^{m} |f_j|\right\} < 1, \tag{2.3}$$

where

$$d_j = \sum_{t=0}^{j} (a_{0,t} - a_{1,t}), \quad e_j = \sum_{t=0}^{j} (a_{1,t} - a_{2,t}), \quad f_j = a_{0,j} - (d_j + e_j), \tag{2.4}$$

then the error bounds between limit curve and its control polygon after k-fold subdivision is

$$\left\|P^k - P^\infty\right\|_\infty \leqslant \gamma\beta\left(\frac{\delta^k}{1-\delta}\right), \tag{2.5}$$

where

$$\begin{aligned} &\gamma = \max\left\{\sum_{j=0}^{m-1}|\tilde{a}_{0,j}|, \sum_{j=0}^{m-1}|\tilde{a}_{1,j}|, \sum_{j=0}^{m-1}|\tilde{a}_{2,j}|\right\}, \quad \tilde{a}_{s,j} = \sum_{t=j+1}^{m} a_{s,t}, \quad 0 \le s \le 2, \\ &\tilde{a}_{1,0} = \sum_{t=1}^{m} a_{1,t} - \frac{1}{3}, \quad \tilde{a}_{2,0} = \sum_{t=1}^{m} a_{2,t} - \frac{2}{3}, \quad \beta = \max_i \left\|p_{i+1}^0 - p_i^0\right\|. \end{aligned} \tag{2.6}$$

Proof. Let $\|.\|_\infty$ denote the maximum norm. Since the maximum difference between P^{k+1} and P^k is attained at a point on the $(k+1)$-th mesh, we have

$$\left\|P^{k+1} - P^k\right\|_\infty \leqslant \max\{\aleph_k^0, \aleph_k^1, \aleph_k^2\}, \tag{2.7}$$

where

$$\begin{cases} \aleph_k^0 = \max\limits_i \left\|p_{3i}^{k+1} - p_i^k\right\|, \quad \aleph_k^1 = \max\limits_i \left\|p_{3i+1}^{k+1} - \frac{1}{3}(2p_i^k + p_{i+1}^k)\right\|, \\ \aleph_k^2 = \max\limits_i \left\|p_{3i+2}^{k+1} - \frac{1}{3}(p_i^k + 2p_{i+1}^k)\right\|. \end{cases} \tag{2.8}$$

From (2.1) and (2.2) we obtain

$$p_{3i}^{k+1} - p_i^k = \sum_{j=0}^{m-1} \tilde{a_{0,j}}(p_{i+j+1}^k - p_{i+j}^k), \tag{2.9}$$

$$p_{3i+1}^{k+1} - \frac{1}{3}(2p_i^k + p_{i+1}^k) = \sum_{j=0}^{m-1} \tilde{a_{1,j}}(p_{i+j+1}^k - p_{i+j}^k), \tag{2.10}$$

$$p_{3i+2}^{k+1} - \frac{1}{3}(p_i^k + 2p_{i+1}^k) = \sum_{j=0}^{m-1} \tilde{a}_{2,j}(p_{i+j+1}^k - p_{i+j}^k), \tag{2.11}$$

where $\tilde{a}_{s,j}, 0 \le s \le 2$ is defined by (2.6). From (2.1), (2.2) and induction on m we can get

$$p_{3i+1}^k - p_{3i}^k = \sum_{j=0}^{m} d_j(p_{i+j+1}^{k-1} - p_{i+j}^{k-1}), \tag{2.12}$$

$$p_{3i+2}^k - p_{3i+1}^k = \sum_{j=0}^{m} e_j(p_{i+j+1}^{k-1} - p_{i+j}^{k-1}), \tag{2.13}$$

$$p_{3i+3}^k - p_{3i+2}^k = \sum_{j=0}^{m} f_j(p_{i+j+1}^{k-1} - p_{i+j}^{k-1}), \tag{2.14}$$

where d_j, e_j and f_j are defined by (2.4). It follows from (2.8)–(2.11) that

$$\aleph_k^s \leqslant \left(\sum_{j=0}^{m-1}|\tilde{a}_{s,j}|\right) \max_i \left\|p_{i+1}^k - p_i^k\right\|, \quad s = 0, 1, 2. \tag{2.15}$$

Using (2.12)–(2.14) recursively gives

$$\max_i \left\|p_{i+1}^k - p_i^k\right\| \leqslant \left(\sum_{j=0}^{m} |d_j|\right)^k \max_i \left\|p_{i+1}^0 - p_i^0\right\|, \tag{2.16a}$$

$$\max_i \left\|p_{i+1}^k - p_i^k\right\| \leqslant \left(\sum_{j=0}^{m} |e_j|\right)^k \max_i \left\|p_{i+1}^0 - p_i^0\right\|, \tag{2.16b}$$

$$\max_i \left\|p_{i+1}^k - p_i^k\right\| \leqslant \left(\sum_{j=0}^{m} |f_j|\right)^k \max_i \left\|p_{i+1}^0 - p_i^0\right\|. \tag{2.16c}$$

If δ is defined by (2.3), then from (2.16) we have

$$\max_i \left\|p_{i+1}^k - p_i^k\right\| \leqslant (\delta)^k \max_i \left\|p_{i+1}^0 - p_i^0\right\|. \tag{2.17}$$

If γ and β are defined by (2.6), then it follows from (2.7), (2.15) and (2.17) that

$$\left\|P^{k+1} - P^k\right\|_\infty \leqslant \gamma\beta\delta^k. \tag{2.18}$$

Using the triangle inequality yields (2.5). This completes the proof. □

Remark 2.1. We point out that the famous binary subdivision schemes satisfy condition (2.3). Our claim is supported by the following corollaries.

Corollary 2.1. *Given $p_i^0 = p_i, i \in \mathbb{Z}$, let the values $p_i^k, k \geqslant 0$ be defined recursively by the 3-point interpolating ternary subdivision scheme* [3]. *Suppose P^k be the piecewise linear interpolant to the values p_i^k and P^∞ be the limit curve of the subdivision process. Then*

$$\left\|P^k - P^\infty\right\|_\infty \leqslant \gamma\beta\left(\frac{\delta^k}{1-\delta}\right),$$

where

$$\gamma = \max\left\{1, \frac{1}{3} + |\alpha_2 - 1| + |\alpha_2 - \frac{1}{3}|, |\alpha_2 - 1| + |\alpha_2|\right\},$$

$$\delta = \max\left\{|\alpha_2 - \frac{1}{3}| + |\alpha_2|, 2|\alpha_2 - \frac{1}{3}| + |2\alpha_2 - 1|\right\},\ \beta = \max_i \left\|p_{i+1}^0 - p_i^0\right\|.$$

Corollary 2.2. *Given $p_i^0 = p_i, i \in \mathbb{Z}$, and let the values $p_i^k, k \geqslant 0$ be defined recursively by the 4-point interpolating ternary subdivision scheme* [4]. *Suppose P^k be the piecewise linear interpolant to the values p_i^k and P^∞ be the limit curve of the subdivision process. Then*

$$\left\|P^k - P^\infty\right\|_\infty \leqslant \gamma\beta\left(\frac{\delta^k}{1-\delta}\right),$$

where

$$\beta = \max_i \left\|p_{i+1}^0 - p_i^0\right\|,\ \delta = \max\left\{\frac{1}{6}(|\mu - \frac{1}{3}| + 2|\mu + 2| + |\mu + \frac{1}{3}|), \frac{1}{3}\left(|2\mu - 1| + 2|\mu|\right)\right\}$$

$$\gamma = \max\left\{1, |\frac{1}{6}\mu - \frac{13}{18}| + |\frac{1}{3}\mu + \frac{2}{3}| + |\frac{1}{6}\mu + \frac{1}{18}|, |\frac{1}{6}\mu + \frac{7}{18}| + |\frac{1}{3}\mu - \frac{1}{3}| + |\frac{1}{6}\mu - \frac{1}{18}|\right\}.$$

Remark 2.2. For Corollary 2.1 the range of α_2 for which condition (2.3) is satisfied is approximately $0.1679 \leqslant \alpha_2 \leqslant 0.6666$ while for Corollary 2.2 the range of μ is approximately $-0.9999 \leqslant \mu \leqslant 0.4999$.

3. The Error Bounds for Subdivision Surfaces

In this section, first we define basic concepts and settle some notations required for fair reading and better understanding. Then we will present our main result to estimate error bounds for ternary subdivision surfaces.

3.1. Definition and notations

Let $p^k_{i,j} \in \mathbb{R}^N$, $i \in \mathbb{Z}$, denote a sequence of points in $\mathbb{R}^N$, $N \geqslant 2$, where k is a nonnegative integer. A tensor product of ternary subdivision process (2.1) is defined by

$$p^{k+1}_{3i+\alpha,3j+\beta} = \sum_{r=0}^{m}\sum_{s=0}^{m} a_{\alpha,r}a_{\beta,s}p^k_{i+r,j+s}, \quad \alpha,\beta = 0,1,2, \tag{3.1}$$

where $a_{\alpha,r}$ satisfies (2.2). Given initial values $p^0_{i,j} \in \mathbb{R}^N, i,j \in \mathbb{Z}$. Then as $k \to \infty$, the process defines an infinite set of points in $\mathbb{R}^{\mathbb{N}}$. The sequence of values $\{p^k_{i,j}\}$ is related, in a natural way, with the diadic mesh points $(\frac{i}{3^k}, \frac{j}{3^k})$, $i,j \in \mathbb{Z}$. The process (3.1) then defines a scheme whereby $p^{k+1}_{3i,3j}$, $p^{k+1}_{3i+3,3j}$, $p^{k+1}_{3i,3j+3}$ and $p^{k+1}_{3i+3,3j+3}$ replaces the values $p^k_{i,j}$, $p^k_{i+1,j}$, $p^k_{i,j+1}$ and $p^k_{i+1,j+1}$ at the mesh points $(\frac{i}{3^k}, \frac{j}{3^k})$, $(\frac{i+1}{3^k}, \frac{j}{3^k})$, $(\frac{i}{3^k}, \frac{j+1}{3^k})$ and $(\frac{i+1}{3^k}, \frac{j+1}{3^k})$ respectively. The values $p^{k+1}_{3i+1,3j}$, $p^{k+1}_{3i+2,3j}$, $p^{k+1}_{3i,3j+1}$, $p^{k+1}_{3i+1,3j+1}$,$p^{k+1}_{3i+2,3j+1}$, $p^{k+1}_{3i+3,3j+1}$, $p^{k+1}_{3i,3j+2}$, $p^{k+1}_{3i+1,3j+2}$, $p^{k+1}_{3i+2,3j+2}$, $p^{k+1}_{3i+3,3j+2}$, $p^{k+1}_{3i+1,3j+3}$, and $p^{k+1}_{3i+2,3j+3}$ are inserted at the new mesh points $(\frac{i+1}{3^{k+1}}, \frac{j}{3^{k+1}})$, $(\frac{i+2}{3^{k+1}}, \frac{j}{3^{k+1}})$, $(\frac{i}{3^{k+1}}, \frac{j+1}{3^{k+1}})$, $(\frac{i+1}{3^{k+1}}, \frac{j+1}{3^{k+1}})$, $(\frac{i+2}{3^{k+1}}, \frac{j+1}{3^{k+1}})$, $(\frac{i+3}{3^{k+1}}, \frac{j+1}{3^{k+1}})$, $(\frac{i}{3^{k+1}}, \frac{j+2}{3^{k+1}})$, $(\frac{i+1}{3^{k+1}}, \frac{j+2}{3^{k+1}})$, $(\frac{i+2}{3^{k+1}}, \frac{j+2}{3^{k+1}})$, $(\frac{i+3}{3^{k+1}}, \frac{j+2}{3^{k+1}})$, $(\frac{i+1}{3^{k+1}}, \frac{j+3}{3^{k+1}})$ and $(\frac{i+2}{3^{k+1}}, \frac{j+3}{3^{k+1}})$, respectively. The sequence of values $\{p^{k+1}_{i,j}\}$ can be inserted into the new mesh at the mesh point $(\frac{i}{3^{k+1}}, \frac{j}{3^k+1})$ by bilinear interpolation. For example, the value $p^{k+1}_{3i+1,3j+1}$ can be inserted at the mesh point $(\frac{i+1}{3^{k+1}}, \frac{j+1}{3^{k+1}})$ as

$$\begin{aligned} p^{k+1}_{3i+1,3j+1}\left(\frac{i+1}{3^{k+1}}, \frac{j+1}{3^{k+1}}\right) &= \left(1-\frac{i+1}{3^{k+1}}\right)\left(1-\frac{j+1}{3^{k+1}}\right)p^k_{i,j} + \frac{i+1}{3^{k+1}}\left(1-\frac{j+1}{3^{k+1}}\right)p^k_{i+1,j} \\ &\quad + \left(1-\frac{i+1}{3^{k+1}}\right)\frac{j+1}{3^{k+1}}p^k_{i,j+1} + \frac{i+1}{3^{k+1}}\frac{j+1}{3^{k+1}}p^k_{i+1,j+1}. \end{aligned}$$

Let us suppose

$$\delta_\alpha = \max\left\{\sum_{s=0}^{m}|a_{\alpha,s}|\sum_{r=0}^{m}|d_r|, \sum_{s=0}^{m}|a_{\alpha,s}|\sum_{r=0}^{m}|e_r|, \sum_{s=0}^{m}|a_{\alpha,s}|\sum_{r=0}^{m}|f_r|\right\}, \quad \alpha = 0,1,2, \tag{3.2}$$

where d_r, e_r and f_r are defined by (2.4). Suppose

$$\bar\eta_1 = \sum_{t=1}^{m}|a_{0,t}| + \sum_{s=1}^{m-1}|\tilde a_{0,s}|,\ \bar\eta_2 = \sum_{t=1}^{m}|a_{1,t}| + \sum_{s=1}^{m-1}|\tilde a_{1,s}|,\ \bar\eta_3 = \sum_{t=1}^{m}|a_{2,t}| + \sum_{s=1}^{m-1}|\tilde a_{2,s}|, \tag{3.3}$$

and let

$$\eta_1 = |a_{0,0}|\bar\eta_1,\ \ \eta_2 = |a_{0,0}|\bar\eta_2 + \frac{1}{3},\ \ \eta_3 = |a_{0,0}|\bar\eta_3 + \frac{2}{3}, \tag{3.4a}$$

$$\eta_4 = |a_{1,0}|\bar\eta_1,\ \ \eta_5 = |a_{2,0}|\bar\eta_1,\ \ \eta_6 = |a_{1,0}|\bar\eta_2 + \frac{2}{8}, \tag{3.4b}$$

$$\eta_7 = |a_{1,0}|\bar\eta_3 + \frac{4}{9},\ \ \eta_8 = |a_{2,0}|\bar\eta_2 + \frac{1}{9},\ \ \eta_9 = |a_{2,0}|\bar\eta_3 + \frac{2}{9}, \tag{3.4c}$$

where $\tilde{a}_{s,j}, 0 \le s \le 2$ is defined by (2.6). Assume further that

$$\tau_\alpha = \sum_{t=1}^{m} |a_{0,t}| + \sum_{r=0}^{m} |a_{\alpha-1,r}| \sum_{s=1}^{m-1} |\tilde{a}_{0,s}|, \quad \alpha = 1,2,3; \tag{3.5a}$$

$$\tau_\alpha = \sum_{t=1}^{m} |a_{1,t}| + \sum_{r=0}^{m} |a_{\alpha-5,r}| \sum_{s=1}^{m-1} |\tilde{a}_{1,s}| + \frac{1}{3}, \quad \alpha = 4,6,7; \tag{3.5b}$$

$$\tau_\alpha = \sum_{t=1}^{m} |a_{2,t}| + \sum_{r=0}^{m} |a_{\alpha-7,r}| \sum_{s=1}^{m-1} |\tilde{a}_{2,s}| + \frac{2}{3}, \quad \alpha = 5,8,9, \tag{3.5c}$$

where we have set $a_{\alpha,t} = a_{0,t}$ for $\alpha < 0$. We also let

$$\begin{aligned}
&\xi_\alpha = \sum_{t=1}^{m} |a_{0,t}|\bar{\xi}_\alpha, \quad \alpha = 1,2,3,\\
&\xi_4 = \sum_{t=1}^{m} |a_{1,t}|\bar{\xi}_1, \quad \xi_5 = \sum_{t=1}^{m} |a_{2,t}|\bar{\xi}_1, \quad \xi_6 = \sum_{t=1}^{m} |a_{1,t}|\bar{\xi}_2 + \frac{1}{9},\\
&\xi_7 = \sum_{t=1}^{m} |a_{1,t}|\bar{\xi}_3 + \frac{2}{9}, \quad \xi_8 = \sum_{t=1}^{m} |a_{2,t}|\bar{\xi}_2 + \frac{2}{9}, \quad \xi_9 = \sum_{t=1}^{m} |a_{2,t}|\bar{\xi}_3 + \frac{4}{9},
\end{aligned} \tag{3.6}$$

where

$$\bar{\xi}_1 = \sum_{t=1}^{m} |a_{0,t}| + \sum_{s=1}^{m-1} |\tilde{a}_{0,s}|, \quad \bar{\xi}_2 = \sum_{t=1}^{m} |a_{1,t}| + \sum_{s=1}^{m-1} |\tilde{a}_{1,s}|, \quad \bar{\xi}_3 = \sum_{t=1}^{m} |a_{2,t}| + \sum_{s=1}^{m-1} |\tilde{a}_{2,s}|.$$

We define forward difference operators $\{\Delta^k_{i,j,t}\}, t = 1,2,3$, along the mesh directions as

$$\Delta^k_{i,j,1} = p^k_{i+1,j} - p^k_{i,j}, \quad \Delta^k_{i,j,2} = p^k_{i,j+1} - p^k_{i,j}, \quad \Delta^k_{i,j,3} = p^k_{i+1,j+1} - p^k_{i,j+1}. \tag{3.7}$$

3.2. Main result

In this section, we will present main result. The following result will support to accomplish the proof of the result.

Lemma 3.1. *Given initial control polygon $p^0_{i,j} = p_{i,j}$, $i,j \in \mathbb{Z}$, and let the values $p^k_{i,j}$, $k \geqslant 0$ be defined recursively by subdivision process* (3.1) *together with* (2.2). *If*

$$\delta = \max\{\delta_1, \delta_2, \delta_3\} < 1, \tag{3.8}$$

where δ_1, δ_2 and δ_3 are defined in (3.2), *then*

$$\max_{i,j} \left\|\Delta^k_{i,j,t}\right\| \leqslant (\delta)^k \max_{i,j} \left\|\Delta^0_{i,j,t}\right\|, \tag{3.9}$$

where $\{\Delta^k_{i,j,t}\}, t = 1,2,3$, are defined in (3.7).

Proof. From (3.1), (2.2) and using similar approach as we did for (2.12)–(2.14) we obtain

$$p^k_{3i+1,3j+\alpha} - p^k_{3i,3j+\alpha} = \sum_{s=0}^{m} a_{\alpha,s} \left(\sum_{r=0}^{m} d_r \left(p^{k-1}_{i+r+1,j+s} - p^{k-1}_{i+r,j+s}\right)\right), \tag{3.10a}$$

$$p^k_{3i+2,3j+\alpha} - p^k_{3i+1,3j+\alpha} = \sum_{s=0}^{m} a_{\alpha,s} \left(\sum_{r=0}^{m} e_r \left(p^{k-1}_{i+r+1,j+s} - p^{k-1}_{i+r,j+s}\right)\right), \tag{3.10b}$$

$$p^k_{3i+3,3j+\alpha} - p^k_{3i+2,3j+\alpha} = \sum_{s=0}^{m} a_{\alpha,s} \left(\sum_{r=0}^{m} f_r \left(p^{k-1}_{i+r+1,j+s} - p^{k-1}_{i+r,j+s}\right)\right), \tag{3.10c}$$

for $\alpha = 0, 1, 2$, where d_r, e_r and f_r are defined by (2.4). Furthermore,

$$p^k_{3i+1,3j+3} - p^k_{3i,3j+3} = \sum_{s=0}^{m} a_{0,s} \left(\sum_{r=0}^{m} d_r \left(p^{k-1}_{i+r+1,j+s+1} - p^{k-1}_{i+r,j+s+1} \right) \right), \tag{3.11}$$

$$p^k_{3i+2,3j+3} - p^k_{3i+1,3j+3} = \sum_{s=0}^{m} a_{0,s} \left(\sum_{r=0}^{m} e_r \left(p^{k-1}_{i+r+1,j+s+1} - p^{k-1}_{i+r,j+s+1} \right) \right), \tag{3.12}$$

$$p^k_{3i+3,3j+3} - p^k_{3i+2,3j+3} = \sum_{s=0}^{m} a_{0,s} \left(\sum_{r=0}^{m} f_r \left(p^{k-1}_{i+r+1,j+s+1} - p^{k-1}_{i+r,j+s+1} \right) \right), \tag{3.13}$$

$$p^k_{3i+\alpha,3j+1} - p^k_{3i+\alpha,3j} = \sum_{s=0}^{m} a_{\alpha,s} \left(\sum_{r=0}^{m} d_r \left(p^{k-1}_{i+s,j+r+1} - p^{k-1}_{i+s,j+r} \right) \right), \tag{3.14}$$

$$p^k_{3i+\alpha,3j+2} - p^k_{3i+\alpha,3j+1} = \sum_{s=0}^{m} a_{\alpha,s} \left(\sum_{r=0}^{m} e_r \left(p^{k-1}_{i+s,j+r+1} - p^{k-1}_{i+s,j+r} \right) \right), \tag{3.15}$$

$$p^k_{3i+\alpha,3j+3} - p^k_{3i+\alpha,3j+2} = \sum_{s=0}^{m} a_{\alpha,s} \left(\sum_{r=0}^{m} f_r \left(p^{k-1}_{i+s,j+r+1} - p^{k-1}_{i+s,j+r} \right) \right), \tag{3.16}$$

for $\alpha = 0, 1, 2$. Using (3.10) recursively and using the notations defined in (3.7) we get

$$\max_{i,j} \left\| \Delta^k_{i,j,1} \right\| \leqslant \left(\sum_{s=0}^{m} |a_{\alpha,s}| \sum_{r=0}^{m} |d_r| \right)^k \max_{i,j} \left\| \Delta^0_{i,j,1} \right\|,$$

$$\max_{i,j} \left\| \Delta^k_{i,j,1} \right\| \leqslant \left(\sum_{s=0}^{m} |a_{\alpha,s}| \sum_{r=0}^{m} |e_r| \right)^k \max_{i,j} \left\| \Delta^0_{i,j,1} \right\|,$$

$$\max_{i,j} \left\| \Delta^k_{i,j,1} \right\| \leqslant \left(\sum_{s=0}^{m} |a_{\alpha,s}| \sum_{r=0}^{m} |f_r| \right)^k \max_{i,j} \left\| \Delta^0_{i,j,1} \right\|,$$

for $\alpha = 0, 1, 2$. From (3.8) and the above inequalities we get

$$\max_{i,j} \left\| \Delta^k_{i,j,1} \right\| \leqslant (\delta)^k \max_{i,j} \left\| \Delta^0_{i,j,1} \right\|.$$

Similarly, using (3.14)–(3.16) recursively together with (3.7) and (3.8), we can obtain

$$\max_{i,j} \left\| \Delta^k_{i,j,2} \right\| \leqslant (\delta)^k \max_{i,j} \left\| \Delta^0_{i,j,2} \right\|.$$

Moreover, using (3.11)–(3.13) recursively together with (3.7) and (3.8) we have

$$\max_{i,j} \left\| \Delta^k_{i,j,3} \right\| \leqslant (\delta)^k \max_{i,j} \left\| \Delta^0_{i,j,3} \right\|.$$

This completes the proof of the lemma. □

We now present our main result to estimate error bounds for subdivision surfaces.

Theorem 3.2. *Given initial control polygon $p^0_{i,j} = p_{i,j}$, $i, j \in \mathbb{Z}$, and let the values $p^k_{i,j}$, $k \geqslant 0$ be defined recursively by the subdivision process* (3.1) *together with* (2.2)*. Suppose P^k be the piecewise linear interpolant to the values $p^k_{i,j}$ and P^∞ be the limit surface of the subdivision process* (3.1)*. If* (3.8) *holds, then the error bounds between the limit surface and its control polygon after k-fold subdivision satisfy*

$$\left\| P^k - P^\infty \right\|_\infty \leqslant (\eta\beta_1 + \tau\beta_2 + \xi\beta_3) \left(\frac{\delta^k}{1-\delta} \right), \tag{3.17}$$

where $\eta = \max_{1\le j\le 9}\{\eta_j\}, \tau = \max_{1\le j\le 9}\{\tau_j\}, \xi = \max_{1\le j\le 9}\{\xi_j\}$, *with* η_t, τ_t, ξ_t *defined by* (3.4)–(3.6), $\delta = \max\{\delta_1, \delta_2, \delta_3\}$, *with* $\delta_t, t = 1, 2, 3$ *defined by* (3.2), $\beta_t = \max\limits_{i,j} \left\|\Delta^0_{i,j,t}\right\|$, *with* $\{\Delta^0_{i,j,t}\}$ *defined by* (3.7).

Proof. Let $\|.\|_\infty$ denote the uniform norm. Since the maximum difference between P^{k+1} and P^k is attained at a point on the $(k+1)$-th mesh, we have

$$\left\|P^{k+1} - P^k\right\|_\infty \leqslant \max\{M_k^1, M_k^2, M_k^3, M_k^4, M_k^5, M_k^6, M_k^7, M_k^8, M_k^9\}, \tag{3.18}$$

where

$$\begin{cases} M_k^1 = \max\limits_{i,j} \left\|p^{k+1}_{3i,3j} - p^k_{i,j}\right\|, \quad M_k^2 = \max\limits_{i,j} \left\|p^{k+1}_{3i+1,3j} - \frac{1}{3}(2p^k_{i,j} + p^k_{i+1,j})\right\|, \\ M_k^3 = \max\limits_{i,j} \left\|p^{k+1}_{3i+2,3j} - \frac{1}{3}(p^k_{i,j} + 2p^k_{i+1,j})\right\|, \\ M_k^4 = \max\limits_{i,j} \left\|p^{k+1}_{3i,3j+1} - \frac{1}{3}(2p^k_{i,j} + p^k_{i,j+1})\right\|, \\ M_k^5 = \max\limits_{i,j} \left\|p^{k+1}_{3i,3j+2} - \frac{1}{3}(p^k_{i,j} + 2p^k_{i,j+1})\right\|, \\ M_k^6 = \max\limits_{i,j} \left\|p^{k+1}_{3i+1,3j+1} - \frac{1}{9}(4p^k_{i,j} + 2p^k_{i+1,j} + 2p^k_{i,j+1} + p^k_{i+1,j+1})\right\|, \\ M_k^7 = \max\limits_{i,j} \left\|p^{k+1}_{3i+2,3j+1} - \frac{1}{9}(2p^k_{i,j} + 4p^k_{i+1,j} + p^k_{i,j+1} + 2p^k_{i+1,j+1})\right\|, \\ M_k^8 = \max\limits_{i,j} \left\|p^{k+1}_{3i+1,3j+2} - \frac{1}{9}(2p^k_{i,j} + p^k_{i+1,j} + 4p^k_{i,j+1} + 2p^k_{i+1,j+1})\right\|, \\ M_k^9 = \max\limits_{i,j} \left\|p^{k+1}_{3i+2,3j+2} - \frac{1}{9}(p^k_{i,j} + 2p^k_{i+1,j} + 2p^k_{i,j+1} + 4p^k_{i+1,j+1})\right\|. \end{cases} \tag{3.19}$$

From (3.1) and (2.2) we get

$$p^{k+1}_{3i,3j} - p^k_{i,j} = \sum_{r=0}^m a_{0,r} \left(\sum_{s=0}^m a_{0,s}(p^k_{i+r,j+s} - p^k_{i,j})\right). \tag{3.20}$$

Since

$$\begin{aligned} & \sum_{s=0}^m a_{0,s}(p^k_{i+r,j+s} - p^k_{i,j}) \\ = \; & a_{0,0}(p^k_{i+r,j} - p^k_{i,j}) + a_{0,1}(p^k_{i+r,j+1} - p^k_{i,j}) + a_{0,2}(p^k_{i+r,j+2} - p^k_{i+r,j+1} + p^k_{i+r,j+1} - p^k_{i,j}) \\ & +a_{0,3}(p^k_{i+r,j+3} - p^k_{i+r,j+2} + p^k_{i+r,j+2} - p^k_{i+r,j+1} + p^k_{i+r,j+1} - p^k_{i,j}) + \dots \\ & +a_{0,m}(p^k_{i+r,j+m} - \dots + p^k_{i+r,j+3} - p^k_{i+r,j+2} + p^k_{i+r,j+2} - p^k_{i+r,j+1} + p^k_{i+r,j+1} - p^k_{i,j}), \end{aligned}$$

we have

$$\begin{aligned} & \sum_{s=0}^m a_{0,s}(p^k_{i+r,j+s} - p^k_{i,j}) \\ = \; & a_{0,0}(p^k_{i+r,j} - p^k_{i,j}) + \sum_{t=1}^m a_{0,t}(p^k_{i+r,j+1} - p^k_{i,j}) + \sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+r,j+s+1} - p^k_{i+r,j+s}), \end{aligned}$$

where $\tilde{a}_{0,s}$ is defined in (2.6). Taking sum on both side of above equation we get

$$\begin{aligned} \sum_{r=0}^m a_{0,r} \left(\sum_{s=0}^m a_{0,s}(p^k_{i+r,j+s} - p^k_{i,j})\right) &= a_{0,0} \sum_{r=0}^m a_{0,r}(p^k_{i+r,j} - p^k_{i,j}) \\ + \sum_{t=1}^m a_{0,t} \left(\sum_{r=0}^m a_{0,r}(p^k_{i+r,j+1} - p^k_{i,j})\right) &+ \sum_{r=0}^m a_{0,r} \left(\sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+r,j+s+1} - p^k_{i+r,j+s})\right). \end{aligned} \tag{3.21}$$

Since

$$\sum_{r=0}^{m} a_{0,r}(p^k_{i+r,j} - p^k_{i,j}) = a_{0,1}(p^k_{i+1,j} - p^k_{i,j}) + a_{0,2}(p^k_{i+2,j} - p^k_{i+1,j} + p^k_{i+1,j} - p^k_{i,j})$$
$$+a_{0,3}(p^k_{i+3,j} - p^k_{i+2,j} + p^k_{i+2,j} - p^k_{i+1,j} + p^k_{i+1,j} - p^k_{i,j}) + \dots$$
$$+a_{0,m}(p^k_{i+m,j} - \dots + p^k_{i+3,j} - p^k_{i+2,j} + p^k_{i+2,j} - p^k_{i+1,j} + p^k_{i+1,j} - p^k_{i,j}),$$

we have

$$\sum_{r=0}^{m} a_{0,r}(p^k_{i+r,j} - p^k_{i,j}) = \sum_{t=1}^{m} a_{0,t}(p^k_{i+1,j} - p^k_{i,j}) + \sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+s+1,j} - p^k_{i+s,j}).$$

Similarly,

$$\sum_{r=0}^{m} a_{0,r}(p^k_{i+r,j+1} - p^k_{i,j})$$
$$= a_{0,0}(p^k_{i,j+1} - p^k_{i,j}) + \sum_{t=1}^{m} a_{0,t}(p^k_{i+1,j+1} - p^k_{i,j}) + \sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+s+1,j+1} - p^k_{i+s,j+1}).$$

Substituting these sums into (3.21), we obtain from (3.20) that

$$\begin{aligned} p^{k+1}_{3i,3j} - p^k_{i,j} &= \left(a_{0,0}\sum_{t=1}^{m} a_{0,t}\right)(p^k_{i+1,j} - p^k_{i,j}) + \left(\sum_{t=1}^{m} a_{0,t}\right)^2 (p^k_{i+1,j+1} - p^k_{i,j+1}) \\ &+ \sum_{t=1}^{m} a_{0,t}(p^k_{i,j+1} - p^k_{i,j}) + a_{0,0}\sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+s+1,j} - p^k_{i+s,j}) \\ &+ \sum_{t=1}^{m} a_{0,t} \sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+s+1,j+1} - p^k_{i+s,j+1}) \\ &+ \sum_{r=0}^{m} a_{0,r}\left(\sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+r,j+s+1} - p^k_{i+r,j+s})\right). \end{aligned} \tag{3.22}$$

Similarly from (3.1) and (2.2) we obtain

$$\begin{aligned} &p^{k+1}_{3i+\alpha,3j+\beta} - \frac{1}{3}(2p^k_{i,j} + p^k_{i+\alpha,j+\beta}) \\ &= \left(a_{\beta,0}\sum_{t=1}^{m} a_{\alpha,t} - \frac{\alpha}{3}\right)(p^k_{i+1,j} - p^k_{i,j}) + \left(\sum_{t=1}^{m} a_{\beta,t}\sum_{t=1}^{m} a_{\alpha,t}\right)(p^k_{i+1,j+1} - p^k_{i,j+1}) \\ &+ \left(\sum_{t=1}^{m} a_{\beta,t} - \frac{\beta}{3}\right)(p^k_{i,j+1} - p^k_{i,j}) + a_{\beta,0}\sum_{s=1}^{m-1} \tilde{a}_{\alpha,s}(p^k_{i+s+1,j} - p^k_{i+s,j}) \\ &+ \left(\sum_{t=1}^{m} a_{\beta,t}\right)\sum_{s=1}^{m-1} \tilde{a}_{\alpha,s}(p^k_{i+s+1,j+1} - p^k_{i+s,j+1}) \\ &+ \sum_{r=0}^{m} a_{\alpha,r}\left(\sum_{s=1}^{m-1} \tilde{a}_{\beta,s}(p^k_{i+r,j+s+1} - p^k_{i+r,j+s})\right), \end{aligned} \tag{3.23}$$

for $(\alpha,\beta)=(1,0)$ or $(0,1)$. Similarly, for $(\alpha,\beta)=(2,0)$ or $(0,2)$, we have

$$\begin{aligned}
&p^{k+1}_{3i+\alpha,3j+\beta}-\frac{1}{3}(p^k_{i,j}+2p^k_{i+\alpha/2,j+\beta/2})\\
&=\left(a_{\beta,0}\sum_{t=1}^{m}a_{\alpha,t}-\frac{\alpha}{3}\right)(p^k_{i+1,j}-p^k_{i,j})+\left(\sum_{t=1}^{m}a_{\alpha,t}\sum_{t=1}^{m}a_{\beta,t}\right)(p^k_{i+1,j+1}-p^k_{i,j+1})\\
&\quad+\left(\sum_{t=1}^{m}a_{\beta,t}-\frac{\beta}{3}\right)(p^k_{i,j+1}-p^k_{i,j})+a_{\beta,0}\sum_{s=1}^{m-1}\tilde a_{\alpha,s}(p^k_{i+s+1,j}-p^k_{i+s,j})\\
&\quad+\left(\sum_{t=1}^{m}a_{\beta,t}\right)\sum_{s=1}^{m-1}\tilde a_{\alpha,s}(p^k_{i+s+1,j+1}-p^k_{i+s,j+1})\\
&\quad+\sum_{r=0}^{m}a_{\alpha,r}\left(\sum_{s=1}^{m-1}\tilde a_{\beta,s}(p^k_{i+r,j+s+1}-p^k_{i+r,j+s})\right).
\end{aligned}\tag{3.24}$$

Similarly, we have, for $\alpha=1,2$,

$$\begin{aligned}
&p^{k+1}_{3i+\alpha,3j+1}-\frac{1}{9}\left(2(3-\alpha)p^k_{i,j}+2\alpha p^k_{i+1,j}+(3-\alpha)p^k_{i,j+1}+\alpha p^k_{i+1,j+1}\right)\\
&=\left(a_{1,0}\sum_{t=1}^{m}a_{\alpha,t}-\frac{2\alpha}{9}\right)(p^k_{i+1,j}-p^k_{i,j})+\left(\sum_{t=1}^{m}a_{1,t}\sum_{t=1}^{m}a_{\alpha,t}-\frac{\alpha}{9}\right)(p^k_{i+1,j+1}-p^k_{i,j+1})\\
&\quad+\left(\sum_{t=1}^{m}a_{1,t}-\frac{1}{3}\right)(p^k_{i,j+1}-p^k_{i,j})+\left(\sum_{t=1}^{m}a_{1,t}\right)\sum_{s=1}^{m-1}\tilde a_{\alpha,s}(p^k_{i+s+1,j+1}-p^k_{i+s,j+1})\\
&\quad+a_{1,0}\sum_{s=1}^{m-1}\tilde a_{\alpha,s}(p^k_{i+s+1,j}-p^k_{i+s,j})+\sum_{r=0}^{m}a_{\alpha,r}\left(\sum_{s=1}^{m-1}\tilde a_{1,s}(p^k_{i+r,j+s+1}-p^k_{i+r,j+s})\right),
\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}
&p^{k+1}_{3i+\alpha,3j+2}-\frac{1}{9}\left((3-\alpha)p^k_{i,j}+\alpha p^k_{i+1,j}+2(3-\alpha)p^k_{i,j+1}+2\alpha p^k_{i+1,j+1}\right)\\
&=\left(a_{2,0}\sum_{t=1}^{m}a_{\alpha,t}-\frac{\alpha}{9}\right)(p^k_{i+1,j}-p^k_{i,j})+\left(\sum_{t=1}^{m}a_{\alpha,t}\sum_{t=1}^{m}a_{2,t}-\frac{2\alpha}{9}\right)(p^k_{i+1,j+1}-p^k_{i,j+1})\\
&\quad+\left(\sum_{t=1}^{m}a_{2,t}-\frac{2}{3}\right)(p^k_{i,j+1}-p^k_{i,j})+\left(\sum_{t=1}^{m}a_{2,t}\right)\sum_{s=1}^{m-1}\tilde a_{\alpha,s}(p^k_{i+s+1,j+1}-p^k_{i+s,j+1})\\
&\quad+a_{2,0}\sum_{s=1}^{m-1}\tilde a_{\alpha,s}(p^k_{i+s+1,j}-p^k_{i+s,j})+\sum_{r=0}^{m}a_{\alpha,r}\left(\sum_{s=1}^{m-1}\tilde a_{2,s}(p^k_{i+r,j+s+1}-p^k_{i+r,j+s})\right).
\end{aligned}\tag{3.26}$$

From (3.9), (3.19) and (3.22)–(3.26) we have, for $\alpha=0,1,2$,

$$\begin{aligned}
M_k^{1+\alpha}\leqslant(\delta)^k\Big\{&\left(|a_{0,0}|\sum_{t=1}^{m}|a_{\alpha,t}|+|a_{0,0}|\sum_{s=1}^{m-1}|\tilde a_{\alpha,s}|+\frac{\alpha}{3}\right)\max_{i,j}\left\|\Delta^0_{i,j,1}\right\|\\
&+\left(\sum_{t=1}^{m}|a_{0,t}|+\sum_{r=0}^{m}|a_{\alpha,r}|\sum_{s=1}^{m-1}|\tilde a_{0,s}|\right)\max_{i,j}\left\|\Delta^0_{i,j,2}\right\|\\
&+\sum_{t=1}^{m}|a_{0,t}|\left(\sum_{t=1}^{m}|a_{\alpha,t}|+\sum_{s=1}^{m-1}|\tilde a_{\alpha,s}|\right)\max_{i,j}\left\|\Delta^0_{i,j,3}\right\|\Big\};
\end{aligned}\tag{3.27}$$

for $\alpha = 1, 2$,

$$M_k^{3+\alpha} \leqslant (\delta)^k \Big\{ |a_{\alpha,0}| \left(\sum_{t=1}^{m} |a_{0,t}| + \sum_{s=1}^{m-1} |\tilde{a}_{0,s}| \right) \max_{i,j} \left\| \Delta_{i,j,1}^0 \right\| + \left(\sum_{t=1}^{m} |a_{\alpha,t}| + \sum_{r=0}^{m} |a_{0,r}| \sum_{s=1}^{m-1} |\tilde{a}_{\alpha,s}| + \frac{\alpha}{3} \right) \max_{i,j} \left\| \Delta_{i,j,2}^0 \right\| + \sum_{t=1}^{m} |a_{\alpha,t}| \left(\sum_{t=1}^{m} |a_{0,t}| + \sum_{s=1}^{m-1} |\tilde{a}_{0,s}| \right) \max_{i,j} \left\| \Delta_{i,j,3}^0 \right\| \Big\}, \tag{3.28}$$

$$M_k^{5+\alpha} \leqslant (\delta)^k \Big\{ \left(|a_{1,0}| \sum_{t=1}^{m} |a_{\alpha,t}| + |a_{1,0}| \sum_{s=1}^{m-1} |\tilde{a}_{\alpha,s}| + \frac{2\alpha}{9} \right) \max_{i,j} \left\| \Delta_{i,j,1}^0 \right\| + \left(\sum_{t=1}^{m} |a_{1,t}| + \sum_{r=0}^{m} |a_{\alpha,r}| \sum_{s=1}^{m-1} |\tilde{a}_{1,s}| + \frac{1}{3} \right) \max_{i,j} \left\| \Delta_{i,j,2}^0 \right\| + \left(\sum_{t=1}^{m} |a_{1,t}| \sum_{t=1}^{m} |a_{\alpha,t}| + \sum_{t=1}^{m} |a_{1,t}| \sum_{s=1}^{m-1} |\tilde{a}_{\alpha,s}| + \frac{\alpha}{9} \right) \max_{i,j} \left\| \Delta_{i,j,3}^0 \right\| \Big\}, \tag{3.29}$$

and

$$M_k^{7+\alpha} \leqslant (\delta)^k \Big\{ \left(|a_{2,0}| \sum_{t=1}^{m} |a_{\alpha,t}| + |a_{2,0}| \sum_{s=1}^{m-1} |\tilde{a}_{\alpha,s}| + \frac{\alpha}{9} \right) \max_{i,j} \left\| \Delta_{i,j,1}^0 \right\| + \left(\sum_{t=1}^{m} |a_{2,t}| + \sum_{r=0}^{m} |a_{\alpha,r}| \sum_{s=1}^{m-1} |\tilde{a}_{2,s}| + \frac{2}{3} \right) \max_{i,j} \left\| \Delta_{i,j,2}^0 \right\| + \left(\sum_{t=1}^{m} |a_{\alpha,t}| \sum_{t=1}^{m} |a_{2,t}| + \sum_{t=1}^{m} |a_{2,t}| \sum_{s=1}^{m-1} |\tilde{a}_{\alpha,s}| + \frac{2\alpha}{9} \right) \max_{i,j} \left\| \Delta_{i,j,3}^0 \right\| \Big\}. \tag{3.30}$$

If $\eta = \max_{1 \le j \le 9}\{\eta_j\}$, $\tau = \max_{1 \le j \le 9}\{\tau_j\}$, and $\xi = \max_{1 \le j \le 9}\{\xi_j\}$, where $\eta_t, \tau_t, \xi_t, t = 1, \cdots, 9$ are defined by (3.4)–(3.6), then from (3.18) and (3.27)–(3.30) we have

$$\left\| P^{k+1} - P^k \right\|_\infty \leqslant (\eta\beta_1 + \tau\beta_2 + \xi\beta_3)\, \delta^k, \tag{3.31}$$

where $\beta_t = \max_{i,j} \left\| \Delta_{i,j,t}^0 \right\|$, $t = 1, 2, 3$. Using triangle inequality we get

$$\left\| P^k - P^\infty \right\|_\infty \leqslant (\eta\beta_1 + \tau\beta_2 + \xi\beta_3) \left(\frac{\delta^k}{1 - \delta} \right). \tag{3.32}$$

This completes the proof of this theorem. □

4. Concluding Remarks

We have estimated error bounds for ternary subdivision curves/surfaces in terms of the maximal differences of the initial control point sequences and constants that depend on the subdivision mask. The bounds are independent of the process of subdivision and can be evaluated without recursive subdivision. Since our technique is independent of parameterizations, it can be easily and efficiently implemented. Estimation of error bounds for binary subdivision curves/surfaces is our forthcoming work. It is yet to be investigated whether we can use above technique for estimating error bounds for ternary subdivision surfaces on arbitrary topological meshes.

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