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CONVERGENCE ANALYSIS OF MORLEY ELEMENT ON ANISOTROPIC MESHES *1)

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Abstract

The main aim of this paper is to study the convergence of a nonconforming triangular plate element-Morley element under anisotropic meshes. By a novel approach, an explicit bound for the interpolation error is derived for arbitrary triangular meshes (which even need not satisfy the maximal angle condition and the coordinate system condition), the optimal consistency error is obtained for a family of anisotropically graded finite element meshes.

Mathematics subject classification: 65N30, 65N15.
Key words: Anisotropic meshes, Interpolation error, Consistency error, Morley element.

1. Introduction

It is well-known that regular assumption or quasi-uniform assumption[9,12] of finite element meshes is a basic condition in the convergence analysis of finite element approximation both for conventional conforming and nonconforming elements. However, with the development of the finite element methods and its applications to more fields and more complex problems, the above conventional meshes conditions become a severe restriction for the finite element methods. For example, the solution may have anisotropic behavior in parts of the domain. This means that the solution varies significantly only in certain directions. In such cases, it is an obvious idea to reflect this anisotropy in the discretization by using anisotropic meshes with a small mesh size in the direction of the rapid variation of the solution and a larger mesh size in the perpendicular direction.

Indeed, some early papers have been written to prove error estimates under more general conditions (refer to [7, 15]). Recently, much attention is paid to FEMs under anisotropic meshes. In particular, for second order problems and rectangular meshes, we refer to Acosta [1,2], Apel[3−6], Chen[10,11], Duran[13,14], Shenk[22] and references therein. Above all, it is now well known that the regularity assumption is not needed. As to fourth order problems, the plate bending problem for example, only some rectangular elements have been concerned, interested reader can refer to [11] for Adini's element and [19] for bicubic Hermite element. However, up to now, there are no papers on anisotropic triangular plate elements, especially for nonconforming ones. This paper is devoted to fill the gap of it.

It is known that the nonconforming Morley element is an effective element for the plate bending problem. This quadratic triangular element is particularly attractive, because of its simple structure and low degrees of freedom. However, since the continuity of Morley element is very weak (nonconforming non-C^0 element), even under quasi-uniform meshes, the error

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estimate of it is not easy and has been explored a long way (refer to [17, 20, 6, 21]). In this paper, we consider the plate bending problem discretized with the nonconforming Morley element under anisotropic triangular meshes. Since the technique developed to estimate the local interpolation error (refer to [4, 10]) is not convenient to be applied for triangular elements, we turn to other tricks. By using of the special properties of the shape function space of Morley element and the results of Poincaré inequality (refer to [8, 18]), we derive an explicit bound of its interpolation error under arbitrary triangular meshes. The consistency error is even more hard to be treated. In order to obtain the optimal consistency error, we have to consider a special type of product anisotropic triangular meshes, namely, tensor product meshes. As to more general anisotropic triangular meshes, we are still work on them.

The outline of the paper is as follows. In the next section, after introducing the nonconforming Morley element approximation to the plate bending problem, we derive the interpolation error of it under arbitrary triangular meshes. In section 3, the optimal anisotropic consistency error of Morley element is obtained by a novel approach under a family of anisotropically graded finite element meshes . In order to verify the validity of theoretical analysis, some numerical experiments are carried out in section 4.

2. The Interpolation Error Estimate on Arbitrary Triangular Meshes

We consider the plate bending problem[12]:

$$\begin{cases} \triangle^2 u = f, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω denotes a plane polygonal domain, $f \in L^2(\Omega)$ is the applied force, n is the unit outward normal along the boundary $\partial\Omega$. The related variational form is :

$$\begin{cases} \text{Find } \ u \in H_0^2(\Omega), \ \text{ such that} \\ a(u,v) = (f,v), \qquad \forall v \in H_0^2(\Omega), \end{cases} \tag{2.2}$$

where

$$\begin{aligned} &a(u,v) = \int_\Omega A(u,v)dxdy, \\ &A(u,v) = \triangle u \triangle v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}), \\ &(f,v) = \int_\Omega fv dxdy, \\ &H_0^2(\Omega) = \{v \in H^2(\Omega), v = \frac{\partial v}{\partial n} = 0, \text{ on } \partial\Omega\} \end{aligned}$$

and σ is the Poisson ratio, $0 < \sigma < \frac{1}{2}$, $u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$, etc.

Clearly, the above bilinear form $a(\cdot,\cdot)$ is bounded and coercive :

$$\begin{cases} |a(v,w)| \le (1+\sigma)|v|_{2,\Omega}|w|_{2,\Omega}, \quad v,w \in H_0^2(\Omega) \\ \qquad\qquad a(v,v) \ge (1-\sigma)|v|^2_{2,\Omega}, \quad v \in H_0^2(\Omega). \end{cases} \tag{2.3}$$

Throughout this paper, we adopt the standard conventions for Sobolev norms and seminorms of a function v defined on an open set G:

$$\|v\|_{m,G} = \left(\int_G \sum_{|\alpha| \le m} |D^\alpha v|^2 \right)^{\frac{1}{2}},$$

$$|v|_{m,G}=\left(\int_G\sum_{|\alpha|=m}|D^\alpha v|^2\right)^{\frac{1}{2}}.$$

We shall also denote by $P_l(G)$ the space of polynomials on G of degrees no more than l.

Let $\mathcal{J}_h$ be an arbitrary triangulation of Ω, with each element K being an open triangle of size h_K, and $h=\max\limits_{K\in\mathcal{J}_h}h_K$. On this triangulation we construct the so-called Morley element (cf. [17]):

$$\begin{aligned}V_h=\{v_h\in L^2(\Omega):v_h|_K\in P_2(K),v_h\ \text{ is continuous at each vertex}\\ a\in K,\int_F[\frac{\partial v_h}{\partial n}]ds=0,\forall\ \ F\subset K,K\in\mathcal{J}_h,v_h(a)=0,a\in\partial\Omega\}\end{aligned}\tag{2.4}$$

where we denote faces of elements by F and by $[v]$ the jump of the function v on the faces F. For boundary faces we identify $[v]$ with v.

We note that V_h is not a subspace of $H^1(\Omega)$ (non C^0 nonconforming element). The discrete problem of (2.2) then reads as

$$\begin{cases}\text{Find }\ u_h\in V_h,\ \text{ such that}\\ a_h(u_h,v_h)=(f,v_h),\qquad\forall v_h\in V_h,\end{cases}\tag{2.5}$$

where $a_h(u_h,v_h)=\sum\limits_{K\in\mathcal{J}_h}\int_K A(u_h,v_h)dxdy$.

Put

$$\|\cdot\|_h=\left(\sum_{K\in\mathcal{J}_h}|\cdot|_{2,K}^2\right)^{\frac{1}{2}}.$$

It is easy to prove that $\|\cdot\|_h$ is a norm of V_h, so the discrete problem (2.5) has unique solution by Lax-Milgram Lemma [9,12].

Let u and u_h be the solutions of (2.1) and (2.5) , respectively, by Strang's Lemma [9,12],

$$\|u-u_h\|_h\le C\left(\inf_{v_h\in V_h}\|u-v_h\|_h+\sup_{v_h\in V_h}\frac{|a_h(u,v_h)-(f,v_h)|}{\|v_h\|_h}\right),\tag{2.6}$$

where the first term is the approximation error and the second one is the consistency error. Throughout this paper, the positive constant C will be used as a generic constant,which is independent of h_K and of $\frac{h_K}{\rho_K}$. In this section we only consider the approximation error, the consistency error will be discussed in the next section.

The Morley's interpolant Π_h, $\Pi_h:H^2(\Omega)\longrightarrow V_h$ is defined by $\Pi_h|_K=\Pi_K$ with

$$\begin{cases}\Pi_Ku(a)=u(a),\qquad\forall\ \text{ vertex }\ a\in K,\\ \displaystyle\int_F\frac{\partial\Pi_Ku}{\partial n}ds=\int_F\frac{\partial u}{\partial n}ds,\qquad\forall F\subset\partial K.\end{cases}\tag{2.7}$$

The following result is the classic Poincaré inequality can be found in [18].

Lemma 2.1. *Let G be a bounded convex domain and let $w\in H^1(G)$ be a function with vanishing average, then*

$$\|w\|_{0,G}\le\frac{d}{\pi}|w|_{1,G}\tag{2.8}$$

where d is the diameter of G.

Remark 2.1. It is very interesting to remark that the constant in the Poincaré inequality can be taken explicitly and independent of the shape (i.e., depending only on the diameter) for a general convex domain. However, the proof in [18] contains a mistake, and recently [8] gives a modification proof, fortunately, the optimal constant $\frac{d}{\pi}$ in the Poincaré inequality remains valid.

Now, we will derive the optimal interpolation error estimate under arbitrary triangular meshes.

Theorem 2.1. *Under the above hypothesis, let $u \in H^3(\Omega)$, then there holds*

$$\inf_{v_h \in V_h} \|u - v_h\|_h \le \|u - \Pi_h u\|_h \le \frac{2}{\pi} h |u|_{3,\Omega}. \tag{2.9}$$

Proof. We only need to prove the the following result

$$|u - \Pi_K u|_{2,K} \le \frac{2}{\pi} h_K |u|_{3,K}, \forall K \in \mathcal{J}_h. \tag{2.10}$$

Firstly, let us consider $\alpha = (2,0)$, since $D^\alpha \Pi_h u = const$, then by Green's formula and the definition of Morley's interpolant, we have

$$\begin{aligned} D^\alpha \Pi_K u &= \frac{1}{|K|} \int_K D^\alpha \Pi_K u dxdy = \frac{1}{|K|} \sum_{F \subset \partial K} \int_F \frac{\partial \Pi_K u}{\partial x} n_x ds \\ &= \frac{1}{|K|} \sum_{F \subset \partial K} \int_F \left(\frac{\partial \Pi_K u}{\partial n} n_x - \frac{\partial \Pi_K u}{\partial s} n_y \right) n_x ds \\ &= \frac{1}{|K|} \sum_{F \subset \partial K} \int_F \left(\frac{\partial u}{\partial n} n_x - \frac{\partial u}{\partial s} n_y \right) n_x ds \\ &= \frac{1}{|K|} \sum_{F \subset \partial K} \int_F \frac{\partial u}{\partial x} n_x ds \\ &= \frac{1}{|K|} \int_K D^\alpha u dxdy. \end{aligned} \tag{2.11}$$

Therefore, $D^\alpha u - D^\alpha \Pi_K u$ has vanishing mean value on the element K, it follows form Lemma 2.1 that

$$\|D^\alpha u - D^\alpha \Pi_K u\|_{0,K} \le \frac{h_K}{\pi} |D^\alpha u|_{1,K}. \tag{2.12}$$

By the same argument, we can obtain the same result of (2.11) for $\alpha = (0,2)$ and $\alpha = (1,1)$, which implies (2.10) and completes the proof of the theorem.

3. The Consistency Error Estimate on Anisotropic Triangular Meshes

In this section, we will focus on explain the ideas for the estimation of the consistency error. For the sake of simplicity, let Ω be a union of rectangles with sides parallel to the axes of the Cartesian coordinate system (x,y). Firstly, assume Ω is decomposed as a union of rectangular elements K with length h_{K1}, h_{K2} in x and y direction respectively, then $\mathcal{J}_h$ is obtained by dividing each rectangle into two triangles.

In the sense of (2.6), it is our aim to derive an estimate for

$$\sup_{v_h \in V_h} \frac{|a_h(u, v_h) - (f, v_h)|}{\|v_h\|_h}.$$

If we start in the usual way, the well known result[16] gives

$$a_h(u, v_h) = - \sum_{K \in \mathcal{J}_h} \int_K \nabla \triangle u \cdot \nabla v_h + E_1(u, v_h) + E_2(u, v_h), \tag{3.1}$$

where

$$\begin{cases} E_1(u, v_h) = \displaystyle\sum_{K \in \mathcal{J}_h} \int_{\partial K} [\triangle u - (1-\sigma) u_{ss}] v_{hn} ds, \\ E_2(u, v_h) = \displaystyle\sum_{K \in \mathcal{J}_h} \int_{\partial K} (1-\sigma) u_{sn} v_{hs} ds \end{cases} \tag{3.2}$$

and $(\cdot)_s = \frac{\partial}{\partial s}, (\cdot)_n = \frac{\partial}{\partial n}$, are tangential and normal derivatives along element boundaries, respectively.

The classical method to estimate the consistence error[16] is directly based on the estimate of the following identity:

$$\int_F (v - P_{0,F} v)(w - P_{0,F} w) ds, \quad F \subset \partial K, v, w \in H^1(K), \tag{3.3}$$

where $P_{0,F} v = \frac{1}{|F|} \int_F v ds$, using coordinate transformation, interpolation theory and trace theorem, through $\partial K \to \partial \hat{K} \to \hat{K} \to K$, then we have

$$\begin{aligned} &\left| \int_F (v - M_F v)(w - M_F w) ds \right| \\ &\le \|v - M_F v\|_{0,F} \|w - M_F w\|_{0,F} \\ &\le C \frac{|F|}{|K|} \times \left(\sum_{i=1,2} h_{Ki}^2 \|\partial_i v\|_{0,K}^2 \right)^{\frac{1}{2}} \left(\sum_{i=1,2} h_{Ki}^2 \|\partial_i w\|_{0,K}^2 \right)^{\frac{1}{2}}, \end{aligned} \tag{3.4}$$

where $\partial_1 = \frac{\partial}{\partial x}, \partial_2 = \frac{\partial}{\partial y}$.

In fact, the estimate is all right for a small side of an element, but we can not get the desire convergence result of (3.4) as usual. Thus it is more difficult for us to estimate anisotropic nonconforming consistency error than conventional one.

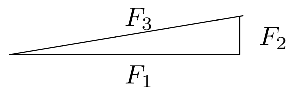

Figure 1. a narrow triangle element K

Let us consider a narrow triangle element K illustrated in Figure 1, $h_{K1} \gg h_{K2}$, for the two long edges F_1, F_3, we have the factor $(\frac{h_{K1}}{h_{K2}})^{\frac{1}{2}}$ (which is unbounded) in the estimate (3.4). So, something must be done for the two long edges.

For the later use, we define an operator $T : H^1(K) \longrightarrow P, P = span\{1, y\}$ as follows:

$$\int_{F_i} T v ds = \int_{F_i} v ds, \quad i = 1, 3. \tag{3.5}$$

It can be checked easily that the operator T is well-posed.

Now, we are in a position to prove an estimate for the consistency error.

Theorem 3.1. *Assume u , u_h to be the solution of (2.2) and (2.5), respectively, further assume $u \in H^3(\Omega) \cap H_0^2(\Omega), f \in L^2(\Omega)$, then we have*

$$\sup_{v_h \in V_h} \frac{|a_h(u, v_h) - (f, v_h)|}{\|v_h\|_h} \le Ch \left(|u|_{3,\Omega} + h \|f\|_{0,\Omega} \right). \tag{3.6}$$

Proof. Firstly, we consider the following term,

$$\int_K \triangle u \triangle v_h dxdy = \int_K \triangle u (v_{hxx} + v_{hyy}) dxdy.$$

Noticed $v_{hyy} = const, n_y|_{F_2} = 0$, by Green's formula, we have

$$\begin{aligned} v_{hyy} &= \frac{1}{|K|}\int_K v_{hyy}dxdy = \frac{1}{|K|}\sum_{i=1,3}\int_{F_i} v_{hy}n_y ds \\ &= \frac{1}{|K|}\sum_{i=1,3}\int_{F_i} Tv_{hy}n_y ds = \frac{1}{|K|}\int_K (Tv_{hy})_y dxdy \\ &= (Tv_{hy})_y. \end{aligned} \tag{3.7}$$

So,

$$\begin{aligned} \int_K \triangle u \triangle v_h dxdy = \int_K &\triangle u(v_{hxx} + (Tv_{hy})_y)ddxdy \\ = -\int_K &[(\triangle u)_x v_{hx} + (\triangle u)_y Tv_{hy}]\, dxdy \\ + \sum_{i=1}^{3}\int_{F_i} &\triangle u(v_{hx}n_x + Tv_{hy}n_y)ds. \end{aligned} \tag{3.8}$$

Green's formula gives

$$\begin{aligned} \int_K u_{xx}v_{hyy}dxdy &= \int_K u_{xx}(Tv_{hy})_y dxdy \\ &= -\int_K u_{xxy}Tv_{hy}dxdy + \sum_{i=1}^{3}\int_{F_i} u_{xx}Tv_{hy}n_y ds \\ &= -\int_K u_{xxy}(Tv_{hy} - v_{hy})dxdy \\ &- \int_K u_{xxy}v_{hy}dxdy + \sum_{i=1}^{3}\int_{F_i} u_{xx}Tv_{hy}n_y ds \\ &= -\int_K u_{xxy}(Tv_{hy} - v_{hy})dxdy + \int_K u_{xy}v_{hxy}dxdy \\ &- \sum_{i=1}^{3}\int_{F_i} u_{xy}v_{hy}n_x ds + \sum_{i=1}^{3}\int_{F_i} u_{xx}Tv_{hy}n_y ds, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \int_K u_{yy}v_{hxx}dxdy &= -\int_K u_{xyy}v_{hx}dxdy + \sum_{i=1}^{3}\int_{F_i} u_{yy}v_{hx}n_x ds \\ &= -\int_K u_{xyy}(v_{hx} - Tv_{hx})dxdy - \int_K u_{xyy}Tv_{hx}dxdy + \sum_{i=1}^{3}\int_{F_i} u_{yy}v_{hx}n_x ds \\ &= -\int_K u_{xyy}(v_{hx} - Tv_{hx})dxdy + \int_K u_{xy}(Tv_{hx})_y dxdy \\ &- \sum_{i=1}^{3}\int_{F_i} u_{xy}Tv_{hx}n_y ds + \sum_{i=1}^{3}\int_{F_i} u_{yy}v_{hx}n_x ds \\ &= -\int_K u_{xyy}(v_{hx} - Tv_{hx})dxdy + \int_K u_{xy}v_{hxy}dxdy \\ &- \sum_{i=1}^{3}\int_{F_i} u_{xy}Tv_{hx}n_y ds + \sum_{i=1}^{3}\int_{F_i} u_{yy}v_{hx}n_x ds. \end{aligned} \tag{3.10}$$

Note that the proof of (3.10) has exploited the property $(Tv_{hx})_y = v_{hxy}$, which can be obtained by the same argument as (3.7).

Let I_h be piecewise linear interpolation operator on Ω, $I_h|_K = I_K, I_K$ is the linear interpolation operator on K. Apparently, $I_h v_h \in H_0^1(\Omega)$, then

$$\begin{aligned}(f, I_h v_h) &= (\triangle^2 u, I_h v_h)\\ &= -\sum_{K\in\mathcal{J}_h}\int_K \nabla\triangle u\cdot\nabla I_h v_h dxdy\\ &\quad -\sum_{K\in\mathcal{J}_h}\int_K [(\triangle u)_x (I_h v_h)_x + (\triangle u)_y (I_h v_h)_y] dxdy.\end{aligned}\tag{3.11}$$

By (3.8), (3.9), (3.10) and (3.11), we have

$$\begin{aligned}a_h(u,v_h) - (f,v_h) &= (f, I_h v_h - v_h) + \sum_{K\in\mathcal{J}_h}\int_K (\triangle u)_x (I_K v_h - v_h)_x dxdy\\ &+ \sum_{K\in\mathcal{J}_h}\int_K (\triangle u)_y ((I_K v_h)_y - Tv_{hy}) dxdy\\ &+ \sum_{K\in\mathcal{J}_h}\sum_{i=1}^{3}\int_{F_i} \triangle u (v_{hx} n_x + Tv_{hy} n_y) ds + (1-\sigma)\{\\ &\sum_{K\in\mathcal{J}_h}\int_K u_{xxy}(Tv_{hy} - v_{hy}) dxdy + \sum_{K\in\mathcal{J}_h}\int_K u_{xyy}(v_{hx} - Tv_{hx}) dxdy\\ &+ \sum_{K\in\mathcal{J}_h}\sum_{i=1}^{3}\int_{F_i} u_{xy} v_{hy} n_x ds - \sum_{K\in\mathcal{J}_h}\sum_{i=1}^{3}\int_{F_i} u_{xx} Tv_{hy} n_y ds\\ &+ \sum_{K\in\mathcal{J}_h}\sum_{i=1}^{3}\int_{F_i} u_{xy} Tv_{hx} n_y ds - \sum_{K\in\mathcal{J}_h}\sum_{i=1}^{3}\int_{F_i} u_{yy} v_{hx} n_x ds\}\\ &= \sum_{i=1}^{4} I_i + (1-\sigma)\sum_{i=5}^{10} I_i.\end{aligned}\tag{3.12}$$

Now we will estimate the above terms one by one.
From classical interpolation theory [9,12], we have

$$\begin{aligned}I_1 = (f, I_h v_h - v_h) &\le \sum_{K\in\mathcal{J}_h}\left|\int_K f(I_K v_h - v_h)\right|\\ &\le \sum_{K\in\mathcal{J}_h}\|f\|_{0,K}\|I_K v_h - v_h\|_{0,K}\\ &\le \sum_{K\in\mathcal{J}_h} Ch_K^2\|f\|_{0,K}|v_h|_{2,K}\\ &\le Ch^2\|f\|_{0,\Omega}\|v_h\|_h.\end{aligned}\tag{3.13}$$

By [3, 4], the interpolation I_h is an anisotropic interpolation, and have the following estimate

$$|I_K v - v|_{1,K} \le Ch_K|v|_{2,K}, \quad \forall v\in H^2(K),\tag{3.14}$$

then

$$\begin{aligned} I_2 &= \sum_{K\in\mathcal{J}_h}\int_K (\triangle u)_x (I_K v_h - v_h)_x dxdy \\ &\le \sum_{K\in\mathcal{J}_h} |u|_{3,K} |I_K v_h - v_h|_{1,K} \\ &\le Ch|u|_{3,\Omega}\|v_h\|_h. \end{aligned} \tag{3.15}$$

I_3 can be decomposed as

$$\begin{aligned} I_3 &= \sum_{K\in\mathcal{J}_h}\int_K (\triangle u)_y (I_K v_h - v_h)_y dxdy \\ &+ \sum_{K\in\mathcal{J}_h}\int_K (\triangle u)_y (v_{hy} - Tv_{hy}) dxdy \\ &= I_{31} + I_{32}. \end{aligned} \tag{3.16}$$

Similar to I_2, I_{31} can be estimated as

$$I_{31} \le Ch|u|_{3,\Omega}\|v_h\|_h. \tag{3.17}$$

Since the operator T is exact for constant, by the interpolation theory we have

$$\begin{aligned} I_{32} &\le \sum_{K\in\mathcal{J}_h} |u|_{3,K}\|v_{hy} - Tv_{hy}\|_{0,K} \\ &\le \sum_{K\in\mathcal{J}_h} Ch_K|u|_{3,K}|v_{hy}|_{1,K} \\ &\le Ch|u|_{3,\Omega}\|v_h\|_h. \end{aligned} \tag{3.18}$$

By the same argument, we can obtain

$$I_5 \le Ch|u|_{3,\Omega}\|v_h\|_h, \quad I_6 \le Ch|u|_{3,\Omega}\|v_h\|_h. \tag{3.19}$$

I_4 can be decomposed as

$$\begin{aligned} I_4 &= \sum_{K\in\mathcal{J}_h}\sum_{i=1}^{3}\int_{F_i} \triangle u v_{hx} n_x ds + \sum_{K\in\mathcal{J}_h}\sum_{i=1}^{3}\int_{F_i} \triangle u T v_{hy} n_y ds \\ &= I_{41} + I_{42}. \end{aligned} \tag{3.20}$$

Employing the properties of the Morley's finite element space, we get

$$I_{41} = \sum_{K\in\mathcal{J}_h}\sum_{i=1}^{3}\int_{F_i} (\triangle u - P_{0,F_i}\triangle u)(v_{hx} - P_{0,F_i} v_{hx}) n_x ds, \tag{3.21}$$

and

$$I_{42} = \sum_{K\in\mathcal{J}_h}\sum_{i=1}^{3}\int_{F_i} (\triangle u - P_{0,F_i}\triangle u)(Tv_{hy} - P_{0,F_i}(Tv_{hy})) n_y ds. \tag{3.22}$$

Thanks to the fact that $n_x|_{F_1} = 0, n_x|_{F_2} = 1, n_x|_{F_3} = -\frac{h_{K2}}{\sqrt{h_{K1}^2 + h_{K2}^2}}$ (refer to Figure 1), then by (3.4),

$$\begin{aligned} I_{41} &\le \sum_{K\in\mathcal{J}_h}\sum_{j=1}^{3} C\frac{|F_j|n_x}{|K|} \times \left(\sum_{i=1,2} h_{Ki}^2\|\partial_i \triangle u\|_{0,K}^2\right)^{\frac{1}{2}} \left(\sum_{i=1,2} h_{Ki}^2\|\partial_i v_{hx}\|_{0,K}^2\right)^{\frac{1}{2}} \\ &\le \sum_{K\in\mathcal{J}_h}\sum_{i=1}^{3} Ch_K|u|_{3,K}|v_h|_{2,K} \\ &\le Ch|u|_{3,\Omega}\|v_h\|_h. \end{aligned} \tag{3.23}$$

Noticed that $n_y|_{F_2} = 0$ and $Tv_{hy} \in span\{1, y\}$, by (3.4) we have

$$\begin{aligned} I_{42} &= \sum_{K\in\mathcal{J}_h}\sum_{j=1,3}\int_{F_j}\left(\triangle u - P_{0,F_j}\triangle u)(Tv_{hy} - P_{0,F_j}(Tv_{hy})\right)n_y ds \\ &\leq \sum_{K\in\mathcal{J}_h}\sum_{j=1,3}\frac{|F_j|n_y}{|K|}\times\left(\sum_{i=1,2}{h_{Ki}}^2\|\partial_i\triangle u\|_{0,K}^2\right)^{\frac{1}{2}}\left(\sum_{i=1,2}{h_{Ki}}^2\|\partial_i(Tv_{hy})\|_{0,K}^2\right)^{\frac{1}{2}} \\ &= \sum_{K\in\mathcal{J}_h}\sum_{j=1,3}\frac{|F_j|n_y}{|K|}\times\left(\sum_{i=1,2}h_{Ki}^2\|\partial_i\triangle u\|_{0,K}^2\right)^{\frac{1}{2}} h_{K2}\|(Tv_{hy})_y\|_{0,K} \\ &= \sum_{K\in\mathcal{J}_h}\sum_{j=1,3}\frac{|F_j|n_y}{h_{K1}}\times\left(\sum_{i=1,2}h_{Ki}^2\|\partial_i\triangle u\|_{0,K}^2\right)^{\frac{1}{2}}\|v_{hyy}\|_{0,K} \\ &\leq \sum_{K\in\mathcal{J}_h}\sum_{j=1,3}Ch_K|u|_{3,K}|v_h|_{2,K} \\ &\leq Ch|u|_{3,\Omega}\|v_h\|_h. \end{aligned} \tag{3.24}$$

Following the lines of I_{41}, there holds

$$I_7 \leq Ch|u|_{3,\Omega}\|v_h\|_h, \quad I_{10} \leq Ch|u|_{3,\Omega}\|v_h\|_h. \tag{3.25}$$

By the same argument of I_{42}, we can show that

$$I_8 \leq Ch|u|_{3,\Omega}\|v_h\|_h, \quad I_9 \leq Ch|u|_{3,\Omega}\|v_h\|_h. \tag{3.26}$$

Thus we have obtain that

$$a_h(u, v_h) - (f, v_h) \leq Ch\left(|u|_{3,\Omega} + h\|f\|_{0,\Omega}\right)\|v_h\|_h, \tag{3.27}$$

which implies the desired result of (3.6) directly.

A combination of Theorem 2.1 and Theorem 3.1 gives the following optimal error estimate.

Theorem 3.2. *Under the hypothesis of Theorem 3.1, we have*

$$\|u - u_h\|_h \leq Ch(|u|_{3,\Omega} + h\|f\|_{0,\Omega}). \tag{3.28}$$

4. Numerical Experiment

In order to examine the numerical performance of Morley element for narrow triangular meshes, we consider the unit square plate bending problem[23] with clamped supported boundaries under a uniform load. Let the Poisson ratio $\sigma = 0.3, f = 1$. The analytic values of deflection and bending moment at the center are 0.00126532 and 0.0229051 respectively.

The unit square $\Omega = [0, 1] \times [0, 1]$ is subdivided in the following two fashions:

mesh 1: Each edge of Ω is divided into n segments with $n + 1$ points $(1 - cos(\frac{i\pi}{n}))/2, i = 0, 1, ..., \frac{n}{2}, (1 + sin(\frac{i\pi}{n} - \frac{\pi}{2}))/2, i = \frac{n}{2} + 1, ..., n$. The mesh obtained in this way for $n = 16$ is illustrated at left Figure 2, and the anisotropic triangular mesh is obtained by dividing each rectangular into two triangles.

mesh 2: Each edge of Ω is divided into n segments with $n + 1$ points $\sin(\frac{i\pi}{n})/2, i = 0, 1, \cdots, n/2, (1 - \cos(\frac{i\pi}{n} - \frac{\pi}{2}))/2, i = n/2 + 1, \cdots, n$. The mesh obtained in this way for $n = 16$ is shown at right Figure 2. Then the anisotropic triangular mesh is obtained by dividing each rectangular into two triangles.

The error of the deflection $|(u - u_h)(O)|$ and the error of bending moment $|(M - M_h)(O)|$ at the center of the unit square are shown in Table 4.1 and Table 4.2, from which the optimal convergence of the element for unregular subdivisions can be seen.

Furthermore, in order to present the advantages of the anisotropic meshes over the regular meshes, we carry our another experiment by solving a biharnomic differential equation with $\Omega = [0,1] \times [0,1]$, $\sigma = 0.3$, and the right hand side $f(x,y)$ is taken such that $u(x,y) = (1-e^{-x(1-x)/\varepsilon})^2(1-e^{-y(1-y)/\varepsilon})^2$ (refer to the left of Figure 2) is the exact solution, which varies significantly near the boundary of Ω for small ε. A comparison of the errors $\|u-u_h\|_h/\|u\|_h$ between square triangular mesh and mesh 1 (please refer to Figure 3), which shows that the anisotropic meshes are more attractive than the regular meshes for some special cases.

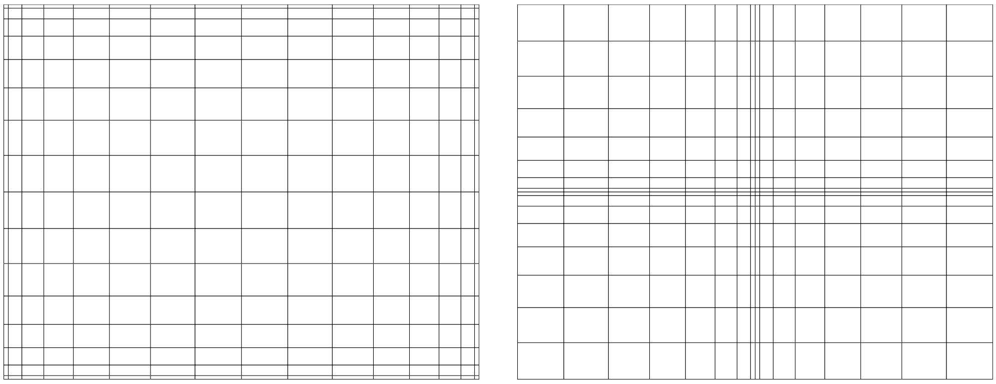

Figure 2. The initial rectangular meshes of Ω for case $n = 16$, mesh 1 (left) and mesh 2 (right)

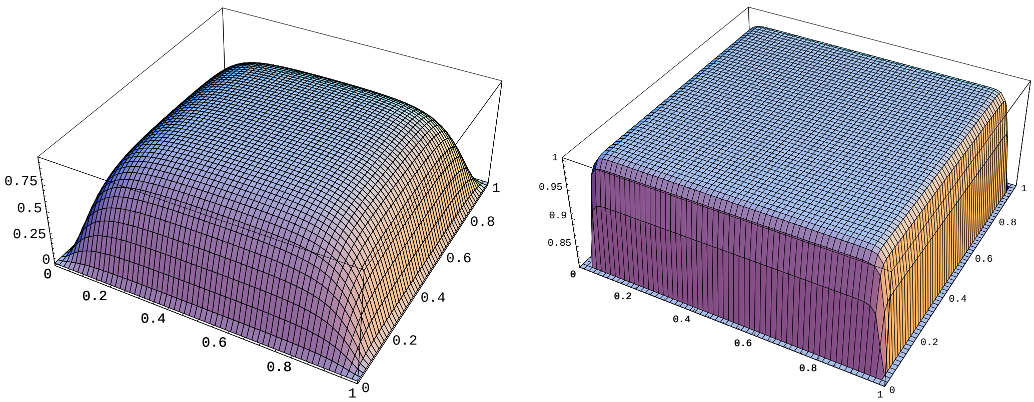

Figure 3. the solution u for case $\varepsilon = 0.05$ (left) and for case $\varepsilon = 0.01$ (right)

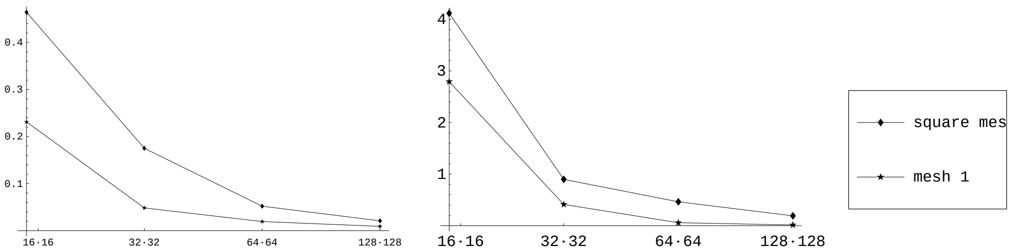

Figure 4. the error $\|u-u_h\|_h/\|u\|_h$ for case $\varepsilon = 0.05$ (left) and for case $\varepsilon = 0.01$ (right)

Table 4.1. The errors $|(u-u_h)(O)|$ and $|(M-M_h)(O)|$ (*mesh 1*)

$n \times n$	8×8	16×16	32×32	64×64	128×128
$\|(u-u_h)(O)\|$	0.00192686	0.00147450	0.00131584	0.00127800	0.00126849
$\|(M-M_h)(O)\|$	0.02123944	0.02253187	0.02281558	0.02288296	0.02289956
$\max\limits_{K\in J_h} h_K$	0.270598	0.137950	0.069309	0.034696	0.017353
$\max\limits_{K\in J_h} \{h_K/\rho_K\}$	7.109732	14.358751	28.786978	57.608674	115.234703

Table 4.2. The errors $|(u-u_h)(O)|$ and $|(M-M_h)(O)|$ (*mesh 2*)

$n \times n$	8×8	16×16	32×32	64×64	128×128
$\|(u-u_h)(O)\|$	0.00185503	0.00141956	0.00130444	0.00127514	0.00126778
$\|(M-M_h)(O)\|$	0.02164107	0.02262186	0.02283717	0.022888306	0.02290090
$\max\limits_{K\in J_h} h_K$	0.270598	0.137950	0.069309	0.034696	0.017353
$\max\limits_{K\in J_h} \{h_K/\rho_K\}$	7.109732	14.358751	28.786978	57.608674	115.234703

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