Journal of Computational Mathematics, Vol.21, No.5, 2003, 569–584.

AN INVERSE EIGENVALUE PROBLEM FOR JACOBI MATRICES *1)

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Abstract

Let $T_{1,n}$ be an $n \times n$ unreduced symmetric tridiagonal matrix with eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n.$$

and

$$W_k = \begin{pmatrix} T_{1,k-1} & 0 \\ 0 & T_{k+1,n} \end{pmatrix}$$

is an $(n-1) \times (n-1)$ submatrix by deleting the k^{th} row and k^{th} column, $k = 1, 2, \cdots, n$ from T_n.
Let

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k-1}$$

be the eigenvalues of $T_{1,k-1}$ and

$$\mu_k \leq \mu_{k+1} \leq \cdots \leq \mu_{n-1}$$

be the eigenvalues of $T_{k+1,n}$.
A new inverse eigenvalues problem has put forward as follows: How do we construct an unreduced symmetric tridiagonal matrix $T_{1,n}$, if we only know the spectral data: the eigenvalues of $T_{1,n}$, the eigenvalues of $T_{1,k-1}$ and the eigenvalues of $T_{k+1,n}$?
Namely if we only know the data: $\lambda_1, \lambda_2, \cdots, \lambda_n$, $\mu_1, \mu_2, \cdots, \mu_{k-1}$ and $\mu_k, \mu_{k+1}, \cdots, \mu_{n-1}$ how do we find the matrix $T_{1,n}$? A necessary and sufficient condition and an algorithm of solving such problem, are given in this paper.

Key words: Symmetric tridiagonal matrix, Jacobi matrix, Eigenvalue problem, Inverse eigenvalue problem.

1. Introduction

Let

$$T_n = \begin{pmatrix} \alpha_1 & \beta_1 & & & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ 0 & & & \beta_{n-1} & \alpha_n \end{pmatrix}$$

* Received June 12, 2001.
1) Project 19771020 supported by National Science Foundation of China.

be an $n \times n$ unreduced symmetric tridiagonal matrix, and denote its submatrix $T_{p,q}, (p < q)$ as follows

$$T_{p,q} = \begin{pmatrix} \alpha_p & \beta_p & & & 0 \\ \beta_p & \alpha_{p+1} & \beta_{p+1} & & \\ & \beta_{p+1} & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{q-1} \\ 0 & & & \beta_{q-1} & \alpha_q \end{pmatrix} \quad p < q$$

We call an unreduced symmetric tridiagonal matrix with $\beta_i > 0$ as a Jacobi matrix. Consider $T_{1,n}$ and $T_{p,q}$ to be Jacobi matrices. The matrix

$$W_k = \begin{pmatrix} T_{1,k-1} & 0 \\ 0 & T_{k+1,n} \end{pmatrix}$$

is gained by deleting the k^{th} row and the k^{th} column $(k = 1, 2, ..., n)$ from T_n. We put forward an inverse eigenvalue problem to be that: If we don't know the matrix $T_{1,n}$, but we know all eigenvalues of matrix $T_{1,k-1}$, all eigenvalues of matrix $T_{k+1,n}$, and all eigenvalues of matrix $T_{1,n}$, could we construct the matrix $T_{1,n}$. Let $\mu_1, \mu_2, \cdots, \mu_{k-1}, \mu_k, \mu_{k+1}, \cdots, \mu_{n-1}$, and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of matrices $T_{1,k-1}, T_{k+1,n}$ and $T_{1,n}$ respectively.Our problem is that from above 2n-1 data to find other 2n-1 data:

$$\alpha_1, \alpha_2, \cdots, \alpha_n, and\ \beta_1, \beta_2, \cdots, \beta_{n-1}$$

Obviously, when k=1 or k=n this problem has been solved and there are many algorithms to construct $T_{1,n}$ [1],[2],[4],[5],[10] . When we delet the k^{th} row and the k^{th} column from $T_{1,n}$, in cases $k = 2, 3, \cdots, n-1$, it means to delet three numbers α_k, β_{k-1},and β_k ,while in case k=1,or n, it only delets two numbers α_1, β_1 or α_n, β_{n-1}. So there is a difference betweem them. For simplicity, we call the case $k = 2, 3, \cdots, n-1$, above inverse eigenvalue problem as (k) jacobi matrix inverse eigenvalue problem.We also call (1) Jacobi matrix inverse eigenvalue problem, (n) Jacobi matrix inverse eigenvalue problem when $k = 1, k = n$ respectively.More simple we call them as (k)problem, (1)problem and (n)problem, respectively. In sectin 2, some basic theorems such as secular equation, separation theorem are discussed, and the sufficient and necessary condition for (k) problem has an unique solution, when $T_{1,k-1}$ and $T_{k+1,n}$ have no common eigenvalue, are given. In section 3, a discussion of the special case, when $T_{1,k-1}$ and $T_{k+1,n}$ have common eigenvalues, is given. There is a sufficient and necessary condition for (k) problem. The interesting fact is that in this case, if (k) problem has a solution, then there are infinite solutions. In section 4, an algorithm and numerical examples are put forward.

2. The Basic Theorems

Theorem 1. *Let $T_{1,n} = T_n$ be $n \times n$ unreduced symmetric tridiagonal matrix, whose eigenvalues are $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. The matrix*

$$W_k = \begin{pmatrix} T_{1,k-1} & 0 \\ 0 & T_{k+1,n} \end{pmatrix}$$

is gained by deleting the k^{th} row and the k^{th} column from T_n, for $k = 1, 2, \cdots, n$. Let μ_i,$i = 1, 2, \cdots, k-1$ are the eigenvalues of $T_{1,k-1}$ and the corresponding unit eigenvectors are $S_i^{(1)}$, $i = 1, 2 \cdots, k-1$. Let μ_i, $i = k, k+1, \cdots, n-1$ are the eigenvalues of $T_{k+1,n}$ and the corresponding unit eigenvectors are $S_i^{(2)}$,$i = k, k+1, \cdots, n-1$. Denote the $(k-1)^{th}$ component of $S_i^{(1)}$ to be

$S^{(1)}_{k-1,i}$ *and the first component of* $S^{(2)}_i$ *to be* $S^{(2)}_{1,i}$, *then*

$$\det(\lambda I - T_n) = \prod_{i=1}^{n-1}(\lambda-\mu_i)\left(\lambda - \alpha_k - \sum_{i=1}^{k-1}\frac{(\beta_{k-1}\times S^{(1)}_{k-1,i})^2}{\lambda-\mu_i} - \sum_{i=k}^{n-1}\frac{(\beta_k \times S^{(2)}_{1,i})^2}{\lambda-\mu_i}\right). \qquad (1)$$

Proof. Permute the rows and columns of T_n, such that the k^{th} row and the k^{th} column are changed into the n^{th} row and the n^{th} column. Denote the matrix of the permutation

$$\begin{pmatrix} 1, & 2, & \cdots, & k-1, & k, & k+1, & \cdots, & n \\ 1, & 2, & \cdots, & k-1, & n, & k, & \cdots, & n-1 \end{pmatrix}$$

by P_n, then

$$P_nT_nP_n = \begin{array}{c} \left(\begin{array}{ccccccccc} & & & & & & & & 0 \\ & T_{1,k-1} & & & & & & & \vdots \\ & & & & & & & & 0 \\ & & & & & & & & \beta_{k-1} \\ & & & & & & & & \beta_k \\ & & & & & & & & 0 \\ & & & & & T_{k+1,n} & & & \vdots \\ & & & & & & & & 0 \\ 0 & \cdots & 0 & \beta_{k-1} & \beta_k & 0 & \cdots & 0 & \alpha_k \end{array}\right) \\ \begin{array}{ccccccccc} & & & k-1 & k & & & & \end{array} \end{array} \begin{array}{l} \\ \\ \\ k-1 \\ k \\ \\ \\ \\ \\ \end{array}$$

is denoted by A_n, which is similar to the matrix T_n and whose $n-1$ order leading principal submatrix is just W_k. Let

$$y^T = (0, 0, \cdots, 0, \underset{k-1}{\beta_{k-1}}, \underset{k}{\beta_k}, 0, \cdots, 0)$$

then

$$A_n = \begin{pmatrix} W_k & y \\ y^T & \alpha_k \end{pmatrix}$$

$$\begin{aligned} \det(\lambda I - A_n) &= \det\left[\begin{pmatrix} I & 0 \\ y^T(\lambda I - W_k)^{-1} & I \end{pmatrix}\begin{pmatrix} \lambda I - W_k & -y \\ -y^T & \lambda - \alpha_k \end{pmatrix}\right] \\ &= \det\begin{pmatrix} \lambda I - W_k & -y \\ 0 & \lambda - \alpha_k - y^T(\lambda I - W_k)^{-1}y \end{pmatrix} \end{aligned}$$

therefore

$$\begin{aligned} \det(\lambda I - T_n) &= \det(\lambda I - A_n) = \det(\lambda I - W_k)(\lambda - \alpha_k - y^T(\lambda I - W_k)^{-1}y) \\ &= \prod_{j=1}^{n-1}(\lambda - \mu_j)(\lambda - \alpha_k - y^T(\lambda I - W_k)^{-1}y). \end{aligned}$$

If the $n-1$ linear independent unit eigenvectors of W_k are $x_1, x_2, \cdots, x_{n-1}$,then due to the construction of W_k, for $i = 1, 2, \cdots, k-1, x_i$ are the vectors corresponding to eigenvalue μ_i with the following forms

$$\begin{pmatrix} S^{(1)}_i \\ 0 \end{pmatrix},$$

and for $i = k, k+1, \cdots, n-1$, x_i are the vectors corresponding to eigenvalue μ_i with the following forms

$$\begin{pmatrix} 0 \\ S_i^{(2)} \end{pmatrix}.$$

$(\lambda I - W_k)^{-1}$ can be expressed as

$$(\lambda I - W_k)^{-1} = \sum_{i=1}^{n-1} \frac{1}{\lambda - \mu_i} x_i x_i^T,$$

therefore, $y^T(\lambda I - W_k)^{-1} y = \sum_{i=1}^{n-1} \frac{(x_i^T y)^2}{\lambda - \mu_i}$. Note that $(x_i^T y) = \beta_{k-1} \times S_{k-1,i}^{(1)}$, when $i = 1, 2, \cdots, k-1$, and $(x_i^T y) = \beta_k \times S_{1,i}^{(2)}$, when $i = k, k+1, \cdots, n-1$, so the Theorem is proved.

Theorem 2. *If $T_{1,k-1}$ and $T_{k+1,n}$ have no common eigenvalue,then any root of following equation:*

$$F(\lambda) = \left(\lambda - \alpha_k - \sum_{i=1}^{k-1} \frac{(\beta_{k-1} \times S_{k-1,i}^{(1)})^2}{\lambda - \mu_i} - \sum_{i=k}^{n-1} \frac{(\beta_k \times S_{1,i}^{(2)})^2}{\lambda - \mu_i} \right) = 0. \tag{2}$$

is an eigenvalue of T_n. On the other side any eigenvalue of T_n is a root of the above equation (2). If $T_{1,k-1}$ and $T_{k+1,n}$ have common eigenvalues,then each of such common eigenvalues is an eigenvalue of T_n,and each of the rest eigenvalues of T_n,if and only if is a root of the above equation (2).

For simplicity we call the case 1 that $T_{1,k-1}$ and $T_{k+1,n}$ have no common eigenvalue, while the case 2 that they have common eigenvalues.

Proof. As $T_{1,k-1}, T_{k+1,n}$ are both unreduced symmetric tridiagonal matrices, both of the first elements and the last elements of whose eigenvectors cannot be zero (see [9] and [6], p.55), hence $\beta_{k-1} \times S_{k-1,i}^{(1)} \neq 0$, and $\beta_k \times S_{1,i}^{(2)} \neq 0$. Thus from (1), we know

$$\det(\mu_j I - T_n) = \prod_{i=1 i\neq j}^{n-1} (\mu_j - \mu_i) \times \begin{cases} (\beta_{k-1} \times S_{k-1,j}^{(1)})^2 & \text{when } j < k, \\ (\beta_k \times S_{1,j}^{(2)})^2 & \text{otherwise.} \end{cases} \tag{3}$$

In case 1,obviously,

$$\det(\mu_j I - T_n) \neq 0.$$

It means that any μ_j is not an eigenvalue of T_n. So

$$\det(\lambda I - T_n) = 0$$

if and only if

$$\left(\lambda - \alpha_k - \sum_{i=1}^{k-1} \frac{(\beta_{k-1} \times S_{k-1,i}^{(1)})^2}{\lambda - \mu_i} - \sum_{i=k}^{n-1} \frac{(\beta_k \times S_{1,i}^{(2)})^2}{\lambda - \mu_i} \right) = 0. \tag{4}$$

Hence the first half part of this Theorem has been proved.
In case 2, if μ_j is a common eigenvalue, then $\prod_{i=1 i\neq j}^{n-1} (\mu_j - \mu_i) = 0$, so from (3)

$$\det(\mu_j I - T_n) = 0.$$

namely μ_j is an eigenvalue of T_n. Further more if μ_j is not such common eigenvalue, then $\prod_{i=1 i\neq j}^{n-1} (\mu_j - \mu_i) \neq 0$,so from (3)

$$\det(\mu_j I - T_n) \neq 0.$$

They are not the eigenvalues of T_n. For any eigenvalue λ_j of T_n has to

$$P(\lambda_j) = \prod_{i=1}^{n-1}(\lambda_j - \mu_i) = 0$$

or

$$F(\lambda_j) = 0$$

If $P(\lambda_j) = 0$, it means there is a μ_i, such that $\lambda_j = \mu_i$ and μ_i is a common eigenvalue of $T_{1,k-1}$ and $T_{k+1,n}$. So if $\lambda_j \neq \mu_i, i = 1, 2, \cdots, n-1$, then $F(\lambda_j) = 0$. On the other hand for any λ, if $F(\lambda) = 0$, from Theorem 1, λ is an eigenvalue of T_n, from the construction of $F(\lambda)$ $\lambda \neq \mu_i, i = 1, 2, \cdots, n-1$. So the Theorem is proved.

Let $\mu_i, i = 1, 2, \cdot, n-1$ be reordered as $\mu_{j_i}, i = 1, 2, \cdots, n-1$, such that

$$\mu_{j_1} \leq \mu_{j_2} \leq \mu_{j_3} \cdots \leq \mu_{j_{n-2}} \leq \mu_{j_{n-1}},$$

and if $\mu_{j_i} = \mu_{j_{i+1}}$,then let μ_{j_i} is an eigenvalue of $T_{1,k-1}$ and $\mu_{j_{i+1}}$ is an eigenvalue of $T_{k+1,n}$. So that $j_1, j_2, \cdots, j_{n-1}$, is a unique permutation of $1, 2, \cdots, n-1$. We have following roots separation theorem for Jacobi matrices.

Theorem 3. *For k=1,2,...,n, in the case 1, the following inequalities hold:*

$$\lambda_1 < \mu_{j_1} < \lambda_2 < \mu_{j_2} < \cdots < \mu_{j_{n-1}} < \lambda_n,$$

and in the case 2, above inequalities also hold except for any common eigenvalue $\mu_{j_i} = \mu_{j_{i+1}}$,there is $\mu_{j_i} = \lambda_{i+1} = \mu_{j_{i+1}}$, instead of above inequalities $\mu_{j_i} < \lambda_{i+1} < \mu_{j_{i+1}}$.

Proof. We first consider the case 1, where $\mu_1, \mu_2, \cdots, \mu_{n-1}$ are distinct each other. The eigenvalue of T_n satisfies equation (2)

$$F(\lambda) = 0.$$

We can rewrite $F(\lambda)$ as

$$F(\lambda) = \left(\lambda - \alpha_k - \sum_{i=1}^{n-1} \frac{c_i}{\lambda - \mu_{j_i}}\right), \tag{5}$$

where $c_i > 0, i = 1, 2, \cdots, n-1$.

For sufficient small positive number ϵ,

$$F(\mu_{j_i} - \epsilon) > 0, \quad F(\mu_{j_i} + \epsilon) < 0, \quad i = 1, 2, \cdots, n-1 \tag{6}$$

$$F(-\infty) < 0, \quad \textit{and } F(+\infty) > 0, \tag{7}$$

hence it holds

$$\lambda_1 < \mu_{j_1} < \lambda_2 < \mu_{j_2} < \cdots < \mu_{j_{n-1}} < \lambda_n,$$

Now consider the case 2, if the least common eigenvalue is μ_{j_i} since $T_{1,k-1}$, $T_{k+1,n}$ are both unreduced, the multiple number is at most two.So $\mu_{j_i} = \mu_{j_{i+1}} < \mu_{j_{i+2}}$.
Because

$$F(\mu_{j_s} - \epsilon) > 0, \quad F(\mu_{j_s} + \epsilon) < 0, \quad s = 1, 2, \cdots, i \tag{8}$$

and (4),we have

$$\lambda_1 < \mu_{j_1} < \lambda_2 < \mu_{j_2} < \cdots < \lambda_i < \mu_{j_i}.$$

By Theorem (2),we know $\mu_{j_i} = \mu_{j_{i+1}}$ is a eigenvalue of $T_{1,n}$,and this one must be λ_{i+1}.So we have

$$\lambda_1 < \mu_{j_1} < \cdots < \lambda_i < \mu_{j_i} = \lambda_{i+1} = \mu_{j_{i+1}}.$$

By same argument, we can prove this theorem completely.

Remark: This is a roots separation theorem. When $k = 1$ or $k = n$ this is well known ([6], and [12],p.300). For$k = 2, 3, \cdots, n-1$, this is new and more usefull and we can specially use it in bisection method and divide and conquer method for finding eigenvalues of a Jacobi matrix parallelly. We have discussed this in others papers([7]), ([8]).

Now let's turn to the (k) problem. First

$$\alpha_k = trace(T_{1,n}) - trace(W_k) = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i \tag{9}$$

. So α_k,a diagonal element of T_n, is known. Look at equation $F(\lambda) = 0$ in Theorem 2, where $\lambda_i, i = 1, 2, \cdots, n; \mu_i, i = 1, 2, \cdots, n-1$ and $\beta_{k-1} S^{(1)}_{k-1,j}, j = 1, 2, \cdots, k-1; \beta_k S^{(2)}_{1,j}, j = k, ..., n-1$ are connected. Thus we have following basic theorem for solving (k) prblem.

Theorem 4. *Let*

$$\lambda_1 < \mu_{j_1} < \lambda_2 < \mu_{j_2} < \cdots < \mu_{j_{n-1}} < \lambda_n, \tag{10}$$

then the following linear algebric equations system:

$$\frac{x_1}{\lambda_i - \mu_1} + \frac{x_2}{\lambda_i - \mu_2} + \cdots + \frac{x_{n-1}}{\lambda_i - \mu_{n-1}} = \lambda_i - \alpha_k \tag{11}$$

$$i = 1, 2, \cdots, n,$$

has unique solution $x = (x_1, x_2, \cdots, x_{n-1})$ *and*

$$x_j = -\frac{\prod_{i=1}^{n} (\lambda_i - \mu_j)}{\prod_{i=1, i\neq j}^{n-1} (\mu_i - \mu_j)} > 0 \tag{12}$$

$$j = 1, 2, \cdots, n-1,$$

Proof. The above equations system (11) has n equations and only n-1 unknown,so it is an overdetermined system. We consider the system of the first n-1 equations:

$$\frac{x_1}{\lambda_i - \mu_1} + \frac{x_2}{\lambda_i - \mu_2} + \cdots + \frac{x_{n-1}}{\lambda_i - \mu_{n-1}} = \lambda_i - \alpha_k \tag{13}$$

$$i = 1, 2, \cdots, n-1,$$

Denot A to be the coefficent matrix of system (13),$A = ((\lambda_i - \mu_j)^{-1})$. It is easy to prove

$$det(A) = (\prod_{1\leq i,j\leq n-1} (\lambda_i - \mu_j))^{-1} \times \prod_{1\leq j<i\leq n-1} (\lambda_i - \lambda_j) \times \prod_{1\leq i<j\leq n-1} (\mu_i - \mu_j) \tag{14}$$

From condition (10), $det(A) \neq 0$. So the system (13) has unique solution x. Now we need to prove: this solution x satisfies the n-th equation of (11). For this aim, we will prove the determinant of following $n \times n$ matrix G is equal 0:

$$G = \begin{pmatrix} (\lambda_1-\mu_1)^{-1}, & (\lambda_1-\mu_2)^{-1}, & \cdots, & (\lambda_1-\mu_{n-1})^{-1}, & \lambda_1-\alpha_k \\ (\lambda_2-\mu_1)^{-1}, & (\lambda_2-\mu_2)^{-1}, & \cdots, & (\lambda-2-\mu_{n-1})^{-1}, & \lambda_2-\alpha_k \\ & & \cdots, & & \\ (\lambda_{n-1}-\mu_1)^{-1}, & (\lambda_{n-1}-\mu_2)^{-1}, & \cdots, & (\lambda_{n-1}-\mu_{n-1})^{-1}, & \lambda_{n-1}-\alpha_k \\ (\lambda_n-\mu_1)^{-1}, & (\lambda_n-\mu_2)^{-1}, & \cdots, & (\lambda_n-\mu_{n-1})^{-1}, & \lambda_n-\alpha_k \end{pmatrix}$$

Firstly,

$$det(G) = (\prod_{\substack{1\le i\le n \\ 1\le j\le n-1}} (\lambda_i-\mu_j))^{-1} \times det(G_1),$$

and

$$G_1 = \begin{pmatrix} f_1(\lambda_1) & f_2(\lambda_1) & \cdots & f_{n-1}(\lambda_1) & g(\lambda_1) \\ f_1(\lambda_2) & f_2(\lambda_2) & \cdots & f_{n-1}(\lambda_2) & g(\lambda_2) \\ & & \cdots & & \\ f_1(\lambda_n) & f_2(\lambda_n) & \cdots & f_{n-1}(\lambda_n) & g(\lambda_n) \end{pmatrix}$$

where $f_i(\lambda) = \prod_{j=1}^{n-1}(\lambda-\mu_j)/(\lambda-\mu_i)$ and $g(\lambda) = \prod_{i=1}^{n-1}(\lambda-\mu_i) \times (\lambda-\alpha_k)$. Let $n \times n$ bidiagonal matrix

$$E_1 = \begin{pmatrix} 1, & -1 & 0 & \cdots, & 0 & 0 \\ 0, & 1 & -1 & \cdots, & 0 & 0 \\ , & & & \cdots, & & \\ 0, & 0 & 0 & \cdots, & 1 & -1 \\ 0, & 0 & 0 & \cdots, & 0 & 1 \end{pmatrix}$$

andịagonal matrix

$$D_1 = \begin{pmatrix} (\lambda_1-\lambda_2)^{-1} & 0 & \cdots & 0 & 0 \\ 0 & (\lambda_2-\lambda_3)^{-1} & \cdots & 0 & 0 \\ & & \cdots & & \\ & & \cdots & (\lambda_{n-1}-\lambda_n)^{-1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

then

$$G_2 = D_1E_1G_1 = \begin{pmatrix} f_1(\lambda_1,\lambda_2) & \cdots & f_{n-1}(\lambda_1,\lambda_2) & g(\lambda_1,\lambda_2) \\ f_1(\lambda_2,\lambda_3) & \cdots & f_{n-1}(\lambda_2,\lambda_3) & g(\lambda_2,\lambda_3) \\ & \cdots & & \\ f_1(\lambda_{n-1},\lambda_n) & \cdots & f_{n-1}(\lambda_{n-1},\lambda_n) & g(\lambda_{n-1},\lambda_n) \\ f_1(\lambda_n) & \cdots & f_{n-1}(\lambda_n) & g(\lambda_n) \end{pmatrix}$$

where

$$f_i(\lambda_p,\lambda_q) = (f_i(\lambda_p) - f_i(\lambda_q))/(\lambda_p-\lambda_q)$$

is a difference quotient. By this argument,similarly denote $E_i, D_i, i = 2,3,\cdots,n-1$, as follows:

$$E_i = \begin{pmatrix} E_{1,1}^{(i)} & E_{1,2}^{(i)} \\ E_{2,1}^{(i)} & E_{2,2}^{(i)} \end{pmatrix}$$

where $E_{1,1}^{(i)}$ is an $(n-i)\times(n-i)$ upper biagonal matrix

$$E_{1,1}^{(i)} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ & 1 & -1 & \cdots & 0 & 0 \\ & & & \cdots & & \\ & & & \cdots & 1 & -1 \\ & & & \cdots & & 1 \end{pmatrix}$$

$E_{1,2}^{(i)}$ is an $(n-i)\times i$ matrix which all elements of are 0 except element (n-i,1) is -1.$E_{2.1}^{(i)} = 0$ and $E_{2,2}^{(i)}$ is a $i \times i$ identity matrix I.

$$D_i = \begin{pmatrix} D_{1,1}^{(i)} & 0 \\ O & I \end{pmatrix},$$

where $D_{1,1}^{(i)}$ is an $(n-i)\times(n-i)$ diagonal matrix

$$D_{1,1}^{(i)} = diag((\lambda_1 - \lambda_{1+i})^{-1}, (\lambda_2 - \lambda_{2+i})^{-1}, \cdots, (\lambda_{n-i} - \lambda_n)^{-1}).$$

Then we have

$$G_n = D_{n-1}E_{n-1}\cdots D_1E_1G_1 = \begin{pmatrix} f_1(\lambda_1, \lambda_2 \cdots \lambda_n) & \cdots & g(\lambda_1, \lambda_2 \cdots \lambda_n) \\ f_1(\lambda_2, \lambda_3 \cdots \lambda_n) & \cdots & g(\lambda_2, \lambda_3 \cdots \lambda_n) \\ & & \\ f_1(\lambda_{n-1}, \lambda_n) & & g(\lambda_{n-1}, \lambda_n) \\ f_1(\lambda_n) & & g(\lambda_n) \end{pmatrix}$$

So

$$det(G_1) = \prod_{1\le i<j\le n} (\lambda_i - \lambda_j) \times det(G_n).$$

Let $p_m(\lambda) = \lambda^m$,it is easy to prove that

$$p_m(\lambda_1, \lambda_2, \cdots \lambda_n) = \begin{cases} 0 & \text{if } m < n-1 \\ 1 & \text{if } m = n-1 \\ \sum_{i=1}^n \lambda_i & \text{if } m = n. \end{cases}$$

Hence

$$f_i(\lambda_1, \lambda_2, \cdots, \lambda_n) = 0, i = 1, 2, ..., n-1,$$

and

$$g(\lambda_1, \lambda_2, \cdots, \lambda_n) = \sum_{i=1}^n \lambda_i - \alpha_k - \sum_{i=1}^{n-1} \mu_i = 0.$$

All elements of the first row of G_n are 0,so

$$det(G_n) = 0 \quad and \quad det(G_1) = 0,$$

and $x_i, i = 1, 2, \cdots, n-1$ satify the last equation of system (11) is proved.
Now turn to prove expression (12) of x_i.By Cramer's rule

$$x_1 = det(B)/det(A)$$

where

$$B = \begin{pmatrix} \lambda_1 - \alpha_k & (\lambda_1 - \mu_2)^{-1} & \cdots & (\lambda_1 - \mu_{n-1})^{-1} \\ \lambda_2 - \alpha_k & (\lambda_2 - \mu_2)^{-1} & \cdots & (\lambda_2 - \mu_{n-1})^{-1} \\ & & \cdots & \\ \lambda_{n-1} - \alpha_k & (\lambda_{n-1} - \mu_2)^{-1} & \cdots & (\lambda_{n-1} - \mu_{n-1})^{-1} \end{pmatrix}.$$

$$det(B) = \prod_{\substack{1 \leq \; i \; \leq n-1 \\ 2 \leq \; j \; \leq n-1}} (\lambda_i - \mu_j)^{-1} \times det(B_1)$$

where

$$B_1 = \begin{pmatrix} g(\lambda_1) & f_2(\lambda_1) & \cdots & f_{n-1}(\lambda_1) \\ g(\lambda_2) & f_2(\lambda_2) & \cdots & f_{n-1}(\lambda_2) \\ & & \cdots & \\ g(\lambda_{n-1}) & f_2(\lambda_{n-1}) & \cdots & f_{n-1}(\lambda_{n-1}) \end{pmatrix},$$

and

$$\begin{aligned} g(\lambda) &= \prod_{i=2}^{n-1} (\lambda - \mu_i) \times (\lambda - \alpha_k), \\ f_j(\lambda) &= \prod_{i=2}^{n-1} (\lambda - \mu_i)/(\lambda - \mu_j). \end{aligned}$$

We do difference quotient process as before,then

$$det(B_1) = c_1 \times det(B_{n-1}),$$

where

$$B_{n-1} = \begin{pmatrix} g(\lambda_1, \lambda_2, \cdots, \lambda_{n-1}) & \cdots, & f_j(\lambda_1, \lambda_2, \cdots, \lambda_{n-1}) & \cdots \\ g(\lambda_2, \lambda_3, \cdots, \lambda_{n-1}) & \cdots, & f_j(\lambda_2, \lambda_3, \cdots, \lambda_{n-1}) & \cdots \\ & \cdots, & & \\ g(\lambda_{n-1}) & \cdots, & f_j(\lambda_{n-1}) & \cdots \end{pmatrix}$$

and

$$c_1 = \prod_{1 \leq i < j \leq n-1} (\lambda_i - \lambda_j).$$

Because

$$\begin{aligned} g(\lambda_1, \lambda_2, \cdots, \lambda_{n-1}) &= \sum_{i=1}^{n-1} \lambda_i - \sum_{j=2}^{n-1} \mu_j - \alpha_k = \mu_1 - \lambda_n, \\ f_j(\lambda_1, \lambda_2, \cdots, \lambda_{n-1}) &= 0, \\ j &= 2, 3, \cdots, n-1 \end{aligned}$$

so

$$det(B_1) = c_1 \times (\mu_1 - \lambda_n) \times det(C_1),$$

where

$$C_1 = \begin{pmatrix} f_2(\lambda_2, \lambda_3, \cdots, \lambda_{n-1}) & , \cdots, & f_{n-1}(\lambda_2, \lambda_3, \cdots, \lambda_{n-1}) \\ f_2(\lambda_3, \lambda_4, \cdots, \lambda_{n-1}) & , \cdots, & f_{n-1}(\lambda_3, \lambda_4, \cdots, \lambda_{n-1}) \\ & , \cdots, & \\ f_2(\lambda_{n-1}) & , \cdots, & f_{n-1}(\lambda_{n-1}) \end{pmatrix}$$

Using

$$(\lambda_i - \lambda_{n-1})f_j(\lambda_i, \lambda_{i+1}, \cdots, \lambda_{n-1}) + f_j(\lambda_{i+1}, \lambda_{i+2}, \cdots, \lambda_{n-1}) = f_j(\lambda, \cdots, \lambda_{n-2})$$

we have

$$C_2 = H_1 \times C_1 = \begin{pmatrix} f_2(\lambda_2, \lambda_3, \cdots, \lambda_{n-2}) & , \cdots, & f_{n-1}(\lambda_2, \lambda_3, \cdots, \lambda_{n-2}) \\ f_2(\lambda_3, \lambda_4, \cdots, \lambda_{n-2}) & , \cdots, & f_{n-1}(\lambda_3, \lambda_4, \cdots, \lambda_{n-2}) \\ & , \cdots, & \\ f_2(\lambda_{n-2}) & , \cdots, & f_{n-1}(\lambda_{n-2}) \\ f_2(\lambda_{n-1} & , \cdots, & f_{n-1}(\lambda_{n-1}) \end{pmatrix}$$

where

$$H_1 = \begin{pmatrix} (\lambda_2 - \lambda_{n-1}) & 1 & 0 & \cdots & 0 & 0 \\ 0 & (\lambda_3 - \lambda_{n-1}) & 1 & \cdots & 0 & 0 \\ & & \ddots & & & \\ & & & & (\lambda_{n-2} - \lambda_{n-1}) & 1 \\ & & & & 0 & 1 \end{pmatrix}.$$

By the same argument,at last we have

$$det(C_1) = \prod_{2 \le j < i \le n-1} (\lambda_i - \lambda_j) \times \prod_{2 \le i < j \le n-1} (\mu_i - \mu_j) / \prod_{2 \le i < j \le n-1} (\lambda_i - \lambda_j)$$

and

$$det(B) = (-1)^n \times \prod_{1 \le j < i \le n-1} (\lambda_i - \lambda_j) \times \prod_{1 \le i < j \le n-1} (\mu_i - \mu_j) \times (\mu_1 - \lambda_n) / \prod_{\substack{1 \le i \le n-1 \\ 2 \le j \le n-1}} (\lambda_i - \mu_j)$$

Therefore

$$x_1 = det(B)/det(A) = -\prod_{i=1}^{n} (\lambda_i - \mu_1) / \prod_{i=2}^{n} (\mu_i - \mu_1).$$

Similarly,we can obtain

$$\begin{aligned} x_j &= -\prod_{i=1}^{n} (\lambda_i - \mu_j) / \prod_{i=1, \neq j}^{n} (\mu_i - \mu_j). \\ j &= 1, 2 \cdots, n-1 \end{aligned}$$

It is easy to verify

$$\begin{aligned} x_j &> 0 \\ j &= 1, 2, \cdots, n-1. \end{aligned}$$

Theorem 5. *If there is no common number between* $\mu_1, \mu_2, \cdots, \mu_{k-1}$ *and* $\mu_k, \mu_{k+1}, \cdots, \mu_{n-1}$*, then the necessary and sufficient condition of the (k) problem having a solution is*

$$\lambda_1 < \mu_{j_1} < \lambda_2 < \mu_{j_2} < \cdots < \mu_{j_{n-1}} < \lambda_n, \tag{15}$$

Furthermore, if a given (k) problem has a solution, then the solution is unique.

Proof. The necessity has been proved in Theorem 3. Now we prove that it is sufficient too. Under this condition the equations system (11) has a unique solution $x_i i = 1, 2, \cdots, n-1$. Let

$$\beta_{k-1} = \sqrt{\sum_{i=1}^{k-1} x_i}, \tag{16}$$

$$\beta_k = \sqrt{\sum_{i=k}^{n-1} x_i}, \tag{17}$$

then

$$S^{(1)}_{k-1,j} = \sqrt{x_j}/\beta_{k-1}, j = 1, 2, \cdots, k-1 \tag{18}$$

$$S^{(2)}_{1,i} = \sqrt{x_i}/\beta_k, i = k, k+1, \cdots, n-1 \tag{19}$$

where $S^{(1)}_{k-1,j}$ is the last element of the unit eigenvector of $T_{1,k-1}$, corresponding to the eigenvalue μ_j,and $S^{(2)}_{1,i}$ is the first element of the unit eigenvector of $T_{k+1,n}$, corresponding to eigenvalue μ_i.

Let g_{k-1} be a $(k-1) \times 1$ vector ,its j-th element is $S^{(1)}_{k-1,j}$, then it is well known that by $\mu_1, \mu_2, \cdots, \mu_{k-1}$ and g_{k-1} can construct the matix $T_{1,k-1}$ uniquely [10] [3].

Similarly let h_1 be an $(n-k) \times 1$ vector,its i-k+1-th element is $S^{(2)}_{1,i}$, then by $\mu_k, \mu_{k+1}, \cdots, \mu_{n-1}$ and h_1 we can construct the matrix $T_{k+1,n}$ [10]. So after remembering

$$\alpha_k = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i$$

the matrix $T_{1,n}$ is constructed completely.

Since (13) has unique solution, it is easy to know that the solution of (k) problem is also unique.

3. Solve (k) Problem in Case 2

Given three real numbers sets

$$\begin{aligned} S1 &= \{\lambda_1, \lambda_2, \cdots, \lambda_n\}, \\ S2 &= \{\mu_1, \mu_2, \cdots, \mu_{k-1}\} \\ \textit{and} \quad S3 &= \{\mu_k, \mu_{k+1}, \cdots, \mu_{n-1}\} \end{aligned}$$

each set $Si, i = 1, 2, 3$ has diferent elements, if $S2$ and $S3$ have common elements, we want to find a matrix $T_{1,n}$ such that the eigenvalues set of $T_{1,n}$ is $S1$, and the eigenvalues sets of submatrix $T_{1,k-1}$ and $T_{k+1,n}$ are $S2$ and $S3$ respectively. For simplicity we consider only one common element for example $\mu_1 = \mu_k$. By Theorem 2, one and only one eigenvalue of $T_{1,n}$,λ_q, is equal to μ_1. The elements of $S1, S2, S3$ must have the separation property of Theorem 3. Firstly $\alpha_k = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i$ is same with the case 1. Furthermore we have

$$F(\lambda_i) = 0, i = 1, 2, \cdots, q-1, q+1, \cdots, n.$$

Consider equation (11) for $i \neq q$,because the coefficient of x_1 is equal to the coefficient of x_k, namely

$$1/(\lambda_i - \mu_1) = 1/(\lambda_i - \mu_k),$$

so the equation can rewrite as

$$\frac{x1+xk}{\lambda_i-\mu_1}+\sum_{p=2}^{k-1}\frac{x_p}{\lambda_i-\mu_p}+\sum_{p=k+1}^{n-1}\frac{x_p}{\lambda_i-\mu_p} = \lambda_i-\alpha_k$$
$$i=1,2,\cdots,q-1,q+1,\cdots,n$$

Let $y_1=x_1+x_k$ and $y_i=x_i, i=2,3,\cdots,k-1,k+1,\cdots,n-1$, then we have equations system

$$\frac{y_1}{\lambda_i-\mu_1}+\sum_{p=2}^{k-1}\frac{y_p}{\lambda_i-\mu_p}+\sum_{p=k+1}^{n-1}\frac{y_p}{\lambda_i-\mu_p} = \lambda_i-\alpha_k \tag{20}$$
$$i=1,2,\cdots,q-1,q+1,\cdots,n. \tag{21}$$

This is a system of n-1 equations, with n-2 unknowns and it is reduced to equation(11), in the case 1. By Theorem 4 it has unique solution $(y_1,y_2,\cdots,y_{k-1},y_{k+1},\cdots,y_{n-1})$ and all its elements are positive.
Now for any $\theta\in(0,1)$, let $x_1=\theta y_1, x_k=(1-\theta)y_1$ and $x_i=y_i, i=2,3,\cdots,k-1,$ $k+1,\cdots,n-1$. We compute

$$(\beta_{k-1})^2 = \sum_{i=1}^{k-1}x_i \tag{22}$$
$$(\beta_k)^2 = \sum_{i=k}^{n-1}x_i \tag{23}$$
$$S^{(1)}_{k-1,i} = \sqrt{x_i/(\beta_{k-1})^2}, i=1,\cdot,k-1, \tag{24}$$
$$S^{(2)}_{1,i} = \sqrt{x_i/(\beta_k)^2}, i=k,\cdots,n-1. \tag{25}$$

By $\mu_1,\mu_2,\cdots,\mu_{k-1}$ and $S^{(1)}_{k-1,i}i=1,\cdots,k-1$,we can construct a matrix $T_{1,k-1}$ and by $\mu_k,\mu_{k+1},\cdots,\mu_{n-1}$ and $S^{(2)}_{1,i}, i=k,\cdots,n-1$, we can construct a matrix $T_{k+1,n}$. From $T_{1,k-1}$, $T_{k+1,n}$, α_k,β_{k-1}, and β_k we obtain $T_{1,n}$.
Obviousely, the eigenvalues set of $T_{1,n},T_{1,k-1}$ and $T_{k+1,n}$ are $S1,S2$, and $S3$ respectively.
So we have proved the following theorem:

Theorem 6. *Given three real numbers sets*

$$\begin{aligned} S1 &= \{\lambda_1,\lambda_2,\cdots,\lambda_n\},\\ S2 &= \{\mu_1,\mu_2,\cdots,\mu_{k-1}\}\\ \textit{and}\qquad S3 &= \{\mu_k,\mu_{k+1},\cdots,\mu_{n-1}\}\end{aligned}$$

that each set has diferent elements and

$$\lambda_1<\lambda_2<\cdots<\lambda_{n-1}<\lambda_n,$$

$$\mu_{j_1}\le\mu_{j_2}\le\mu_{j_3}\cdots\le\mu_{j_{n-2}}\le\mu_{j_{n-1}},$$

where $(j_1,j_2,\cdots,j_{n-1})$ *is a permutation of* $(1,2,\cdots,n-1)$, *if* $\mu_{j_q}=\mu_{j_{q+1}}$, *then the sufficient and necessary condition which the (k) problem has a solution is that the following strict separation:*

$$\lambda_1<\mu_{j_1}<\lambda_2<\mu_{j_2}<\cdots<\mu_{j_{n-1}}<\lambda_n, \tag{26}$$

holds except $\mu_{j_q} = \lambda_{q+1} = \mu_{j_{q+1}}$*,instead of above* $\mu_{j_q} < \lambda_{q+1} < \mu_{j_{q+1}}$*. Furthermore if the (k) problem has a solution then there are infinite solutions.*

4. Algorithm and Numerical Examples

Summarize above discussion,we can give following algorithm for solving (k) problem:

Step 1 Find α_k from (9).
Step 2 Find $x = (x_1, x_2, \cdots, x_{n-1})$ by solving the system (13) or from (12) in case 1.
In case 2, first merge some x_i to be a vector y,and find y by (13) or (12).
Then split corresponding elements of y and obtain x as show in section 3.
Step 3 computeβ_{k-1} and $\beta_k$from (16),(17).
Step 4 compute $S^{(1)}_{k-1,i}, i = 1, 2, \cdots, k-1$ and $S^{(2)}_{1,i} i = k, k+1, \cdots, n-1$.
from (18),(19).
Step 5 Compute $T_{1,k-1}$ from $S^{(1)}_{k-1,i}, i = 1, 2, \cdots, k-1$ and $\mu_1, \mu_2, \cdots, \mu_{k-1}$
by Lanczos Process or Givens Orthogonal Reduction Process [3],[5] .
Compute $T_{k+1,n}$ from $S^{(2)}_{1,i}, i = k, k+1, \cdots, n-1$ and $\mu_k, \mu_{k+1}, \cdots, \mu_{n-1}$
by Lanczos Process or Givens Orthogonal Reduction Process.

Example 1.
Let

$$T_{1,9} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 8 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 \end{pmatrix}$$

Its eigenvalues are

$$\begin{array}{lll} \lambda_1 = 0.25380581710031 & \lambda_2 = 1.78932135473495 \\ \lambda_3 = 2.96105907080106 & \lambda_4 = 3.99605612592861 \\ \lambda_5 = 5.00000000000000 & \lambda_6 = 6.00394387407139 \\ \lambda_7 = 7.03894092919895 & \lambda_8 = 8.21067864526505 \\ \lambda_9 = 9.74619418289969 \end{array}$$

Pick k=5,and delet 5th row and 5th column from $T_{1,9}$. There are two submatrices $T_{1,4}$ and $T_{6,9}$.The eigenvalues of them are

$\mu_1 = 0.25471875982586$ $\mu_2 = 1.82271708088711$
$\mu_3 = 3.17728291911289$ $\mu_4 = 4.74528124017414$
and
$\mu_5 = 5.25471875982586$ $\mu_6 = 6.82271708088711$
$\mu_7 = 8.17728291911289$ $\mu_8 = 9.74528124017414$
respectively.
Now we reconstruct a Jacobi matrix by these eigenvalues according to the above algorithm.
Step 1.$\alpha_5 = \sum_{i=1}^{9} \lambda_i - \sum_{i=1}^{8} \mu_i = 5.00000000000000$

Step 2.After solving the equation system (13),obtain

$$\begin{array}{ll} x_1 = 0.00390189193968 & x_2 = 0.08478217124168 \\ x_3 = 0.30610888364033 & x_4 = 0.60520705317831 \\ x_5 = 0.60520705317831 & x_6 = 0.30610888364033 \\ x_7 = 0.08478217124167 & x_8 = 0.00390189193068 \end{array}$$

Step 3.Compute

$$(\beta_4)^2 = \sum_{i=1}^{4} x_i = 1.00000000000001 \text{and } (\beta_5)^2 = \sum_{i=5}^{8} x_i = 0.99999999999999.$$

Obtain

$$\beta_4 = 1.00000000000000 \quad \beta_5 = 1.00000000000000$$

Step 4.Computed $S_{4,i}^{(1)}$ and $S_{1,i}^{(2)}$ as follows

$$\begin{array}{ll} S_{4,1}^{(1)} = 0.06246512578778 & S_{4,2}^{(1)} = 0.29117378185832 \\ S_{4,3}^{(1)} = 0.55327107609230 & S_{4,4}^{(1)} = 0.77795054674337 \end{array}$$

and

$$\begin{array}{ll} S_{1,5}^{(2)} = 0.77795054674337 & S_{1,6}^{(2)} = 0.55327107609230 \\ S_{1,7}^{(2)} = 0.29117378185831 & S_{1,8}^{(2)} = 0.06246512578775 \end{array}$$

Step 5.From μ_i and $S_{4,i}^{(1)} i = 1, 2, 3, 4$ we compute $T_{1,4}$ and from $\mu_i and S_{1,i}^{(2)}, i = 5, 6, 7, 8$ compute $T_{6,9}$ by Lanczos Process. At last we get $T_{1,9}$ reconstruction as follows:

α_i	β_i	i
0.99999999999997	1.00000000000001	1
2.00000000000004	1.00000000000001	2
2.99999999999994	1.00000000000001	3
4.00000000000002	1.00000000000000	4
5.00000000000000	1.00000000000000	5
5.99999999999994	0.99999999999997	6
6.99999999999996	0.99999999999987	7
7.99999999999963	0.99999999999987	8
9.00000000000051		9

Example 2.
Given spectral data:

$$\begin{aligned} S1 &= \{0.98044571894161, 1.34987354061316, 1.81383673188837, 2.00000000000000 \\ &\quad 2.78435327623025, 3.41147477897885, 5.66001595334776\} \\ S2 &= \{1.00000000000000, 2.00000000000000, 3.00000000000000\} \\ \text{and} & \\ S3 &= \{1.50000000000000, 2.00000000000000, 3.50000000000000\} \end{aligned}$$

we construct a Jacobi matrix $T_{1,7}$.
Because S2,S3 have common element 2.00000000000000, so it is (k) problem in case 2. The

data satisfy the sufficient and necessary condition of Theorem 6.
The algorithm of case 2 is different from that of case 1 only at step 2. In step 2, consider a new (k) problem:

$$\begin{aligned} S'1 &= \{0.98044571894161, 1.34987354061316, 1.81383673188837,\\ &\quad 2.78435327623025, 3.41147477897885, 5.66001595334776\} \\ S'2 &= \{1.00000000000000, 2.00000000000000, 3.00000000000000\} \\ \text{and} \quad S'3 &= \{1.50000000000000, 3.50000000000000\} \end{aligned}$$

Solve the system (13) under S'1,S'2.S'3,obtain the solution

$$\begin{array}{lll} y_1 = 0.04465819873852 & y_2 = 0.66666666666666 & y_3 = 0.62200846792815 \\ y_4 = 0.33333333333333 & y_5 = 0.33333333333334 & \end{array}$$

Then let

$$\begin{array}{lll} x_1 = y_1 & x_2 = \theta * y_2 & x_3 = y_3 \\ x_4 = y_4 & x_5 = (1-\theta) * y_2 & x_6 = y_5. \end{array}$$

We pick 2 kind of θ, one is $\theta = 0.5$ and the other is $\theta = 0.4$. After running Step 3, Step 4 and Step 5, we get 2 Jacobi matrices $Tp_{1,7}$ for $\theta = 0.5$ and $Tq_{1,7}$ for $\theta = 0.4$
The α_i and β_i of $Tp_{1,7}$ are as follows:

α_i	β_i	i
1.42264973081038	0.57735026918963	1
1.99999999999999	0.57735026918963	2
2.57735026918963	1.00000000000000	3
5.00000000000000	1.00000000000000	4
2.33333333333335	0.84983658559880	5
2.78205128205126	0.39970403251589	6
1.88461538461540		7

The α_i and β_i of $Tq_{1,7}$ are as follows:

α_i	β_i	i
1.46706128997881	0.57563959796522	1
1.91434913588945	0.57587555344990	2
2.61858957413174	0.96609178307930	3
5.00000000000000	1.03279555898865	4
2.31250000000000	0.82679728470769	5
2.81607142857142	0.41991252733426	6
1.87142857142857		7

Comments
1. The numerical results are get by Matlab 5.2
2. The algorithm can also use to solve (1) problem and (n) problem.

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