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A STABILITY THEOREM FOR CONSTRAINED OPTIMAL CONTROL PROBLEMS *

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Abstract

This paper presents the stability of difference approximations of an optimal control problem for a quasilinear parabolic equation with controls in the coefficients, boundary conditions and additional restrictions. The optimal control problem has been convered to one of the optimization problem using a penalty function technique. The difference approximations problem for the considered problem is obtained. The estimations of stability of the solution of difference approximations problem are proved. The stability estimation of the solution of difference approximations problem by the controls is obtained.

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1. Introduction

Optimal control problems for partial differential equations are currently of much interest. A large amount of the theoretical concept which governed by quasilinear parabolic equations [1-4] has been investigated in the field of optimal control problems. These problems have dealt with the processes of hydro- and gasdyn amics, heatphysics, filtration, the physics of plasma and others [5-6]. Difference methods of solution of optimal control problems for partial differential equations are investigated comparatively small [7-11]. In this paper, the stability of difference approximations of an optimal control problem for a quasilinear parabolic equation with controls in the coefficients, boundary conditions and additional restrictions. The optimal control problem has been convered to one of the optimization problem using a penalty function technique. The difference approximations problem for the considered problem is obtained. The estimations of stability of the solution of difference approximations problem are proved. The stability estimation of the solution of difference approximations problem by the controls is obtained.

Let $\Omega = \{(x,t) : x \in D = (0,l), t \in (0,T)\}$ where l, T are given positive numbers. Now, we need to introduce some functional spaces as follows:
1) $L_2(D)$ is a Banach space which consisting of all measurable functions on D with the norm $\|z\|_{L_2(D)} = [\int_D \ |z|^2 \ dx \ ]^{\frac{1}{2}}$. 2) $L_2(0,l)$ is a Hilbert space which consisting of all measurable functions on $(0,l)$ with

$$\langle z_1, z_2\rangle_{L_2(0,l)} = \int_0^l z_1(x)z_2(x)dx, \|z\|_{L_2(0,l)} = \sqrt{\langle z, z\rangle_{L_2(0,l)}}.$$

3) $L_2(\Omega)$ is a Hilbert space which consisting of all measurable functions on Ω with

$$\langle z_1, z_2\rangle_{L_2(\Omega)} = \int_0^l \int_0^T z_1(x,t)z_2(x,t)dxdt, \|z\|_{L_2(\Omega)} = \sqrt{\langle z, z\rangle_{L_2(\Omega)}}.$$

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4) $W_2^{1,0}(\Omega)=\{z\in L_2(\Omega)\ \ and\ \ \frac{\partial z}{\partial x}\in L_2(\Omega)\}$ is a Hilbert space with

$$\|z\|_{W_2^{1,0}(\Omega)}=\int_\Omega\ [\ \ z_1z_2+\frac{\partial z_1}{\partial x}\frac{\partial z_2}{\partial x}]\ \ dx\ \ dt$$

$$\langle z_1,z_2\rangle_{W_2^{1,0}(\Omega)}=[\|z\|^2_{L_2(\Omega)}+\|\frac{\partial z}{\partial x}\|^2_{L_2(\Omega)}]^{\frac{1}{2}}.$$

5) $W_2^{1,1}(\Omega)=\{z\in L_2(\Omega)\ \ and\ \ \frac{\partial z}{\partial x}\in L_2(\Omega),\frac{\partial z}{\partial t}\in L_2(\Omega)\}$ is a Hilbert space with

$$\|z\|_{W_2^{1,1}(\Omega)}=\int_\Omega\ [\ \ z_1z_2+\frac{\partial z_1}{\partial x}\frac{\partial z_2}{\partial x}+\frac{\partial z_1}{\partial t}\frac{\partial z_2}{\partial t}]\ \ dx\ \ dt$$

$$\langle z_1,z_2\rangle_{W_2^{1,1}(\Omega)}=[\|z_1\|^2_{L_2(\Omega)}+\|\frac{\partial z}{\partial x}\|^2_{L_2(\Omega)}+\|\frac{\partial z}{\partial t}\|^2_{L_2(\Omega)}]^{\frac{1}{2}}.$$

6) $V_2(\Omega)$ is a Banach space consisting of elements the space $W_2^{1,0}(\Omega)$ with the norm

$$\|z\|_{V_2(\Omega)}=vraimax_{0\le t\le T}\|z(x,t)\|_{L_2(D)}+(\int_\Omega|\frac{\partial z}{\partial x}|^2)^{\frac{1}{2}}.$$

7) $V_2^{1,0}(\Omega)$ is a subspace of $V_2(\Omega)$, the elements of which have in sections $D_t=\{(x,\tau):x\in D,\tau=t\}$ traces from $L_2(D)$ at all $t\in[0,T]$,continuously changing from $t\in[0,T]$ in the norm $L_2(D)$.

2. Problem Formulation

Let $V=\{v:v=(v_1,v_2,...,v_N)\in E_N,\|v\|_{E_N}\le R\}$, where $R>0$ are given numbers. We consider the heat exchange process described by the equation

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}(\lambda(u,v)\frac{\partial u}{\partial x})+B(u,v)\frac{\partial u}{\partial x}=f(x,t),(x,t)\in\Omega \tag{1}$$

with initial and boundary conditions

$$u(x,0)=\phi(x),x\in D \tag{2}$$

$$\lambda(u,v)\frac{\partial u}{\partial x}|_{x=0}=Y_0(t),\lambda(u,v)\frac{\partial u}{\partial x}|_{x=l}=Y_1(t),0\le t\le T \tag{3}$$

where $\phi(x)\in L_2(D)$, $Y_0(t),Y_1(t)\in L_2(0,T)$ and $f(x,t)$ is given function. Besides, the functions $\lambda(u,v),B(u,v)$ are continuous on $(u,v)\in[r_1,r_2]$ x E_N, have continuous derivatives in u and $\forall(u,v)\in[r_1,r_2]$ x E_N, the derivative $\frac{\partial\lambda(u,v)}{\partial u}$, $\frac{\partial B(u,v)}{\partial u}$ are bounded. Here r_1,r_2 are given numbers.

On the set V, under the conditions (1)-(3) and additional restrictions

$$\nu_0\le\lambda(u,v)\le\mu_0,\qquad\nu_0\le B(u,v)\le\mu_0,\qquad r_1\le u(x,t)\le r_2 \tag{4}$$

is required to minimize the function

$$J_\alpha(v)=\beta_0\int_0^T[u(0,t)-f_0(t)]^2dt+\beta_1\int_0^T[u(l,t)-f_1(t)]^2dt+\alpha\|v-\omega\|^2_{E_N} \tag{5}$$

where $f_0(t)$, $f_1(t)\in L_2(0,T)$ are given functions, $\alpha\ge0$, $\nu_0,\mu_0>0$, $\beta_0\ge0$, $\beta_1\ge0$, $\beta_0+\beta_1\ne0$ are given numbers, $\omega\in E_N$ is also given: $\omega=(\omega_1,\omega_2,...,\omega_N)$.

Definition 1. *The problem of finding a function $u=u(x,t)\in V_2^{1,0}(\Omega)$ from conditions (1)-(4) at given $v\in V$ is called the reduced problem.*

Definition 2. *The solution of the reduced problem (1)-(4) corresponding to the $v\in V$ is a function $u(x,t)\in V_2^{1,0}(\Omega)$ and satisfies the integral identity*

$$\int_0^l\int_0^T[u\tfrac{\partial\eta}{\partial t}-\lambda(u,v)\tfrac{\partial u}{\partial x}\tfrac{\partial\eta}{\partial x}-B(u,v)\tfrac{\partial u}{\partial x}\eta+\eta f(x,t)]dxdt=$$
$$-\int_0^l\phi(x)\eta(x,0)dx-\int_0^T\eta(0,t)Y_0(t)dt+\int_0^T\eta(l,t)Y_1(t)dt, \tag{6}$$

$\forall\quad\eta=\eta(x,t)\in W_2^{1,1}(\Omega)\ \ and\ \ \eta(x,T)=0.$

The solution of the reduced problem (1)-(3) explicitly dependes on the control v, therefore we shall also use the notation $u = u(x,t;v)$.

On the basis of adopted assumptions and the results of [12] follows that for every $v \in V$ the solution of the problem (1)-(4) is existed, unique and $|u_x| \le C_0$,$\forall (x,t) \in \Omega$, $\forall v \in V$, where C_0 is a certain constant.

Optimal control problems of the coefficients of differential equations do not always have solution [13]. In [14], we proved the existence and uniquness of the solution of problem (1)-(5) as follows:

Lemma 1. *At above adopted assumptions for the solution of the reduced problem (1)-(5) the following estimation is valid*

$$\|\delta u\|_{V_2^{1,0}(\Omega)} \le C_1[\|\delta\lambda \frac{\partial u}{\partial x}\|^2_{L_2(\Omega)} + \|\delta B \frac{\partial u}{\partial x}\|^2_{L_2(\Omega)}]^{\frac{1}{2}} \tag{7}$$

where $C_1 \ge 0$ is constant not depending on δv.

Lemma 2. *The function $J_0(v)$ is continuous on V.*

Theorem 1. *The problem (1)-(5) at any $\alpha \ge 0$ has at least one solution.*

Theorem 2. *The problem (1)-(5) at $\alpha > 0$, at almost all $\omega \in E_N$ has a unique solution.*

The inequality constrained problem (1) through (5) is converted to a problem without inequality constraints by adding a penalty function [15] to the objective (5) , yielding the following $\Phi_{\alpha,s}(v, A_s)$ function:

$$\Phi_{\alpha,s}(v, A_s) \equiv \Phi(v) = J_\alpha(v) + P_s(v), \tag{8}$$

where

$$DUV(u,v) = [\max\{\nu_0 - \lambda(u,v); 0\}]^2 + [\max\{\lambda(u,v) - \mu_0; 0\}]^2$$

$$BUV(u,v) = [\max\{\nu_0 - B(u,v); 0\}]^2 + [\max\{B(u,v) - \mu_0; 0\}]^2$$

$$Q^1(u) = [\max\{r_1 - u(x,t;v); 0\}]^2, Q^2(u) = [\max\{u(x,t;v) - r_2; 0\}]^2$$

$$P_s(v) = A_s \int_0^l \int_0^T [DUV(u,v) + BUV(u,v) + Q^1(u) + Q^2(u)]dxdt$$

and A_s , s=1,2,... are positive numbers, $\lim_{s\to\infty} A_s = +\infty$.

3. Approximation of the Control Problem

In this section, we shall find the difference approximations problem for (8), (1)-(3). From [16], We give the net norms for the net functions $[Z] = \{(Z_1)_i^j, \cdots, (Z_m)_i^j\}$, $i = \overline{0,N}, j = \overline{0,M}$ with m components:

$$\|Z\|_{L_2(\overline{\omega}_{h\tau})} = [h\tau \sum_{i=0}^{N-1}\sum_{j=1}^{M} (Z_i^j)^2]^{\frac{1}{2}}, \max_j \|Z\|_{L_2(\overline{\omega}_x)} = \max_j [h \sum_{i=0}^{N-1} (Z_i^j)^2]^{\frac{1}{2}},$$

for the net functions $[Z] = \{(Z_1)_i, \cdots, (Z_m)_i\}, i = \overline{0,N}$ and also for the net functions $[Z] = \{(Z_1)_j, \cdots, (Z_m)_j\}, j = \overline{0,M}$ the norms are

$$\|Z\|_{L_2(\overline{\omega}_h)} = [h \sum_{i=0}^{N-1} (Z_i)^2]^{\frac{1}{2}}, \|Z\|_{L_2(\overline{\omega}_\tau)} = [\tau \sum_{j=1}^{M} (Z_j)^2]^{\frac{1}{2}} \qquad \textit{correspondingly}.$$

For discretization the optimal control problem (1)-(3),(8) in $\overline{\Omega}$ we introduce the net $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_\tau$ where

$$\overline{\omega}_h = \{x_i = ih, i = \overline{0,N}, Nh = l\}, \qquad \overline{\omega}_\tau = \{t_j = j\tau, j = \overline{0,M}, M\tau = T\}.$$

Here and further for arbitrary net functions $u = u_i^j = u(x,t) = u(x_i,t_j) = u_i^j, x = x_i \in \overline{\omega}_h, t = t_j \in \overline{\omega}_\tau$ adopt denotations [17]:

$$\hat{u} = u(x_i, t_{j+1}) = u_i^{j+1}, \qquad u^* = u(x_i, t_{j-1}) = u_i^{j-1}$$

$$u^- = u(x_{i-1},t_j) = u^j_{i-1}, \qquad u^+ = u(x_{i+1},t_j) = u^j_{i+1}$$
$$u_x = \frac{u^+ - u}{h}, u_{\overline{x}} = \frac{u - u^-}{h}, u_t = \frac{\hat{u} - u}{\tau}, u_{\overline{t}} = \frac{u - u^*}{\tau}.$$

The given functions in (6) approximate as follows:

$$\lambda^j_i = \frac{1}{h\tau}\int_{t_{j-1}}^{t_j}\int_{x_i}^{x_{i+1}} \lambda(u(x,t),v))dxdt, f^j_i = \frac{1}{h\tau}\int_{t_{j-1}}^{t_j}\int_{x_i}^{x_{i+1}} f(x,t)dxdt,$$
$$B^j_i = \frac{1}{h\tau}\int_{t_{j-1}}^{t_j}\int_{x_i}^{x_{i+1}} B(u(x,t),v))dxdt, \int_{x_i}^{x_{i+1}} f(x,t)dxdt, \qquad i = \overline{0,N-1}, j = \overline{1,M},$$
$$\phi_i = \frac{1}{h}\int_{x_i}^{x_{i+1}} \phi(x)dx, \qquad i = \overline{0,N-1},$$
$$(Y_0)^j = \frac{1}{\tau}\int_{t_j-\frac{\tau}{2}}^{t_j+\frac{\tau}{2}} Y_0(t)dt, \quad (Y_1)^j = \frac{1}{\tau}\int_{t_j-\frac{\tau}{2}}^{t_j+\frac{\tau}{2}} Y_1(t)dt, \qquad j = \overline{1,M-1}.$$

The discrete analogy of the integral identity (6) writes in the form

$$h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M-1} u^j_i(\eta^j_i)_t - \sum_{i=0}^{N-1}\sum_{j=1}^{M}[-\lambda^j_i(u^j_i)_x(\eta^j_i)_x - B^j_i(u^j_i)_x\eta^j_i + \eta^j_i f^j_i] =$$
$$= -h\sum_{i=0}^{N-1}\phi_i\eta^0_i - \tau\sum_{j=1}^{M}\eta^j_0(Y_0)^j - \tau\sum_{j=1}^{M}\eta^j_N(Y_1)^j, \tag{9}$$

for any network function $\eta^j_i, \eta^M_i = 0$.

From [18], we have

$$h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M-1} u^j_i(\eta^j_i)_t = -h\tau\sum_{i=1}^{N-1}\sum_{j=1}^{M}(u^j_i)_{\overline{t}}\eta^j_i + h\sum_{i=0}^{N-1} u^M_i\eta^M_i$$
$$-h\sum_{i=0}^{N-1} u^0_i\eta^0_i + h\tau\sum_{j=1}^{M}(u^j_0)_{\overline{t}}\eta^j_0 \tag{10}$$

$$-h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M}\lambda^j_i(u^j_i)_x(\eta^j_i)_x = h\tau\sum_{i=1}^{N-1}\sum_{j=1}^{M}(\lambda^j_i(u^j_i)_x)_{\overline{x}}\eta^j_i$$
$$-\tau\sum_{j=1}^{M}\lambda^j_{N-1}(u^j_{N-1})_x\eta^j_N + \tau\sum_{j=1}^{M}\lambda^j_0(u^j_0)_x\eta^j_0. \tag{11}$$

Using (10), (11), from (9) we obtain

$$h\tau\sum_{i=1}^{N-1}\sum_{j=1}^{M}[-(u^j_i)_{\overline{t}} + (\lambda^j_i(u^j_i)_x)_{\overline{x}} - B^j_i(u^j_i)_x + f^j_i]\eta^j_i = h\sum_{i=0}^{N-1} u^0_i\eta^0_i$$
$$-h\tau\sum_{j=1}^{M}(u^j_0)_{\overline{t}}\eta^j_0 + \tau\sum_{j=1}^{M}[\lambda^j_0(u^j_0)_x\eta^j_0 + \lambda^j_{N-1}(u^j_{N-1})_x\eta^j_N] - h\sum_{i=0}^{N-1}\phi_i\eta^0_i$$
$$+\tau\sum_{j=1}^{M}[\eta^j_0(Y_0)^j - \eta^j_N(Y_1)^j] + h\tau\sum_{j=1}^{M}[B^j_0(u^j_0)_x\eta^j_0 - \eta^j_0 f^j_0]. \tag{12}$$

Hence, equality to zero the coefficients of η^j_i, we obtain the difference approximations problem for (1)-(3):

$$(u^j_i)_{\overline{t}} - (\lambda^j_i(u^j_i)_x)_{\overline{x}} + B^j_i(u^j_i)_x - f^j_i = 0, i = \overline{1,N-1}, j = \overline{1,M}, \tag{13}$$
$$u^0_i = \phi_i, i = \overline{0,N-1} \tag{14}$$
$$\lambda^j_0(u^j_0)_x - (Y_0)^j - h(u^j_0)_{\overline{t}} - hf^j_0 + hB^j_0(u^j_0)_x = 0, j = \overline{1,M} \tag{15}$$
$$\lambda^j_{N-1}(u^j_{N-1})_x - (Y_1)^j = 0, j = \overline{1,M} \tag{16}$$

But the functional (8) is approximated by the following way [19]:

$$I(v) = \beta_0\sum_{j=1}^{M}[u^j_0 - (f_0)^j]^2 + \beta_1\sum_{j=1}^{M}[u^j_N - (f_1)^j]^2 + \alpha\sum_{k=1}^{L}[v_k - \omega_k]^2$$
$$+A_s\sum_{i=0}^{N-1}\sum_{j=1}^{M}[DUV(u^j_i,v) + BUV(u^j_i,v) + Q^1(u^j_i) + Q^2(u^j_i)] \tag{17}$$

4. The Estimates of Stability for (13)-(16)

In this section, we shall prove the estimates of stability for the difference approximations problem (13)-(16).

Theorem 3. *Suppose that the all functions in the system (1)-(3) satisfy the above enumerated conditions. Besides,* $\lambda(u,v), B(u,v)$ *satisfy the Lipschits condition on* v *with constant* L , $\forall (x,t)\in\Omega$ *and for any* $v\in V$. *Then the estimates of stability for the difference approximation problem (13)-(16) are*

$$\|u\|^2_{L_2(\overline{\Omega}_{h\tau})}\le C_2[\|\phi\|^2_{L_2(\overline{\Omega}_h)}+\|f\|^2_{L_2(\overline{\Omega}_{h\tau})}+\|Y_0\|^2_{L_2(\overline{\Omega}_\tau)}+\|Y_1\|^2_{L_2(\overline{\Omega}_\tau)}] \tag{18}$$

$$\|u_x\|^2_{L_2(\overline{\Omega}_{h\tau})}\le C_2[\|\phi\|^2_{L_2(\overline{\Omega}_h)}+\|f\|^2_{L_2(\overline{\Omega}_{h\tau})}+\|Y_0\|^2_{L_2(\overline{\Omega}_\tau)}+\|Y_1\|^2_{L_2(\overline{\Omega}_\tau)}] \tag{19}$$

$$\max_j\|u^j\|^2_{L_2(\overline{\Omega}_x)}\le C_2[\|\phi\|^2_{L_2(\overline{\Omega}_h)}+\|f\|^2_{L_2(\overline{\Omega}_{h\tau})}+\|Y_0\|^2_{L_2(\overline{\Omega}_\tau)}+\|Y_1\|^2_{L_2(\overline{\Omega}_\tau)}] \tag{20}$$

where C_2 *is a constant depending only on the constants in (1)-(3) and* L.

Proof. Using the arbitrariness of choice η_i^j at every layer $j\ge 1$ and from (9), we obtain

$$-h\sum_{i=0}^{N-1}[(u_i^j)_{\overline{t}}\eta_i^j+\lambda_i^j(u_i^j)_x(\eta_i^j)_x+B_i^j(u_i^j)_x\eta_i^j-\eta_i^jf_i^j]=\eta_0^j(Y_0)^j-\eta_N^j(Y_1)^j. \tag{21}$$

Put $\eta_i^j=2\tau u_i^j$ and $\lambda_i^j\ge\nu_0>0$, $B_i^j\ge\nu_1>0$, we have

$$h\sum_{i=0}^{N-1}(u_i^j)^2+2h\tau\sum_{i=0}^{N-1}\lambda_i^j(u_i^j)_x^2+2h\tau\sum_{i=0}^{N-1}B_i^j(u_i^j)_xu_i^j\le$$
$$2h\tau\sum_{i=0}^{N-1}u_i^jf_i^j+2[u_0^j(Y_0)^j-u_N^j(Y_1)^j]. \tag{22}$$

Applying $\varepsilon-$ inequality, we obtain

$$h\sum_{i=0}^{N-1}(u_i^j)^2+2h\tau\sum_{i=0}^{N-1}\lambda_i^j(u_i^j)_x^2+2h\tau\varepsilon_0\sum_{i=0}^{N-1}(B_i^j)^2(u_i^j)_x^2+$$
$$\tfrac{2h\tau}{\varepsilon_0}\sum_{i=0}^{N-1}(u_i^j)^2\le h\tau\varepsilon_1\sum_{i=0}^{N-1}u_i^j+\tfrac{h\tau}{\varepsilon_1}\sum_{i=0}^{N-1}(f_i^j)^2+$$
$$+\varepsilon_2\tau[(u_0^j)^2+(u_N^j)^2]+\tfrac{\tau}{\varepsilon_2}[((Y_0)^j)^2+((Y_1)^j)^2]. \tag{23}$$

Using the estimate from [20,p. 290], we have

$$h\sum_{i=0}^{N-1}(u_i^j)^2+2h\tau\sum_{i=0}^{N-1}\lambda_i^j(u_i^j)_x^2+2h\tau\varepsilon_0\sum_{i=0}^{N-1}(B_i^j)^2(u_i^j)_x^2+$$
$$\tfrac{2h\tau}{\varepsilon_0}\sum_{i=0}^{N-1}(u_i^j)^2\le h\tau\varepsilon_1\sum_{i=0}^{N-1}u_i^j+\tfrac{h\tau}{\varepsilon_1}\sum_{i=0}^{N-1}(f_i^j)^2+\tfrac{\tau}{\varepsilon_2}[((Y_0)^j)^2-((Y_1)^j)^2]+$$
$$+2\varepsilon_2(\varepsilon_3+h)h\tau\sum_{i=0}^{N-1}(u_i^j)_x^2+2\varepsilon_2(\tfrac{1}{\varepsilon_3}+\tfrac{1}{L})h\tau\sum_{i=0}^{N-1}(u_i^j)^2. \tag{24}$$

Summing the inequality (24) on j from 1 to $M_1\le M$, we have

$$h\sum_{i=0}^{N-1}(u_i^{M_1})^2+2h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M_1}\lambda_i^j(u_i^j)_x^2+2h\tau\varepsilon_0\sum_{i=0}^{N-1}\sum_{j=1}^{M_1}(B_i^j)^2(u_i^j)_x^2+$$
$$\tfrac{2h\tau}{\varepsilon_0}\sum_{i=0}^{N-1}\sum_{j=1}^{M_1}(u_i^j)^2\le h\sum_{i=0}^{N-1}(u_i^0)^2+h\tau\varepsilon_1\sum_{i=0}^{N-1}\sum_{j=1}^{M_1}u_i^j$$
$$+\tfrac{h\tau}{\varepsilon_1}\sum_{i=0}^{N-1}\sum_{j=1}^{M_1}(f_i^j)^2+\tfrac{\tau}{\varepsilon_2}\sum_{j=1}^{M_1}[((Y_0)^j)^2-((Y_1)^j)^2]$$
$$+2\varepsilon_2(\varepsilon_3+h)h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M_1}(u_i^j)_x^2+2\varepsilon_2(\tfrac{1}{\varepsilon_3}+\tfrac{1}{L})h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M_1}(u_i^j)^2. \tag{25}$$

Taking into $h\sum_{i=0}^{N-1}(u_i^0)^2=h\sum_{i=0}^{N-1}\phi^2$ and summing the inequality (25) on M_1 from 1 to $M_2\le M$ and multiply by τ, we have

$$h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M_2}(u_i^j)^2\le -T_2[\nu_0+\nu_1^2\varepsilon_0-2\varepsilon_2(\varepsilon_3+h)]h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M_2}(u_i^j)_x^2$$
$$+2T_2h\sum_{i=0}^{N-1}\phi^2+T_2(\varepsilon_1-\tfrac{1}{\varepsilon_0})h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M_2}(u_i^j)^2+\tfrac{T_2h\tau}{\varepsilon_1}\sum_{i=0}^{N-1}\sum_{j=1}^{M_2}(f_i^j)^2+$$
$$\tfrac{T_2\tau}{\varepsilon_2}\sum_{j=1}^{M_2}[((Y_0)^j)^2-((Y_1)^j)^2]+2\varepsilon_2(\tfrac{1}{\varepsilon_3}+\tfrac{1}{L})T_2h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M_2}(u_i^j)^2, \tag{26}$$

where $T_2=\tau M_2$.

At sufficient small τ,h, we may always choose $M_2,\varepsilon_0,\varepsilon_1,\varepsilon_3$ so that $[\nu_0+\nu_1^2\varepsilon_0-2\varepsilon_2(\varepsilon_3+h)]>0$, $T_2[(\varepsilon_1-\frac{1}{\varepsilon_0})+2\varepsilon_2(\frac{1}{\varepsilon_3}+\frac{1}{L})]<1$, we have

$$h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M_2}(u_i^j)^2\le C_3\{h\sum_{i=0}^{N-1}\phi^2+h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M_2}(f_i^j)^2+$$
$$+\tau\sum_{j=1}^{M_2}[((Y_0)^j)^2-((Y_1)^j)^2]\}^2, \tag{27}$$

where $C_3=(1-T_2C_4)^{-2}\max\{2T_2,\frac{T_2}{\varepsilon_1},C_5T_2\}$, $C_4=[(\varepsilon_1-\frac{1}{\varepsilon_0})+2\varepsilon_2(\frac{1}{\varepsilon_3}+\frac{1}{L})]$, $C_5=\max\{\frac{1}{\varepsilon_2}\}$.

In (25), put $M_1 = M_2$, we have

$$h\tau \sum_{i=0}^{N-1} (u_i^{M_2})^2 \leq C_4 \{ h \sum_{i=0}^{N-1} \phi^2 + h\tau \sum_{i=0}^{N-1} \sum_{j=1}^{M_2} (f_i^j)^2 + \\ + \tau \sum_{j=1}^{M_2} [((Y_0)^j)^2 - ((Y_1)^j)^2] \}, \tag{28}$$

where C_6 is a constant depending on ν_0, ν_1 and T_2.

Summing (27) on j from $M_1 + 1$ to M_3, and on M_3 from $M_2 + 1$ to $2M_2$, we obtain

$$h\tau \sum_{i=0}^{N-1} \sum_{j=M_2+1}^{2M_2} (u_i^j)^2 \leq C_3 \{ h \sum_{i=0}^{N-1} (u_i^{M_2})^2 + h\tau \sum_{i=0}^{N-1} \sum_{j=M_2+1}^{2M_2} (f_i^j)^2 + \\ + \tau \sum_{j=M_2+1}^{2M_2} [((Y_0)^j)^2 - ((Y_1)^j)^2] \}. \tag{29}$$

$$h\tau \sum_{i=0}^{N-1} (u_i^{2M_2})^2 \leq C_6 \{ h \sum_{i=0}^{N-1} (u_i^{M_2})^2 + h\tau \sum_{i=0}^{N-1} \sum_{j=M_2+1}^{2M_2} (f_i^j)^2 + \\ + \tau \sum_{j=M_2+1}^{2M_2} [((Y_0)^j)^2 - ((Y_1)^j)^2] \}. \tag{30}$$

Then from (29) and (30) we have

$$h\tau \sum_{i=0}^{N-1} (u_i^{2M_2})^2 \leq (C_6)^2 \{ h \sum_{i=0}^{N-1} (\phi_i)^2 + h\tau \sum_{i=0}^{N-1} \sum_{j=1}^{M_2} (f_i^j)^2 \\ + \tau \sum_{j=1}^{M_2} [((Y_0)^j)^2 - ((Y_1)^j)^2] \} + C_6 \{ h\tau \sum_{i=0}^{N-1} \sum_{j=M_2+1}^{2M_2} (f_i^j)^2 \\ + \tau \sum_{j=M_2+1}^{2M_2} [((Y_0)^j)^2 - ((Y_1)^j)^2] \}. \tag{31}$$

Not treating the generality for shortness of arfunents, set $\frac{M}{M_2} = K$ is an entre. Then obtaing sucessively the estimations for $h\tau \sum_{i=0}^{N-1} (u_i^j)^2$ at $j = 3M_2, 4M_2, \cdots, M$ analogously as it was made at obtaing of (31), we have

$$h\tau \sum_{i=0}^{N-1} (u_i^M)^2 \leq C_7 T \{ h \sum_{i=0}^{N-1} (\phi_i)^2 + h\tau \sum_{i=0}^{N-1} \sum_{j=1}^{M} (f_i^j)^2 + \\ + \tau \sum_{j=1}^{M} [((Y_0)^j)^2 - (Y_1)^j)^2] \}, \tag{32}$$

where $C_7 = C_6$ if $C_6 \geq 1, C_7 = (C_6)^K$ if $C_6 > 1$.

Then from (32) follows (20).

It is obvious that inequality is valid, if in the left part to substitute M to $M_1 < M$. Summing this inequality on M_1 from 1 to M and multiply by τ, we have

$$h\tau \sum_{i=0}^{N-1} \sum_{j=1}^{M} (u_i^j)^2 \leq C_7 T \{ h \sum_{i=0}^{N-1} (\phi_i)^2 + h\tau \sum_{i=0}^{N-1} \sum_{j=1}^{M} (f_i^j)^2 + \\ + \tau \sum_{j=1}^{M} [((Y_0)^j)^2 - (Y_1)^j)^2] \}. \tag{33}$$

Then from (33) follows (18).

Put $M_1 = M$ in (33), throwing the term at left part and using (25), we have

$$h\tau \sum_{i=0}^{N-1} \sum_{j=1}^{M} (u_i^j)_x^2 \leq C_8 T \{ h \sum_{i=0}^{N-1} (\phi_i)^2 + h\tau \sum_{i=0}^{N-1} \sum_{j=1}^{M} (f_i^j)^2 + \\ + \tau \sum_{j=1}^{M} [((Y_0)^j)^2 - (Y_1)^j)^2] \}. \tag{34}$$

Then from (33) follows (19). The Theorem is proved.

5. An Estimate of Stability on v

In this section, we obtain the stability estimation of the solution of difference approximations problem (13)-(16) on v.

Theorem 4. *Suppose that the all functions in the system (1)-(3) satsify as above enumerated conditions. Besides, $\lambda(u, v), B(u, v)$ satisfy the Lipschits condition on u and v with constant L , $\forall (x, t) \in \Omega$ and for any $v \in V$. Then the stability estimation of the solution of difference approximations problem (13)-(16) on v are*

$$h \sum_{i=0}^{N-1} (\delta u_i^j)^2 + h\tau \sum_{i=0}^{N-1} \sum_{j=0}^{M} (\delta u_i^j)^2 + h\tau \sum_{i=0}^{N-1} \sum_{j=0}^{M} (\delta u_i^j)_x^2 \leq C_9 \|\delta v\|_{E_N}^2 \tag{35}$$

where C_9 is a constant depending only on constants in (1)-(4), L and δu is the increment of u corresponding increment of δv for v.

Proof. Let v obtain an admissible increment of δv. Corresponding increment for u_i^j denote by δu_i^j. It is obvious that increments which are obtained for λ_i^j, B_i^j are equal to $\lambda_i^j(u_i^j+\delta u_i^j, v+\delta v)-\lambda_i^j(u_i^j,v)$,$B_i^j(u_i^j+\delta u_i^j, v+\delta v)-B_i^j(u_i^j,v)$.

The components of the function $\overline{u}_i^j = u_i^j + \delta u_i^j$ satisfy the equality (9) with functions $\overline{\lambda}_i^j = \lambda_i^j+\delta\lambda_i^j$, $\overline{B}_i^j = B_i^j+\delta B_i^j$.

Substraction from (9) for $\overline{u}_i^j$ the equality (9) for u_i^j, we obtain

$$\begin{aligned}&-h\tau\textstyle\sum_{i=0}^{N-1}\sum_{j=1}^{M}(\delta u_i^j)_{\overline{t}}\eta_i^j + h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M}[-\lambda_i^j(u_i^j+\delta u_i^j, v+\delta v)((u_i^j)_x\\ &+(\delta u_i^j)_x)+\lambda_i^j(u_i^j,v)(u_i^j)_x](\eta_i^j)_x + h\tau\textstyle\sum_{i=0}^{N-1}\sum_{j=1}^{M}[-B_i^j(u_i^j+\delta u_i^j, v+\delta v)\\ &((u_i^j)_x+(\delta u_i^j)_x)+B_i^j(u_i^j,v)(u_i^j)_x]\eta_i^j = 0.\end{aligned} \tag{36}$$

Put $\eta_i^j = 2\tau\delta u_i^j$, we obtain

$$\begin{aligned}&-2h\tau\textstyle\sum_{i=0}^{N-1}(\delta u_i^j)_{\overline{t}}\eta_i^j + 2h\tau\sum_{i=0}^{N-1}[-\lambda_i^j(u_i^j+\delta u_i^j, v+\delta v)\Delta_1\\ &+\lambda_i^j(u_i^j,v)(u_i^j)_x](\delta u_i^j)_x + 2h\tau\textstyle\sum_{i=0}^{N-1}[-B_i^j(u_i^j+\delta u_i^j, v+\delta v)\Delta_1\\ &+B_i^j(u_i^j,v)(u_i^j)_x]\delta u_i^j = 0.\end{aligned} \tag{37}$$

where $\Delta_1 = (u_i^j)_x+(\delta u_i^j)_x$. Taking into account that $2\tau(\delta u_i^j)_{\overline{t}}\delta u_i^j = (\delta u_i^j)^2-(u_i^{j-1})^2+(\tau)^2(\delta u_i^j)^2$, we obtain

$$\begin{aligned}&h\textstyle\sum_{i=0}^{N-1}(\delta u_i^j)^2 + h\sum_{i=0}^{N-1}(u_i^{j-1})^2 \le\\ &+2h\tau\textstyle\sum_{i=0}^{N-1}(\delta u_i^j)_x[\lambda_i^j(u_i^j+\delta u_i^j, v+\delta v)-\lambda_i^j(u_i^j+\delta u_i^j, v)](u_i^j)_x\\ &+2h\tau\textstyle\sum_{i=0}^{N-1}(\delta u_i^j)_x[\lambda_i^j(u_i^j+\delta u_i^j, v)-\lambda_i^j(u_i^j,v)](u_i^j)_x^2\\ &+2h\tau\textstyle\sum_{i=0}^{N-1}\delta u_i^j[B_i^j(u_i^j+\delta u_i^j, v+\delta v)B_i^j(u_i^j+\delta u_i^j, v)](u_i^j)_x\\ &+2h\tau\textstyle\sum_{i=0}^{N-1}\delta u_i^j[B_i^j(u_i^j+\delta u_i^j, v)-B_i^j(u_i^j,v)](u_i^j)_x.\end{aligned} \tag{38}$$

Taking into account, that $\lambda(u,v), B(u,v)$ satisfy the Lipschitz condition on v, the second and fourth terms in (38) become

$$\begin{aligned}&2h\tau\textstyle\sum_{i=0}^{N-1}(\delta u_i^j)_x[\lambda_i^j(u_i^j+\delta u_i^j, v)-\lambda_i^j(u_i^j,v)](u_i^j)_x^2\\ &\le 2h\tau\textstyle\sum_{i=0}^{N-1}|(\delta u_i^j)_x(u_i^j)_x|L|\delta u_i^j|\\ &\le 2h\tau L\textstyle\sum_{i=0}^{N-1}|(\delta u_i^j)_x\delta u_i^j|\\ &\le L[h\tau\textstyle\sum_{i=0}^{N-1}(\delta u_i^j)_x^2 + h\tau\sum_{i=0}^{N-1}(\delta u_i^j)^2].\end{aligned} \tag{39}$$

$$2h\tau\sum_{i=0}^{N-1}\delta u_i^j[B_i^j(u_i^j+\delta u_i^j, v)-B_i^j(u_i^j,v)](u_i^j)_x u_i^j| \le 2h\tau L\sum_{i=0}^{N-1}(\delta u_i^j)^2. \tag{40}$$

Appliying the $\varepsilon-$ inequality for the first and third terms in (38), we have

$$\begin{aligned}&2h\tau\textstyle\sum_{i=0}^{N-1}(\delta u_i^j)_x[\lambda_i^j(u_i^j+\delta u_i^j, v+\delta v)\lambda_i^j(u_i^j+\delta u_i^j, v)](u_i^j)_x \le\\ &\tfrac{h\tau}{\varepsilon_1}\textstyle\sum_{i=0}^{N-1}(\delta u_i^j)_x^2 + \varepsilon_1 h\tau\sum_{i=0}^{N-1}[\lambda_i^j(u_i^j+\delta u_i^j, v+\delta v)-\lambda_i^j(u_i^j+\delta u_i^j, v)](u_i^j)_x^2\\ &\le \varepsilon_1 h\tau L\|\delta v\|_{E_N}^2\textstyle\sum_{i=0}^{N-1}(u_i^j)_x^2 + \tfrac{h\tau}{\varepsilon_1}\sum_{i=0}^{N-1}(\delta u_i^j)_x^2.\end{aligned} \tag{41}$$

$$\begin{aligned}&2h\tau\textstyle\sum_{i=0}^{N-1}\delta u_i^j[B_i^j(u_i^j+\delta u_i^j, v+\delta v)B_i^j(u_i^j+\delta u_i^j, v)](u_i^j)_x \le\\ &\tfrac{h\tau}{\varepsilon_2}\textstyle\sum_{i=0}^{N-1}(\delta u_i^j)^2 + \varepsilon_2 h\tau\sum_{i=0}^{N-1}[B_i^j(u_i^j+\delta u_i^j, v+\delta v)-B_i^j(u_i^j+\delta u_i^j, v)](u_i^j)_x^2\\ &\le \varepsilon_2 h\tau L\|\delta v\|_{E_N}^2\textstyle\sum_{i=0}^{N-1}(u_i^j)_x^2 + \tfrac{h\tau}{\varepsilon_2}\sum_{i=0}^{N-1}(\delta u_i^j)^2.\end{aligned} \tag{42}$$

From (39)-(42) and (38), we have

$$\begin{aligned}&h\textstyle\sum_{i=0}^{N-1}(\delta u_i^j)^2 \le h\sum_{i=0}^{N-1}(u_i^{j-1})^2 + (\frac{1}{\varepsilon_1}+L)h\tau\sum_{i=0}^{N-1}(\delta u_i^j)_x^2\\ &+(\tfrac{1}{\varepsilon_2+2L})h\tau\textstyle\sum_{i=0}^{N-1}(\delta u_i^j)^2 + L(\varepsilon_1+\varepsilon_2)\|\delta v\|_{E_N}^2 h\tau\sum_{i=0}^{N-1}(u_i^j)_x^2.\end{aligned} \tag{43}$$

Summing (43) on j from 1 to $M_2 \leq M$ and multiplying to τ, we obtain

$$h\tau \sum_{i=0}^{N-1} (\delta u_i^{M_2})^2 \leq (2L + \tfrac{1}{\varepsilon_2}) h\tau \sum_{i=0}^{N-1} \sum_{j=1}^{M_2} (\delta u_i^j)^2$$
$$+ L(\varepsilon_1 + \varepsilon_2) \|\delta v\|_{E_N}^2 h\tau \sum_{i=0}^{N-1} (u_i^j)_x^2. \qquad (44)$$

The reasoning used in the proof of theorem 5.1 applied here, proves that the estimation (35) is true. The Theorem is proved.

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