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NONCONFORMING QUADRILATERAL ROTATED Q_1 ELEMENT FOR REISSNER-MINDLIN PLATE *1)

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Dedicated to the 80th birthday of Professor Zhou Yulin

Abstract

In this paper, we extend two rectangular elements for Reissner-Mindlin plate [9] to the quadrilateral case. Optimal H^1 and L^2 error bounds independent of the plate hickness are derived under a mild assumption on the mesh partition.

Key words: Reissner-Mindlin Plate, Quadrilateral Rotated Q_1 element, Locking-free.

1. Introduction

We consider the finite element approximation of the solution of Reissner-Mindlin (R-M hereinafter) model, which describes the deformation of a plate subjected to a transverse loading in terms of the transverse displacement of the midplane and the rotations of fibers normal to the midplane. As it is well-known, standard finite element approximation of this model usually fails to yield good results when the plate thickness is small, which is commonly referred to *locking phenomenon*, so some numerical stabilization tricks such as reduced integration or the mixed variational principles are needed to overcome this difficulty. MS elements proposed in [9] seem the simplest rectangular elements in such category [3]. However, quadrilateral elements are far more flexible than rectangular elements, so it is quite important to construct quadrilateral R-M plate elements, or extend the existing rectangular R-M elements to the quadrilateral case. On the other and, it is noticed recently that the extension of rectangular R-M elements to isoparametric quadrilateral R-M elements is not so straightforward [10]. The goal of this paper is to extend MS elements to the quadrilateral case and give a mathematical analysis.

We conclude this section with a list of some basic notations used in the sequel. In §2, the R-M plate model and its varational formulation of Brezzi and Fortin [4, 6] are recalled. In §3, we describe the quadrilateral version of MS elements and the method we used is recast in the variational formulation of Brezzi and Fortin based upon a kind of discrete Helmholtz Decomposition. The error estimates are included in §4.

We use the standard notation and definition for the Sobolev spaces $H^s(\Omega)$ and $H^s(\partial\Omega)$ for $s \geq 0$ [1], the standard associated inner products are denoted by $(\cdot,\cdot)_s$ and $(\cdot,\cdot)_{s,\partial\Omega}$, and their norms by $\|\cdot\|_s$ and $\|\cdot\|_{s,\partial\Omega}$, respectively. For $s=0$, $H^s(\Omega)$ coincides with a $L^2(\Omega)$. In this case, the norm and inner product are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$ respectively. As usual, $H_0^s(\Omega)$ is the subspace of $H^s(\Omega)$ with vanishing trace on Ω. Let $L_0^2(\Omega)$ be the set of all $L^2(\Omega)$ functions with zero integral mean.

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Throughout this paper, the generic constant C is assumed to be independent of the plate thickness t and the mesh size h.

Finally, we use the standard differential operators:

$$\nabla r = \begin{pmatrix} \partial r/\partial x \\ \partial r/\partial y \end{pmatrix}, \qquad \operatorname{curl} p = \begin{pmatrix} \partial p/\partial y \\ -\partial p/\partial x \end{pmatrix},$$

$$\operatorname{div} \boldsymbol{\psi} = \partial\psi_1/\partial x + \partial\psi_2/\partial y, \quad \operatorname{rot} \boldsymbol{\psi} = \partial\psi_2/\partial x - \partial\psi_1/\partial y.$$

We also need the following vector spaces

$$\boldsymbol{H}_0(\operatorname{rot},\Omega) = \{\, \boldsymbol{q} \in \boldsymbol{L}^2(\Omega) \mid \operatorname{rot} \boldsymbol{q} \in L^2(\Omega), \boldsymbol{q}\cdot\boldsymbol{t} = 0 \text{ on } \partial\Omega \,\},$$

where $\boldsymbol{t}$ is denoted as the unit tangent to $\partial\Omega$, and

$$\boldsymbol{H}(\operatorname{div},\Omega) = \{\, \boldsymbol{q} \in \boldsymbol{L}^2(\Omega) \mid \operatorname{div} \boldsymbol{q} \in L^2(\Omega) \,\}.$$

The norm in $\boldsymbol{H}(\operatorname{div},\Omega)$ is given by

$$\|\boldsymbol{\eta}\|_{\boldsymbol{H}(\operatorname{div})} = \left(\|\boldsymbol{\eta}\|_0^2 + \|\operatorname{div} \boldsymbol{\eta}\|_0\right)^{1/2}.$$

2. Reissner-Mindlin Plate Model

Let Ω be a convex polygon representing the mid-surface of the plate. Assume that the plate is clamped along the boundary $\partial\Omega$. Let ω and $\boldsymbol{\phi}$ denote the transverse deflection and the rotations, respectively, which are determined by the following

Problem 2.1. *Find* $(\boldsymbol{\phi},\omega) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega)$ *such that*

$$a(\boldsymbol{\phi},\boldsymbol{\psi}) + (\boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}) = (g, v) \quad \forall(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega). \tag{2.1}$$

The shear strain $\boldsymbol{\gamma}$ *is defined as*

$$\boldsymbol{\gamma} := \lambda t^{-2}(\nabla\omega - \boldsymbol{\phi}).$$

Here g is the scaled transverse loading, t is the plate thickness, $\lambda = E\kappa/2(1+\nu)$ is the shear modulus with Young's modulus E, ν the Poisson ratio, and κ the shear correction factor. The bilinear form a is defined as $a(\boldsymbol{\eta},\boldsymbol{\psi}) = (\mathcal{C}\mathcal{E}\boldsymbol{\eta}, \mathcal{E}\boldsymbol{\psi})$, here $\mathcal{C}\tau$ is defined for any 2×2 symmetric matrix τ as

$$\mathcal{C}\tau := \frac{E}{12(1-\nu^2)}\left[(1-\nu)\tau + \nu \operatorname{tr}(\tau)\boldsymbol{I}\right].$$

Following [4] and [6], Problem 2.1 can be written into the following decoupled system as

Problem 2.2. *Find* $(r,\boldsymbol{\phi},p,\boldsymbol{\alpha},\omega) \in H_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0(\operatorname{rot},\Omega) \times H_0^1(\Omega)$*, such that*

$$\begin{aligned}
(\nabla r, \nabla\mu) &= (g,\mu) \quad \forall \mu \in H_0^1(\Omega),\\
a(\boldsymbol{\phi},\boldsymbol{\psi}) - (p, \operatorname{rot}\boldsymbol{\psi}) &= (\nabla r, \boldsymbol{\psi}) \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega),\\
-(\operatorname{rot}\boldsymbol{\phi}, q) - \lambda^{-1}t^2(\operatorname{rot}\boldsymbol{\alpha}, q) &= 0 \quad \forall q \in L_0^2(\Omega),\\
(\boldsymbol{\alpha},\boldsymbol{\delta}) - (p, \operatorname{rot}\boldsymbol{\delta}) &= 0 \quad \forall \boldsymbol{\delta} \in \boldsymbol{H}_0(\operatorname{rot},\Omega),\\
(\nabla\omega, \nabla s) &= (\boldsymbol{\phi} + \lambda^{-1}t^2\nabla r, \nabla s) \quad \forall s \in H_0^1(\Omega).
\end{aligned}$$

It is not difficult to show that Problem 2.1 is equivalent to Problem 2.2. The existence and uniqueness of the solution of Problem 2.2 and the following regularity result can be found in [4], [9, Theorem 2.1] and references therein.

Lemma 2.3. *If $(r, \boldsymbol{\phi}, p, \boldsymbol{\alpha}, \omega) \in H_0^1 \times \boldsymbol{H}_0^1 \times L_0^2 \times \boldsymbol{H}_0(\mathrm{rot}) \times H_0^1$ are the solutions of Problem 2.2, then we have the following a-priori and regularity estimates*

$$\begin{aligned} \|\boldsymbol{\phi}\|_2 + \|\boldsymbol{\gamma}\|_0 + \|\omega\|_1 \le C\|g\|_{-1}, \\ \|r\|_1 + \|p\|_1 + t\|p\|_2 + \|\boldsymbol{\alpha}\|_0 + t\|\boldsymbol{\alpha}\|_1 + t^2\|\mathrm{rot}\boldsymbol{\alpha}\|_0 \le C\|g\|_{-1}, \end{aligned} \tag{2.2}$$

$$\|\omega\|_2 \le C(\|g\|_{-1} + t^2\|g\|_0), \tag{2.3}$$

$$\|\boldsymbol{\gamma}\|_{\boldsymbol{H}(\mathrm{div})} + \|r\|_2 \le C\|g\|_0, \quad t\|\boldsymbol{\gamma}\|_1 \le C(\|g\|_{-1} + t\|g\|_0). \tag{2.4}$$

3. Finite Element Approximation

Let $\mathcal{T}_h$ be a partition of $\overline{\Omega}$ into convex quadrilaterals K with the mesh size h_K, define $h\colon = \max_{K\in\mathcal{T}_h} h_K$. The usual regularity assumption for $\mathcal{T}_h$ is assumed in the sense of Ciarlet and Raviart [7, pp. 247], the quasi-uniformity of $\mathcal{T}_h$ is also assumed. We denote the distance between the middle points of two diagonals of K as d_K.

Definition 3.1. $(1+\alpha)-$*Section Condition* $(0 \le \alpha \le 1)$ *[12]*

$$d_K = \mathcal{O}(h_K^{1+\alpha}),$$

uniformly for all elements K as $h \to 0$.

In particular, we recover the *Bi-Section Condition* [14] if $\alpha = 1$.

Let $\hat{K} = [-1, 1]^2$ be the reference equare. Then there exists a bilinear mapping $\mathcal{F}_K$ from $\hat{K}$ onto K with Jacobian $\mathcal{DF}_K$ and determinant J_K. Denote $J_0 = J_K(0, 0)$. Let $\mathcal{Q}_{i,j}$ be the space of polynomials of degree no more than i for the first variable and no more than j for the second, and set $\mathcal{Q}_i = \mathcal{Q}_{i,i}$.

We use the conforming bilinear element space

$$W_h\colon = \{\, v \in H_0^1 \mid v_{|_K} \in \mathcal{Q}_1(\hat{K}) \quad \forall K \in \mathcal{T}_h \,\}$$

for the approximation of the deflection. Let

$$\hat{\mathcal{Q}}_1\colon = \{\, q \circ \mathcal{F}_K^{-1} \mid q \in \mathrm{Span}\langle 1, x, y, x^2 - y^2\rangle \,\}.$$

For any edge $\mathcal{F} \subset \partial K$, the edge functional $\mathcal{J}_\mathcal{F}$ is defined as

$$\mathcal{J}_\mathcal{F}(v)\colon = \frac{1}{|\mathcal{F}|}\int_\mathcal{F} v\, ds, \quad \forall v \in L^2(K).$$

A local interpolation operator $\mathcal{J}_K$ is generated by $\mathcal{J}_\mathcal{F}$ with $\mathcal{J}_{K|_\mathcal{F}} = \mathcal{J}_\mathcal{F}$ for all $\mathcal{F} \subset \partial K$. The following NRQ_1 space [13] is used for the rotations:

$$N_h\colon = \{\, v \in L^2(\Omega) \mid v_{|_K} \in \hat{\mathcal{Q}}_1 \;\text{and}\; v \;\text{is continuous regarding}\; \mathcal{J}_\mathcal{F} \,\}.$$

The corresponding homogeneous space is defined as

$$N_{0,h}\colon = \{\, v \in N_h \mid \mathcal{J}_\mathcal{F}(v) = 0, \quad \text{for}\; \mathcal{F} \subset \partial\Omega \,\}.$$

Let

$$\|v\|_{l,h} := \Big(\sum_{K\in\mathcal{T}_h} \|v\|_{l,K}^2\Big)^{1/2} \quad \text{and} \quad |v|_{l,h} := \Big(\sum_{K\in\mathcal{T}_h} |v|_{l,K}^2\Big)^{1/2}, \quad l = 1,2.$$

It is obvious that $|\cdot|_{1,h}$ is a norm on $N_{0,h}$.

Define

$$\boldsymbol{V}_h := N_{0,h} \times W_h.$$

As $\boldsymbol{V}_h$ is nonconforming, so when differential operators such as $\mathcal{E}, \mathrm{div}, \mathrm{curl}, \mathrm{rot}$ and ∇ may be applied to functions in $\boldsymbol{V}_h$, we shall write $\mathcal{E}_h, \mathrm{div}_h, \mathrm{curl}_h, \mathrm{rot}_h$ and ∇_h in all these cases, which are defined in a piecewise manner.

We summarize some interpolation results for $\mathcal{J}_{\mathcal{F}}$ and its global version which is defined as $\Pi_{h|_K} = \mathcal{J}_K$.

Lemma 3.2. *[13] The forgoing defined interpolation operator $\mathcal{J}_{\mathcal{F}}$ and Π_h admit the following estimates*

$$\|v - \mathcal{J}_{\mathcal{F}}(v)\|_{0,\mathcal{F}} \le C h_K^{1/2} |v|_{1,K} \quad \forall v \in H^1(K). \tag{3.1}$$

Moreover, if the $(1+\alpha)-$Section Condition *holds, then*

$$\|v - \Pi_h v\|_0 + h\|v - \Pi_h v\|_{1,h} \le C h^{1+\alpha} \|v\|_2 \quad \forall v \in H_0^1 \cap H^2. \tag{3.2}$$

At last, we use the lowest-order rotated Raviart-Thomas space for approximating the shear force as follows:

$$\boldsymbol{\Gamma}_h := \{\, \boldsymbol{\chi} \in \boldsymbol{H}_0(\mathrm{rot},\Omega) \mid \boldsymbol{\chi}\circ\mathcal{F}_K = \mathcal{D}\mathcal{F}_K^{-T}\hat{\boldsymbol{\chi}},\ \hat{\boldsymbol{\chi}} \in \mathcal{Q}_{0,1}(\hat{K}) \times \mathcal{Q}_{1,0}(\hat{K}) \quad \forall K \in \mathcal{T}_h \,\}.$$

The lowest-order rotated Raviart-Thomas interpolation operator $\boldsymbol{R}_h$ is defined as $\boldsymbol{R}_{h|_K} = \boldsymbol{R}_K$ and

$$\int_{\mathcal{F}} (\boldsymbol{\psi} - \boldsymbol{R}_K\boldsymbol{\psi}) \cdot \boldsymbol{t}\, ds = 0, \quad \forall \mathcal{F} \subset \partial K,$$

for any $\boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathrm{rot},\Omega)$. Clearly, we have

$$\begin{aligned} (\mathrm{rot}\,\boldsymbol{\psi})\circ\mathcal{F}_K &= J_K^{-1}\widehat{\mathrm{rot}}\hat{\boldsymbol{\psi}}, \\ \textstyle\int_K \mathrm{rot}\,\boldsymbol{R}_K\boldsymbol{\psi}\,dx &= \textstyle\int_K \mathrm{rot}\,\boldsymbol{\psi}\,dx. \end{aligned} \tag{3.3}$$

Since $\int_{\mathcal{F}} \boldsymbol{\psi}\cdot\boldsymbol{t}$ is well-defined for $\boldsymbol{\psi} \in \boldsymbol{V}_h$, so does $\boldsymbol{R}_h\boldsymbol{\psi}$. We list some interpolation error estimates for $\boldsymbol{R}_h$ in the following lemma.

Lemma 3.3. *For any $\boldsymbol{\psi} \in \boldsymbol{H}^1 \cap \boldsymbol{H}_0(\mathrm{rot})$, we have*

$$\|\boldsymbol{\psi} - \boldsymbol{R}_h\boldsymbol{\psi}\|_0 \le Ch|\boldsymbol{\psi}|_1. \tag{3.4}$$

Moreover, if the $(1+\alpha)$-Section Condition *holds, then*

$$\|\,\mathrm{rot}(\boldsymbol{\psi} - \boldsymbol{R}_h\boldsymbol{\psi})\|_0 \le C(h|\mathrm{rot}\,\boldsymbol{\psi}|_1 + h^{\alpha}\|\,\mathrm{rot}\,\boldsymbol{\psi}\|_0). \tag{3.5}$$

Proof. Proceeding along the same line of [8, Lemma 7.1], we get (3.4). Noting that (3.5) is already included in [10, Theorem 3.1].

□

Further we denote $\mathcal{Q}_h$ the subspace of $L_0^2(\Omega)$ with piecewise constant on each element and Π_h^0 the corresponding L^2 projection operator. Let Π_h^1 be the standard bilinear interpolation operator and its vector counterpart $\mathbf{\Pi}_1 := (\Pi_h^1, \Pi_h^1)$.

With these notations we have the following finite element approximation:

Problem 3.4. *Find* $(\boldsymbol{\phi}_h, \omega_h) \in \boldsymbol{V}_h \times W_h$ *such that*

$$a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + \lambda t^{-2}(\nabla\omega_h - \boldsymbol{R}_h\boldsymbol{\phi}_h, \nabla v - \boldsymbol{R}_h\boldsymbol{\psi}) = (g, v) \quad \forall(\boldsymbol{\psi}, v) \in \boldsymbol{V}_h \times W_h. \tag{3.6}$$

The shear force is defined locally as

$$\boldsymbol{\gamma}_h := \lambda t^{-2}(\nabla\omega_h - \boldsymbol{R}_h\boldsymbol{\phi}_h).$$

The operator $\mathrm{curl}_h : \mathcal{Q}_h \to \boldsymbol{\Gamma}_h$ is defined by

$$(\mathrm{curl}_h\, q, \boldsymbol{\psi}) = (q, \mathrm{rot}\,\boldsymbol{\psi}) \quad \forall\boldsymbol{\psi} \in \boldsymbol{\Gamma}_h.$$

Our analysis is based upon the following *Discrete Helmholtz Decomposition*. Its rectangular version has been proved in [9, Lemma 3.3].

Lemma 3.5. *For any* $\boldsymbol{q} \in \boldsymbol{\Gamma}_h$, *there exist unique* $r \in W_h, p \in \mathcal{Q}_h$ *and* $\boldsymbol{\alpha} \in \boldsymbol{\Gamma}_h$ *such that*

$$\boldsymbol{q} = \nabla r + \boldsymbol{\alpha}, \tag{3.7}$$

and $(\boldsymbol{\alpha}, \boldsymbol{s}) = (\mathrm{rot}\,\boldsymbol{s}, p) \quad \forall\boldsymbol{s} \in \boldsymbol{\Gamma}_h$.

Proof. Consider the following mixed problem: find $(\boldsymbol{\alpha}, p) \in \boldsymbol{\Gamma}_h \times \mathcal{Q}_h$ such that

$$\begin{aligned}(\boldsymbol{\alpha}, \boldsymbol{s}) - (\mathrm{rot}\,\boldsymbol{s}, p) &= 0 \quad \forall\boldsymbol{s} \in \boldsymbol{\Gamma}_h,\\ (\mathrm{rot}\,\boldsymbol{\alpha}, m) &= (\mathrm{rot}\,\boldsymbol{q}, m) \quad \forall m \in \mathcal{Q}_h.\end{aligned}$$

Notice that $(\boldsymbol{\Gamma}_h, \mathcal{Q}_h)$ is a stable pair for such mixed approximation, so there is a unique pair $(\boldsymbol{\alpha}, p)$ satisfies the above equations. Using $(3.3)_1$ and $J_K > 0$, we have $\mathrm{rot}(\boldsymbol{\alpha} - \boldsymbol{q})_{|K} = 0$ iff $\widehat{\mathrm{rot}}(\widehat{\boldsymbol{\alpha} - \boldsymbol{q}}) = 0$ on $\hat{K}$. Defining m on each element as $m = \widehat{\mathrm{rot}}(\widehat{\boldsymbol{\alpha} - \boldsymbol{q}})/J_0$ and putting it into the second equation of the above formulation, we obtain $\widehat{\mathrm{rot}}(\widehat{\boldsymbol{\alpha} - \boldsymbol{q}}) = 0$, so $\mathrm{rot}(\boldsymbol{\alpha} - \boldsymbol{q}) = 0$. Notice that $\nabla W_h \subset \boldsymbol{\Gamma}_h$, and therefore, there exists $r \in W_h$ such that $\boldsymbol{\alpha} - \boldsymbol{q} = \nabla r_h$, which completes the proof. $\square$

Using the above lemma, we have

$$\lambda t^{-2}(\nabla\omega_h - \boldsymbol{R}_h\boldsymbol{\phi}_h) = \nabla r_h + \boldsymbol{\alpha}_h$$

with $r_h \in W_h$ and $\boldsymbol{\alpha}_h \in \boldsymbol{\Gamma}_h$. Proceeding along the same line of [5, 6], and using the following equations

$$\begin{aligned}(\mathrm{rot}\,\boldsymbol{R}_h\boldsymbol{\psi}, q) &= (\mathrm{rot}\,\boldsymbol{\psi}, q) \qquad \forall\boldsymbol{\psi} \in H_0^1 \quad \text{and} \quad q \in \mathcal{Q}_h,\\ (\mathrm{rot}_h\,\boldsymbol{R}_h\boldsymbol{\psi}, q) &= (\mathrm{rot}_h\,\boldsymbol{\psi}, q) \qquad \forall\boldsymbol{\psi} \in \boldsymbol{V}_h \quad \text{and} \quad q \in \mathcal{Q}_h,\end{aligned} \tag{3.8}$$

we obtain the following alternative variational formulation of Problem 3.4:

Problem 3.6. *Find* $(r_h, \boldsymbol{\phi}_h, p_h, \boldsymbol{\alpha}_h, \omega_h) \in W_h \times \boldsymbol{V}_h \times \mathcal{Q}_h \times \boldsymbol{\Gamma}_h \times W_h$ *such that*

$$(\nabla r_h, \nabla v) = (g, v) \quad \forall v \in W_h, \tag{3.9}$$

$$a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}) - (p_h, \mathrm{rot}_h\,\boldsymbol{\psi}) = (\nabla r_h, \boldsymbol{R}_h\boldsymbol{\psi}) \quad \forall\boldsymbol{\psi} \in \boldsymbol{V}_h, \tag{3.10}$$

$$-(\mathrm{rot}_h\,\boldsymbol{\phi}_h, q) - \lambda^{-1}t^2(\mathrm{rot}\,\boldsymbol{\alpha}_h, q) = 0 \quad \forall q \in \mathcal{Q}_h, \tag{3.11}$$

$$(\boldsymbol{\alpha}_h, \boldsymbol{\delta}) - (p_h, \mathrm{rot}\,\boldsymbol{\delta}) = 0 \quad \forall\boldsymbol{\delta} \in \boldsymbol{\Gamma}_h, \tag{3.12}$$

$$(\nabla\omega_h, \nabla s) = (\boldsymbol{R}_h\boldsymbol{\phi}_h + \lambda^{-1}t^2\nabla r_h, \nabla s) \quad \forall s \in W_h. \tag{3.13}$$

As before, we can rewrite the equations (3.10)-(3.12) into a compact form as

$$A_h(\boldsymbol{\phi}_h,\boldsymbol{\alpha}_h;\boldsymbol{\psi},\boldsymbol{\delta})+B_h(\boldsymbol{\psi},\boldsymbol{\delta};p_h)=(\nabla r_h,\boldsymbol{R}_h\boldsymbol{\psi})\quad\forall(\boldsymbol{\psi},\boldsymbol{\delta})\in\boldsymbol{V}_h\times\boldsymbol{\Gamma}_h.$$
$$B_h(\boldsymbol{\phi}_h,\boldsymbol{\alpha}_h;q)=0\quad\forall q\in\mathcal{Q}_h,$$

where

$$A_h(\boldsymbol{\phi},\boldsymbol{\alpha};\boldsymbol{\psi},\boldsymbol{\delta}):=(\mathcal{C}\mathcal{E}_h\boldsymbol{\phi},\mathcal{E}_h\boldsymbol{\psi})+\lambda^{-1}t^2(\boldsymbol{\alpha},\boldsymbol{\delta}),$$
$$B_h(\boldsymbol{\psi},\boldsymbol{\delta};q):=-(\operatorname{rot}_h\boldsymbol{\psi},q)-\lambda^{-1}t^2(q,\operatorname{rot}\boldsymbol{\delta}).$$

The above two bilinear forms are bounded with respect to the following norm

$$|||\boldsymbol{\psi},\boldsymbol{\delta}|||:=\|\boldsymbol{\psi}\|_1^2+t^2\|\boldsymbol{\delta}\|_0^2+t^4\|\operatorname{rot}\boldsymbol{\delta}\|_0^2.$$

Remark 3.7. (3.8) *is crucial to derive the above variational formulation* (3.9)-(3.13)*. Notice that in the rectangular case, we have the following simpler form (Cf [9, (3.21) and (3.22)])*

$$\operatorname{rot}\boldsymbol{R}_h\boldsymbol{\psi}=\boldsymbol{\Pi}_0\operatorname{rot}\boldsymbol{\psi}\quad\forall\boldsymbol{\psi}\in\boldsymbol{H}_0^1,\qquad\operatorname{rot}\boldsymbol{R}_h\boldsymbol{\psi}=\boldsymbol{\Pi}_0\operatorname{rot}_h\boldsymbol{\psi}\quad\forall\boldsymbol{\psi}\in\boldsymbol{V}_h.$$

3.1 Well-posedness of M-S 1 Element

The well-posedness of Problem 3.6 hangs on the following two assumptions

1. K-ellipticity. There exists a constant $C>0$ such that
$$A_h(\boldsymbol{\psi},\boldsymbol{\delta};\boldsymbol{\psi},\boldsymbol{\delta})\geq C|||\boldsymbol{\psi},\boldsymbol{\delta}|||^2 \tag{3.14}$$
for all
$$\begin{aligned}(\boldsymbol{\psi},\boldsymbol{\delta})\in\boldsymbol{Z}_h:&=\{(\boldsymbol{\psi},\boldsymbol{\delta})\in\boldsymbol{V}_h\times\boldsymbol{\Gamma}_h\mid B_h(\boldsymbol{\psi},\boldsymbol{\delta};q)=0\forall q\in\mathcal{Q}_h\}\\&=\{(\boldsymbol{\psi},\boldsymbol{\delta})\in\boldsymbol{V}_h\times\boldsymbol{\Gamma}_h\mid\operatorname{rot}_h\boldsymbol{R}_h\boldsymbol{\psi}=-\lambda^{-1}t^2\operatorname{rot}\boldsymbol{\delta}\}.\end{aligned} \tag{3.15}$$
2. B-B condition. There exists a constant C such that
$$\sup_{(\boldsymbol{\psi},\boldsymbol{\delta})\in\boldsymbol{V}_h\times\boldsymbol{\Gamma}_h}\frac{B(\boldsymbol{\psi},\boldsymbol{\delta};q)}{|||\boldsymbol{\psi},\boldsymbol{\delta}|||}\geq C\|q\|_0\quad\forall q\in\mathcal{Q}_h. \tag{3.16}$$

Proceeding along the same line of that in [9, Theorem 5.7], we need the *Poincaré inequality* and the *Korn inequality* for $\boldsymbol{V}_h$ to ensure the *K-ellipticity*. The former can be found in [11] and the latter in [9, Theorem 5.1.8]. AS to the *B-B inequality* for the finite element space pair $(\boldsymbol{V}_h,\mathcal{Q}_h)$, a proof is included in [9, Theorem 5.2.7]. Consequently we have

Theorem 3.8. *Problem 3.6 admits a unique solution* $(r_h,\boldsymbol{\phi}_h,p_h,\boldsymbol{\alpha}_h,\omega_h)\in W_h\times\boldsymbol{V}_h\times\mathcal{Q}_h\times\boldsymbol{\Gamma}_h\times W_h$ *such that* (3.9)-(3.13) *hold.*

3.2 Energy error estimate for M-S 1 Element

Here follow the energy error estimates which can be derived literally as [9, Theorem 5.8, Theorem 5.9].

Theorem 3.9. *Let* $(\boldsymbol{\phi},\omega,\boldsymbol{\gamma})$ *and* $(\boldsymbol{\phi}_h,\omega_h,\boldsymbol{\gamma}_h)$ *be the solutions of Problem 2.1 and 3.4, respectively. Moreover, if the* $(1+\alpha)-$Section Condition *holds, then*

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,h}+\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{-1}\leq Ch^\alpha\|g\|_{-1},$$
$$\|\nabla\omega-\nabla\omega_h\|_0\leq C(h^\alpha\|g\|_{-1}+h\max(t^2,h^\alpha)\|g\|_0).$$

Remark 3.10. *Obviously, if the* Bi-Section Condition *holds, i.e.,* $\alpha=1$*, in view of the above theorem, the optimal error bounds for all variables with respect to the energy norm are obtained.*

3.3 L^2 Error estimate for M-S 1 Element

Now we consider the L^2 error estimate for the rotations and the deflection by employing a super-approximation results proved in [10, Lemma 3.6].

Lemma 3.11. *Let* $\boldsymbol{\psi} \in \boldsymbol{H}_0^1 \cap \boldsymbol{H}^2$ *and* $\boldsymbol{\zeta} \in \boldsymbol{H}(\text{div})$. *If the* $(1+\alpha)-$ Section Condition *holds, then*

$$|(\boldsymbol{\zeta}, \boldsymbol{\Pi\psi} - \boldsymbol{R}_h\boldsymbol{\Pi\psi})| \leq Ch^{2\alpha}\|\boldsymbol{\zeta}\|_{\boldsymbol{H}(\text{div})}\|\boldsymbol{\psi}\|_2. \tag{3.17}$$

With the above lemma and Theorem 3.9, proceeding along the same line as that in [9, Theorem 5.10], we obtain

Theorem 3.12. *Let* $(\boldsymbol{\phi}, \omega)$ *and* $(\boldsymbol{\phi}_h, \omega_h)$ *be solutions of Problem 2.1 and 3.4, respectively. If the* $(1+\alpha)-$Section Condition *holds, then*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + \|\omega - \omega_h\|_0 \leq Ch^{2\alpha}\|g\|_0.$$

4. Error estimates for M-S 2 Element

We now turn to error estimate of M-S 2 element. The only difference between M-S 1 and M-S 2 is the approximation of the rotations. We define the approximation space as

$$\boldsymbol{V}_h := N_{0,h} \times N_{0,h}.$$

Proceeding along the same approach of [2], we obtain

Theorem 4.1. *Let* $(\boldsymbol{\phi}, \omega, \boldsymbol{\gamma})$ *and* $(\boldsymbol{\phi}_h, \omega_h, \boldsymbol{\gamma}_h)$ *be the solutions of Problem 2.1 and 3.4, respectively. If the* $(1+\alpha)-$Section Condition *holds, then*

$$\begin{aligned}
\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,h} + \|\nabla(\omega - \omega_h)\|_0 &\leq Ch^{\alpha}\max(1, t^2/h^2)\|g\|_0, \\
\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{-1} &\leq Ch^{\alpha}(1 + \max(1, t/h))\|g\|_0, \\
\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + \|\omega - \omega_h\|_0 &\leq Ch^{2\alpha}\max(1, t^2/h^2)\|g\|_0.
\end{aligned}$$

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