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MODIFIED MORLEY ELEMENT METHOD FOR A FOURTH ORDER ELLIPTIC SINGULAR PERTURBATION PROBLEM *1)

Ming Wang
(*LMAM, School of Mathematical Science, Peking University, Beijing 100871, China*)

Jin-chao Xu
(*School of Mathematical Science, Peking University, Beijing 100871, China and Department of Mathematics, Pennsylvania State University, USA*)

Yu-cheng Hu
(*School of Mathematical Science, Peking University, Beijing 100871, China*)

Abstract

This paper proposes a modified Morley element method for a fourth order elliptic singular perturbation problem. The method also uses Morley element or rectangle Morley element, but linear or bilinear approximation of finite element functions is used in the lower part of the bilinear form. It is shown that the modified method converges uniformly in the perturbation parameter.

Mathematics subject classification: 65N30.
Key words: Morley element, Singular perturbation problem.

1. Introduction

Let Ω be a bounded polygonal domain of R^2. Denote the boundary of Ω by $\partial\Omega$. For $f \in L^2(\Omega)$, we consider the following boundary value problem of fourth order elliptic singular perturbation equation:

$$\begin{cases} \varepsilon^2 \Delta^2 u - \Delta u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

where $\nu = (\nu_1, \nu_2)^\top$ is the unit outer normal to $\partial\Omega$, Δ is the standard Laplacian operator and ε is a real small parameter with $0 < \varepsilon \leq 1$. When $\varepsilon \to 0$ the differential equation formally degenerates to Poisson equation.

To overcome the C^1 difficult, it is prefer to using nonconforming finite element to solve problem (1.1). Since the differential equation degenerates to Poisson equation as $\varepsilon \to 0$, C^0 nonconforming elements seem better to be used. An C^0 nonconforming finite element was proposed in [4], and its uniform convergence in ε was shown.

It is known that Morley element is not C^0 element and it is divergent for Poisson equation (see [6]). When Morley element is applied to solve problem (1.1), it fails when $\varepsilon \to 0$ (see [4]). On the other hand, we have noticed the remark in the end of paper [4]: the best result uniformly in ε seems to be order of $O(h^{1/2})$ for any finite element method to problem (1.1). Here h is the mesh size. Since Morley element has the least number of element degrees of freedom, we prefer to use a method which still uses the degrees of freedom of Morley element to solve problem (1.1).

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In this paper, we will propose a modified Morley element method for problem (1.1). The method also uses Morley element, but the linear approximation of finite element functions is used in the part of the bilinear form corresponding to the second order differential term. The modified method degenerates to the conforming linear element method for Poisson equation when $\varepsilon = 0$, and this is consistent with the degenerate case of problem (1.1). We will show that the modified method converges uniformly in perturbation parameter ε.

The modified rectangle Morley element method is also considered in this paper.

The paper is organized as follows. The rest of this section lists some preliminaries. Section 2 gives the detail descriptions of the modified Morley element method. Section 3 shows the uniform convergence of the method. The last section gives some numerical results.

For nonnegative integer s, $H^s(\Omega)$, $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ denote the usual Sobolev space, norm and semi-norm respectively. Let $H_0^s(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ with respect to the norm $\|\cdot\|_{s,\Omega}$ and $(\cdot,\cdot)$ denote the inner product of $L^2(\Omega)$. Define

$$a(v,w) = \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j} \mathrm{d}x, \quad \forall v,w \in H^2(\Omega). \tag{1.2}$$

$$b(v,w) = \int_\Omega \sum_{i=1}^{2} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \mathrm{d}x, \quad \forall v,w \in H^1(\Omega). \tag{1.3}$$

The weak form of problem (1.1) is: find $u \in H_0^2(\Omega)$ such that

$$\varepsilon^2 a(u,v) + b(u,v) = (f,v), \quad \forall v \in H_0^2(\Omega). \tag{1.4}$$

Let u^0 be the solution of following boundary value problem:

$$\begin{cases} -\Delta u^0 = f, & \text{in } \Omega, \\ u^0|_{\partial\Omega} = 0 \end{cases} \tag{1.5}$$

The following lemma is shown in paper [4].

Lemma 1.1. *If Ω is convex, then there exists a constant C independent of ε such that*

$$|u|_{2,\Omega} + \varepsilon |u|_{3,\Omega} \le C\varepsilon^{-\frac{1}{2}} \|f\|_{0,\Omega} \tag{1.6}$$

$$|u - u^0|_{1,\Omega} \le C\varepsilon^{\frac{1}{2}} \|f\|_{0,\Omega} \tag{1.7}$$

for all $f \in L^2(\Omega)$.

2. Modified Morley Element Method

For a subset $B \subset R^2$ and r a nonnegative integer, let $P_r(B)$ be the space of all polynomials with degree not greater than r.

Morley Element

Given a triangle T, its three vertices is denoted by a_j, $1 \le j \le 3$. The edge of T opposite a_j is denoted by F_j, $1 \le j \le 3$. Denote the measures of T and F_i by $|T|$ and $|F_i|$ respectively. Morley element can be described by (T, P_T, Φ_T) with

1) T is a triangle.

2) $P_T = P_2(T)$.

3) Φ_T is the vector of degrees of freedom whose components are:

$$v(a_j), \ \frac{1}{|F_j|} \int_{F_j} \frac{\partial v}{\partial \nu} \mathrm{d}s, \quad 1 \le j \le 3$$

for $v \in C^1(T)$.

Rectangle Morley Element

Given a rectangle T, its four vertices and edges are denoted by a_j and F_j, $1 \le j \le 4$, respectively. Rectangle Morley element can be described by (T, P_T, Φ_T) with

1) T is a rectangle with its edges parallel to some coordinate axes respectively.

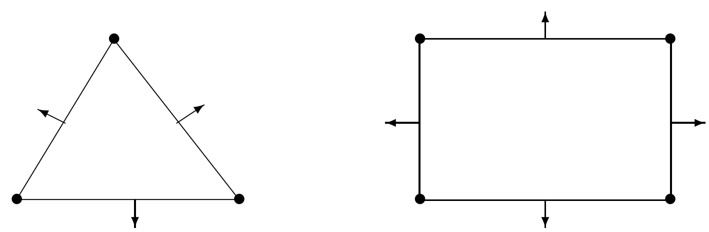
2) $P_T = P_2(T) + \mathrm{span}\{\, x_1^3, x_2^3\}$.

3) Φ_T is the vector of degrees of freedom whose components are:

$$v(a_j),\ \frac{1}{|F_j|}\int_{F_j}\frac{\partial v}{\partial \nu}\mathrm{d}s,\quad 1\le j\le 4$$

for $v \in C^1(T)$.

The degrees of freedom of these two elements are shown in Fig. 1.



Morley element   Rectangle Morley element

Fig. 1

The Morley element and its convergence for biharmonic equations can be found in [1-3,5], while the rectangle Morley element in [7].

For mesh size h, take $\mathcal{T}_h$ a triangulation of Ω. For Morley element $\mathcal{T}_h$ consists of triangles, otherwise $\mathcal{T}_h$ consists of rectangles with their edges parallel to some coordinate axes respectively. For each $T \in \mathcal{T}_h$, let h_T be the diameter of the smallest disk containing T and ρ_T be the diameter of the largest disk contained in T. Let $\{\mathcal{T}_h\}$ be a family of triangulations with $h \to 0$. Throughout the paper, we assume that $\{\mathcal{T}_h\}$ is quasi-uniform, namely it satisfied that $h_T \le h \le \eta\rho_T$, $\forall T \in \mathcal{T}_h$ for a positive constant η independent of h.

For each $\mathcal{T}_h$, let V_h and V_{h0} be the corresponding finite element spaces associated with Morley element or with rectangle Morley element for the discretization of $H^2(\Omega)$ and $H_0^2(\Omega)$ respectively. This defines two families of finite element spaces $\{V_h\}$ and $\{V_{h0}\}$. It is known that $V_h \not\subset H^2(\Omega)$ and $V_{h0} \not\subset H_0^2(\Omega)$. Let Π_h be the interpolation operator corresponding to $\mathcal{T}_h$ and Morley element or the rectangular Morley element.

We define

$$a_h(v,w) = \sum_{T\in\mathcal{T}_h}\int_T\sum_{i,j=1}^{2}\frac{\partial^2 v}{\partial x_i\partial x_j}\frac{\partial^2 w}{\partial x_i\partial x_j}\mathrm{d}x,\ \ v,w\in H^2(\Omega)+V_h \tag{2.1}$$

$$b_h(v,w) = \sum_{T\in\mathcal{T}_h}\int_T\sum_{i=1}^{2}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}\mathrm{d}x,\ \ v,w\in H^1(\Omega)+V_h \tag{2.2}$$

The standard finite element method for problem (1.4) corresponding to Morley element or to rectangle Morley element is: find $u_h \in V_{h0}$ such that

$$\varepsilon^2 a_h(u_h,v_h) + b_h(u_h,v_h) = (f,v_h),\ \forall v_h\in V_{h0}. \tag{2.3}$$

For Morley element, let Π_h^1 be the interpolation operator corresponding to linear conforming element for second order partial differential equation and $\mathcal{T}_h$. For the rectangle Morley element let Π_h^1 be the bilinear interpolation operator. We consider the following modified Morley element method: to find $u_h \in V_{h0}$ such that

$$\varepsilon^2 a_h(u_h, v_h) + b_h(\Pi_h^1 u_h, \Pi_h^1 v_h) = (f, \Pi_h^1 v_h), \quad \forall v_h \in V_{h0} \tag{2.4}$$

Problem (2.4) has unique solution when $\varepsilon > 0$. When $\varepsilon = 0$, the problem degenerates to

$$b_h(\Pi_h^1 u_h, \Pi_h^1 v_h) = (f, \Pi_h^1 v_h), \quad \forall v_h \in V_{h0} \tag{2.5}$$

Although the solution of problem (2.5) is not unique yet, $\Pi_h^1 u_h$ is uniquely determined. Actually, $\Pi_h^1 u_h$ is the exact finite element solution of linear or bilinear conforming elemet for problem (1.5). Hence the modified Morley element method seems to give a more natural way to solve problem (1.1).

We introduce the following mesh dependent norm $\|\cdot\|_{m,h}$ and semi-norm $|\cdot|_{m,h}$:

$$\begin{cases} \|v\|_{m,h} = \Big(\sum_{T\in\mathcal{T}_h} \|v\|_{m,T}^2\Big)^{1/2}, \\ |v|_{m,h} = \Big(\sum_{T\in\mathcal{T}_h} |v|_{m,T}^2\Big)^{1/2}, \end{cases} \qquad \forall v \in V_h + H^m(\Omega).$$

3. Convergence Analysis

In this section, we discuss the convergence properties of the modified Morley element methods in previous section.

Let u and u_h be the solutions of problem (1.4) and (2.4) respectively.

Lemma 3.1. *There exists a constant C independent of h and ε such that* $\forall v_h \in V_{h0}$

$$|\, b_h(\Pi_h^1 u, \Pi_h^1 v_h) + (\Delta u, \Pi_h^1 v_h)| \le Ch|u|_{2,\Omega}|\Pi_h^1 v_h|_{1,h} \tag{3.1}$$

$$|\, a_h(u, v_h) - (\Delta^2 u, \Pi_h^1 v_h)| \le Ch|u|_{3,\Omega}|v_h|_{2,h} \tag{3.2}$$

when $u \in H^3(\Omega)$.

Proof. Let $v_h \in V_{h0}$. Then $\Pi_h^1 v_h \in H_0^1(\Omega)$ and

$$|\, b_h(\Pi_h^1 u, \Pi_h^1 v_h) + (\Delta u, \Pi_h^1 v_h)| = |\, b_h(u - \Pi_h^1 u, \Pi_h^1 v_h)|.$$

By the interpolation theory and Schwarz inequality we obtain (3.1).

Now take $\phi \in H^1(\Omega)$. Given $T \in \mathcal{T}_h$ and an edge F of T, let P_F^0 be the orthogonal projection operator from $L^2(F)$ to $P_0(F)$.

Let $i, j \in \{1, 2\}$. It is known that the integral average of $\frac{\partial}{\partial x_j} v_h$ on F is continuous through F and vanishes when $F \subset \partial\Omega$. Then Green formula gives

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h} \int_T \Big(\phi \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial \phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big) \mathrm{d}x \\
&\quad = \sum_{T\in\mathcal{T}_h} \int_{\partial T} \phi \frac{\partial v_h}{\partial x_j} \nu_i \mathrm{d}s = \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \int_F \phi \frac{\partial v_h}{\partial x_j} \nu_i \mathrm{d}s \\
&\quad = \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \int_F \phi \Big(\frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j}\Big) \nu_i \mathrm{d}s \\
&\quad = \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \int_F (\phi - P_F^0 \phi) \Big(\frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j}\Big) \nu_i \mathrm{d}s
\end{aligned}$$

From Schwarz inequality and the interpolation theory we have

$$\begin{aligned}\Big|\sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F(\phi-P_F^0\phi)\Big(\frac{\partial v_h}{\partial x_j}-P_F^0\frac{\partial v_h}{\partial x_j}\Big)\nu_i\mathrm{d}s\Big|\\
&\le\sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\|\phi-P_F^0\phi\|_{0,F}\Big\|\frac{\partial v_h}{\partial x_j}-P_F^0\frac{\partial v_h}{\partial x_j}\Big\|_{0,F}\\
&\le C\sum_{T\in\mathcal{T}_h}h|\phi|_{1,T}|v_h|_{2,T}\le Ch|\phi|_{1,\Omega}|v_h|_{2,h}.\end{aligned}$$

Consequently, we obtain that $\forall\phi\in H^1(\Omega),\ \forall v_h\in V_{h0},\ i,j\in\{1,2\}$,

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T\Big(\phi\frac{\partial^2 v_h}{\partial x_i\partial x_j}+\frac{\partial\phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big)\mathrm{d}x\Big|\le Ch|\phi|_{1,\Omega}|v_h|_{2,h}. \tag{3.3}$$

We obtain the conclusion of the lemma from (3.3), the interpolation theory and the following equality,

$$\begin{aligned}a_h(u,v_h)&-(\Delta^2u,\Pi_h^1v_h)\\
&=\sum_{i=1}^2\sum_{T\in\mathcal{T}_h}\int_T\frac{\partial\Delta u}{\partial x_i}\frac{\partial(\Pi_h^1v_h-v_h)}{\partial x_i}\mathrm{d}x\\
&\quad+\sum_{i=1}^2\sum_{T\in\mathcal{T}_h}\int_T\Big(\Delta u\frac{\partial^2v_h}{\partial x_i^2}+\frac{\partial\Delta u}{\partial x_i}\frac{\partial v_h}{\partial x_i}\Big)\mathrm{d}x\\
&\quad+\sum_{1\le i\ne j\le 2}\sum_{T\in\mathcal{T}_h}\int_T\Big(\frac{\partial^2u}{\partial x_i\partial x_j}\frac{\partial^2v_h}{\partial x_i\partial x_j}+\frac{\partial^3u}{\partial x_i^2\partial x_j}\frac{\partial v_h}{\partial x_j}\Big)\mathrm{d}x\\
&\quad-\sum_{1\le i\ne j\le 2}\sum_{T\in\mathcal{T}_h}\int_T\Big(\frac{\partial^2u}{\partial x_i^2}\frac{\partial^2v_h}{\partial x_j^2}+\frac{\partial^3u}{\partial x_i^2\partial x_j}\frac{\partial v_h}{\partial x_j}\Big)\mathrm{d}x.\end{aligned} \tag{3.4}$$

From lemma 3.1, we have

Theorem 3.1. *There exists a constant C independent of h and ε such that*

$$\varepsilon\|u-u_h\|_{2,h}+\|u-\Pi_h^1u_h\|_{1,\Omega}\le Ch(\varepsilon|u|_{3,\Omega}+|u|_{2,\Omega}) \tag{3.5}$$

when $u\in H^3(\Omega)$.

Proof. Let $w_h=\Pi_hu$, then

$$\begin{aligned}\varepsilon\|u-u_h\|_{2,h}+\|u-\Pi_h^1u_h\|_{1,\Omega}\le{}&\varepsilon\|u-w_h\|_{2,h}+\|u-\Pi_h^1w_h\|_{1,\Omega}\\
&+\varepsilon\|u_h-w_h\|_{2,h}+\|\Pi_h^1(u_h-w_h)\|_{1,\Omega}\end{aligned} \tag{3.6}$$

Set $v_h=u_h-w_h$. From (2.4) and (1.1), we have

$$\begin{aligned}\varepsilon^2a_h(v_h,v_h)+&b_h(\Pi_h^1v_h,\Pi_h^1v_h)\\
&=\varepsilon^2a_h(u-w_h,v_h)+b_h(\Pi_h^1(u-w_h),\Pi_h^1v_h)\\
&\quad+\varepsilon^2\Big((\Delta^2u,\Pi_h^1v_h)-a_h(u,v_h)\Big)\\
&\quad-\Big((\Delta u,\Pi_h^1v_h)+b_h(\Pi_h^1u,\Pi_h^1v_h)\Big)\end{aligned}$$

From the interpolation theory, (3.1) and (3.2),

$$\varepsilon^2a_h(v_h,v_h)+b_h(\Pi_h^1v_h,\Pi_h^1v_h)\le Ch(\varepsilon|u|_{3,\Omega}+|u|_{2,\Omega})(\varepsilon|v_h|_{2,h}+|\Pi_h^1v_h|_{1,\Omega}).$$

Since

$$\varepsilon^2\|v_h\|_{2,h}^2 + \|\Pi_h^1 v_h\|_{1,\Omega}^2 \le C\Big(\varepsilon^2 a_h(v_h, v_h) + b_h(\Pi_h^1 v_h, \Pi_h^1 v_h)\Big)$$

we obtain that

$$\varepsilon\|u_h - w_h\|_{2,h} + \|\Pi_h^1(u_h - w_h)\|_{1,\Omega} \le Ch(\varepsilon|u|_{3,\Omega} + |u|_{2,\Omega}). \tag{3.7}$$

The theorem follows from the interpolation theory, (3.6) and (3.7).

Theorem 3.2. *If Ω is convex, then there exists a constant C independent of h and ε such that*

$$\varepsilon\|u - u_h\|_{2,h} + \|u - \Pi_h^1 u_h\|_{1,\Omega} \le Ch^{1/2}\|f\|_{0,\Omega}. \tag{3.8}$$

Proof. From the interpolation theory,

$$\|u - \Pi_h u\|_{2,h}^2 \le C|u|_{2,\Omega}\|u - \Pi_h u\|_{2,h} \le Ch|u|_{2,\Omega}|u|_{3,\Omega}$$

By lemma 1.1, we have

$$\varepsilon\|u - \Pi_h u\|_{2,h} \le Ch^{1/2}\|f\|_{0,\Omega}. \tag{3.9}$$

Similar to (4.4) in [4], we can show that

$$\|v - \Pi_h^1 v\|_{1,\Omega}^2 \le Ch|v|_{1,\Omega}|v|_{2,\Omega}, \quad \forall v \in H_0^2(\Omega). \tag{3.10}$$

Using (3.10), we obtain

$$\|u - u^0 - \Pi_h^1(u - u^0)\|_{1,\Omega}^2 \le Ch|u - u^0|_{1,\Omega}|u - u^0|_{2,\Omega}$$

and by the interpolation theory,

$$\|u^0 - \Pi_h^1 u^0\|_{1,\Omega} \le Ch|u^0|_{2,\Omega}.$$

From lemma 1.1 and the following inequalities

$$\|u^0\|_{2,\Omega} \le C\|f\|_{0,\Omega}$$

$$\|u - \Pi_h^1 u\|_{1,\Omega} \le \|u - u^0 - \Pi_h^1(u - u^0)\|_{1,\Omega} + \|u^0 - \Pi_h^1 u^0\|_{1,\Omega}$$

we have

$$\|u - \Pi_h^1 u\|_{1,\Omega} \le Ch^{1/2}\|f\|_{0,\Omega}. \tag{3.11}$$

Set $v_h = u_h - \Pi_h u$. Then $\Pi_h^1 v_h \in H_0^1(\Omega)$ and

$$|\, b_h(\Pi_h^1 u, \Pi_h^1 v_h) + (\Delta u, \Pi_h^1 v_h)| = |\, b_h(u - \Pi_h^1 u, \Pi_h^1 v_h)|.$$

From (3.11) and Schwarz inequality,

$$|\, b_h(\Pi_h^1 u, \Pi_h^1 v_h) + (\Delta u, \Pi_h^1 v_h)| \le Ch^{1/2}\|f\|_{0,\Omega}\|\Pi_h^1 v_h\|_{1,\Omega}. \tag{3.12}$$

Now take $\phi \in H^1(\Omega)$ and $i, j \in \{1, 2\}$. From the proof of lemma 3.1, we have

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T\Big(\phi\frac{\partial^2 v_h}{\partial x_i\partial x_j} + \frac{\partial\phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big)\mathrm{d}x\Big|$$

$$\le \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\|\phi - P_F^0\phi\|_{0,F}\Big\|\frac{\partial v_h}{\partial x_j} - P_F^0\frac{\partial v_h}{\partial x_j}\Big\|_{0,F}.$$

Since

$$\|\phi - P_F^0\phi\|_{0,F} \le 2\|\phi\|_{0,F} \le 2\|\phi\|_{0,\partial T} \le C\|\phi\|_{0,T}^{1/2}\|\phi\|_{1,T}^{1/2} \tag{3.13}$$

we have, by the interpolation theory

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T\Big(\phi\frac{\partial^2 v_h}{\partial x_i\partial x_j} + \frac{\partial\phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big)\mathrm{d}x\Big| \le Ch^{1/2}\|\phi\|_{0,\Omega}^{1/2}\|\phi\|_{1,\Omega}^{1/2}|v_h|_{2,h}. \tag{3.14}$$

If $\varepsilon \le h$, then by Green formula

$$\sum_{T\in\mathcal{T}_h}\int_T\frac{\partial\phi}{\partial x_i}\frac{\partial(\Pi_h^1 v_h - v_h)}{\partial x_i}\mathrm{d}x = \sum_{T\in\mathcal{T}_h}\int_{\partial T}\phi\frac{\partial(\Pi_h^1 v_h - v_h)}{\partial x_i}\nu_i\mathrm{d}s$$

$$-\sum_{T\in\mathcal{T}_h}\int_T \phi\frac{\partial^2(\Pi_h^1 v_h - v_h)}{\partial x_i^2}\mathrm{d}x$$

From Schwarz inequality, the interpolation thoery and (3.13), we obtain

$$\begin{aligned}\Big|\sum_{T\in\mathcal{T}_h}\int_T\frac{\partial\phi}{\partial x_i}\frac{\partial(\Pi_h^1 v_h - v_h)}{\partial x_i}\mathrm{d}x\Big| \le &\sum_{T\in\mathcal{T}_h}\|\phi\|_{0,\partial T}\Big\|\frac{\partial(\Pi_h^1 v_h - v_h)}{\partial x_i}\Big\|_{0,\partial T}\\ &+\sum_{T\in\mathcal{T}_h}\|\phi\|_{0,T}|\Pi_h^1 v_h - v_h|_{2,T}\\ \le &C\big(h^{1/2}\|\phi\|_{0,\Omega}^{1/2}\|\phi\|_{1,\Omega}^{1/2}+\|\phi\|_{0,\Omega}\big)|v_h|_{2,h}.\end{aligned}$$

Hence when $\varepsilon \le h$

$$\varepsilon^2\Big|\sum_{T\in\mathcal{T}_h}\int_T\frac{\partial\phi}{\partial x_i}\frac{\partial(\Pi_h^1 v_h - v_h)}{\partial x_i}\mathrm{d}x\Big| \le Ch^{1/2}\Big(\varepsilon^2\|\phi\|_{0,\Omega}^{1/2}\|\phi\|_{1,\Omega}^{1/2}+\varepsilon^{3/2}\|\phi\|_{0,\Omega}\Big)|v_h|_{2,h}. \tag{3.15}$$

When $\varepsilon > h$, by Schwarz inequality and the interpolation theory we have,

$$\varepsilon^2\Big|\sum_{T\in\mathcal{T}_h}\int_T\frac{\partial\phi}{\partial x_i}\frac{\partial(\Pi_h^1 v_h - v_h)}{\partial x_i}\mathrm{d}x\Big| \le Ch\varepsilon^2|\phi|_{1,\Omega}|v_h|_{2,h} \le Ch^{1/2}\varepsilon^{5/2}|\phi|_{1,\Omega}|v_h|_{2,h}. \tag{3.16}$$

From lemma 1.1, (3.4), (3.14), (3.15) and (3.16) we obtain

$$\varepsilon^2|\,a_h(u,v_h)-(\Delta^2 u,\Pi_h^1 v_h)| \le C\varepsilon h^{1/2}\|f\|_{0,\Omega}|v_h|_{2,h}. \tag{3.17}$$

Combining (3.9), (3.11), (3.12), (3.17) and the proof of theorem 3.1, we obtain the theorem.

4. Numerical Results

In this section, we will show some numerical results of the modified Morley element methods. We will use the same example used in [4] for comparison.

Let $\Omega = [0,1]\times[0,1]$ and $u(x) = \big(\sin(\pi x_1)\sin(\pi x_2)\big)^2$. For $\varepsilon \ge 0$, set $f = \varepsilon^2\Delta^2 u - \Delta u$. Then u is the solution of problem (1.1) when $\varepsilon > 0$, and is the solution of problem (1.5) when $\varepsilon = 0$. For the rectangle Morley element, Ω is divided into $h\times h$ squares, and for Morley element, each square is further divided into two triangles by the diagonal with a negative slash.

Define

$$\|v_h\|_{\varepsilon,h} = \big(\varepsilon^2 a_h(v_h,v_h)+b_h(\Pi_h^1 v_h,\Pi_h^1 v_h)\big)^{1/2},\quad \forall v_h\in V_{h0}.$$

Different values of ε and h are chosen to demonstrate the behaviors of the following relative error of two modified Morley element methods,

$$E_{\varepsilon,h} = \frac{\|u_h^I - u_h\|_{\varepsilon,h}}{\|u_h^I\|_{\varepsilon,h}} \tag{4.1}$$

where u_h is the solution of problem (2.4) and u_h^I denote the interpolant of u by Morley element or rectangular Morley element.

Let $g = \Delta^2 u$, then u is the solution of the following boundary value problem of biharmonic equation,

$$\begin{cases}\Delta^2 u = g, & \text{in } \Omega,\\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial\nu}\Big|_{\partial\Omega} = 0\end{cases} \tag{4.2}$$

For comparison, we also consider the error of finite element solution to problem (4.2). Let $\tilde{u}_h \in V_{h0}$ be the solution of the following problem,

$$a_h(\tilde{u}_h,v_h) = (g,\Pi_h^1 v_h),\quad \forall v_h\in V_{h0}. \tag{4.3}$$

In this situation, the relative error $\tilde{E}_h$ is presented by

$$\tilde{E}_h^2 = \frac{a_h(u_h^I - \tilde{u}_h, u_h^I - \tilde{u}_h)}{a_h(u_h^I, u_h^I)} \tag{4.4}$$

For the modified Morley element method and the modified rectangular Morley element method, $E_{\varepsilon,h}$ and $\tilde{E}_h$, corresponding some ε and h, are listed in Table 1 and Table 2 respectively.

Table 1. Modified Morley Element Method

$\varepsilon \setminus h$	2^{-3}	2^{-4}	2^{-5}	2^{-6}
0	0.0576	0.0145	0.0036	0.0009
2^{-10}	0.0577	0.0146	0.0037	0.0009
2^{-8}	0.0586	0.0156	0.0046	0.0017
2^{-6}	0.0713	0.0266	0.0119	0.0057
2^{-4}	0.1653	0.0832	0.0418	0.0210
2^{-2}	0.3404	0.1749	0.0885	0.0451
2^{0}	0.3869	0.1979	0.1000	0.0512
Biharmonic	0.3908	0.1998	0.1010	0.0517

Table 2. Modified Rectangular Morley Element Method

$\varepsilon \setminus h$	2^{-3}	2^{-4}	2^{-5}	2^{-6}
0	0.0205	0.0046	0.0011	0.0002
2^{-10}	0.0205	0.0047	0.0011	0.0003
2^{-8}	0.0211	0.0052	0.0016	0.0006
2^{-6}	0.0285	0.0107	0.0049	0.0024
2^{-4}	0.0757	0.0360	0.0179	0.0091
2^{-2}	0.1568	0.0770	0.0392	0.0211
2^{0}	0.1774	0.0875	0.0449	0.0246
Biharmonic	0.1791	0.0884	0.0453	0.0249

From Table 1 we see that the modified Morley element method, unlike Morley element method (see Table 1 in [4]), converges for all $\varepsilon \in [0,1]$. More precisely, the result shows that $E_{\varepsilon,h}$ tends to $E_{0,h}$ as ε approaches 0, while $E_{\varepsilon,h}$ is linear with respect to h as well as $\tilde{E}_h$ is when ε is large. That is, the modified Morley element method behaves in the way that the conforming linear element does for Poisson equation when ε is small, while it likes Morley element for the biharmoni equation when ε is large.

We can get the similar discussion about the modified rectangular Morley element method from Table 2.

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