Journal of Computational Mathematics, Vol.20, No.1, 2002, 15–22.

LINEAR FINITE ELEMENT APPROXIMATIONS FOR THE TIMOSHENKO BEAM AND THE SHALLOW ARCH PROBLEMS*1)

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Abstract

In this paper we discuss the linear finite element approximations for the Timoshenko beam and the shallow arch problems with shear dampening and reduced integration. We derive directly the optimal order error estimates uniformly with the small thickness parameter, without relying on the theory of saddle point problems.

Key words: Timoshenko beam, Shallow arch, Shear dampening, Reduced integration.

1. Introduction

In this paper we examine the linear finite element discretization of the Timoshenko beam and the shallow arch problems. The thickness of beam and arch appears parametrically in the model and locking phenomenon may occur due to the small parameter. In [1] Arnold proposed reduced integration formulation for the Timoshenko beam and in [6] Reddy applied the method for the shallow arch problem. In these papers, the problems are analyzed based on the equivalence of reduced integration formulation and a mixed formulation, and thus a major role is played by the theory of saddle point problems.

In this paper we modify the formulation in [1] with shear dampening for the Timoshenko beam and in [6] with shear and axial dampening for the shallow arch problem. We prove directly the uniform convergence with respect to small parameter without relying on the saddle point theory. The proofs of error estimates are elementary, without relying on the theory of the suddle point problem. In the next section, we propose and analyze a family of linear element schemes to solve a Timoshenko beam problem. Additionally, we also derive the explicit formulation of the exact solution, which can be used to evaluate the performance of the numerical methods. In Section 3, we propose and analyze a family of linear element schemes to solve a shallow arch problem.

2. The Timoshenko Beam

2.1. The Model

Following [1], we consider the following variational formulation for a Timoshenko beam model problem: find $(\phi, w) \in H_0^1(I) \times H_0^1(I)$ such that

$$(\phi', \psi') + \frac{1}{\varepsilon}(\phi - w', \psi - v') = (g, v), \quad \forall (\psi, v) \in H_0^1(I) \times H_0^1(I), \tag{2.1}$$

* Received October 24, 1998.

1) The work was supported in part by National Natural Science Foundation (Grand No. 10001029) of China and in part by the Hong Kong RGC Grant.

where $H_0^1(I)$ is the usual Sobolev space, $I = (0, 1)$, $(,\cdot,\cdot)$ is the $L^2(I)$ inner product, and the parameter $0 < \varepsilon \ll 1$. In the model (2.1), ϕ, w and ε are proportional to the rotation, the vertical displacement and the thickness of the beam, respectively. Denote the shear strain variable

$$\sigma = \frac{1}{\varepsilon}(\phi - w'). \tag{2.2}$$

It is proved in [1] that there exists a unique solution of (2.1) and there is a constant C independent of g and ε such that

$$\|\phi\|_2 + \|w\|_2 + \|\sigma\|_1 \leq C\|g\|_0. \tag{2.3}$$

It is well known that the locking phenomenon occurs when the standard finite element method is applied to the equation (2.1) directly. An effective approach to eliminate the locking phenomenon is the use of the reduced integration technique, see [1]. This method is based on the equvalent formulation of the mixed finite element method. Thus the error estimates were obtained by the saddle point theory. Here we will modify the formulation with shear dampening and obtain the uniform convergence with respect to the small parameter directly.

2.2. Linear Element Scheme with Shear Dampening

First we assume that the interval $[0, 1]$ is partitioned into subintervals $I_e = [x_e, x_{e+1}], 0 \leq e \leq N - 1$,

$$0 = x_0 < x_1 < \cdots < x_N = 1.$$

Denote $h_e = x_{e+1} - x_e, h = \max_e h_e$, and suppose that the mesh refinements are quasi-uniform in the sense that there exists a constant $\theta > 0$ such that $\min_e h_e / \max_e h_e \geq \theta$. Define the linear finite element space

$$W_h = \{v \in H_0^1(I) : v|_{I_e} \in P_1(I_e), e = 0, 1, \cdots, N-1\} \tag{2.4}$$

and an auxiliary space

$$Q_h = \{q \in L^2(I) : q|_{I_e} \in P_0(I_e), e = 0, 1, \cdots, N-1\}, \tag{2.5}$$

where $P_k(I_e)$ denotes the space of all the polynomials of degree less than or equal to k on the interval I_e.

The approximation problem is to find $(\phi_h, w_h) \in V_h \times V_h$ such that

$$(\phi_h', \psi_h') + \frac{1}{\varepsilon + \alpha_0 h^2}(\pi_h(\phi_h - w_h'), \pi_h(\psi_h - v_h')) = (g, v_h), \quad \forall (\psi_h, v_h) \in V_h \times V_h, \tag{2.6}$$

where $\pi_h : L^2(I) \to Q_h$ is an orthogonal projection operator, i.e., for $q \in L^2(I)$,

$$\pi_h q|_{I_e} = \frac{1}{h_e}\int_{I_e} q\, dx, \tag{2.7}$$

and $\alpha_0 > 0$ is a constant independent of ε and h. Obviously, (2.6) has an unique solution.

Remark 2.1. The scheme is actually used the one-point Gaussian quadrature to compute the term $(\phi - w', \psi - v')$. We write the operator π_h for convenience of notation.

Remark 2.2. $\alpha_0 = 0$ is the case of the formulation of [1] and $\alpha_0 = \frac{1}{12}$ is the same as Petrov-Galerkin formulation in [5] with a slightly different right-hand side term. Compared with [4], we remove the bubble function.

2.3. Error Estimates

We introduce an approximation of the shear strain by the formula

$$\sigma_h = (\varepsilon + \alpha_0 h^2)^{-1}\pi_h(\phi_h - w_h'). \tag{2.8}$$

Theorem 2.1. *Let (ϕ, w) and (ϕ_h, w_h) be the solution of the problem (2.1) and (2.6), respectively, then*

$$\|\phi' - \phi_h'\|_0 + \|w' - w_h'\|_0 \leq C\, h\, \|g\|_0 \tag{2.9}$$

for some constant C independent of h and ε but dependent on α_0.

Proof. From (2.1), (2.2), (2.6) and (2.8), we get the relations

$$(\phi' - \phi_h', \psi_h') + (\sigma - \sigma_h, \psi_h) = 0, \quad \forall \psi_h \in V_h, \tag{2.10}$$

$$(\sigma - \sigma_h,\, v_h') = 0, \quad \forall v_h \in V_h. \tag{2.11}$$

Denote $I_h : H_0^1(I) \to V_h$ the standard linear interpolation operator, i.e., for $u \in H_0^1(I),\ I_h u \in V_h$ is defined by the interpolation conditions

$$(I_h u)(x_i) = u(x_i), \quad i = 0, 1, \cdots, N.$$

Error estimation of the interpolation is discussed in detail in [2, 3]. Here we need the following interpolation error estimate

$$\|u - I_h u\|_0 + h\|u' - (I_h u)'\|_0 \leq C\, h^2\, \|u\|_2. \tag{2.12}$$

Then

$$\begin{aligned} \|\phi' - \phi_h'\|_0^2 = &\ (\phi' - \phi_h',\, \phi' - (I_h\phi)') + (\phi' - \phi_h', (I_h\phi)' - \phi_h') \\ &\leq C\, h\, \|\phi''\|_0\, \|\phi' - \phi_h'\|_0 + (\phi' - \phi_h',\, (I_h\phi)' - \phi_h'). \end{aligned} \tag{2.13}$$

From (2.10), we have

$$(\phi' - \phi_h', (I_h\phi)' - \phi_h') = (\sigma - \pi_h\sigma,\, \phi_h' - (I_h\phi)') + (\pi_h\sigma - \sigma_h, \pi_h(\phi_h - I_h\phi)). \tag{2.14}$$

By definition (2.2) and (2.8),

$$\begin{aligned} \pi_h\phi_h &= (\varepsilon + \alpha_0\, h^2)\sigma_h + \pi_h w_h', \\ \pi_h(I_h\phi) &= \varepsilon\pi_h\sigma + \pi_h w' + \pi_h(I_h\phi - \phi). \end{aligned} \tag{2.15}$$

It is easy to prove that

$$\begin{aligned} &\pi_h w' = \pi_h\, (I_h w)', \quad \forall w \in H_0^1(I) \\ &\|\pi_h w\|_0 \leq \|w\|_0, \quad \|w - \pi_h w\|_0 \leq C\, h\, |w|_1. \end{aligned} \tag{2.16}$$

Thus

$$\begin{aligned} &(\pi_h\sigma - \sigma_h, \pi_h(\phi_h - I_h\phi)) \\ &\quad = (\pi_h\sigma - \sigma_h, (\varepsilon + \alpha_0 h^2)\sigma_h - \varepsilon\pi_h\sigma) + (\pi_h\sigma - \sigma_h, \pi_h w_h' - \pi_h w') + (\pi_h\sigma - \sigma_h,\, \pi_h(I_h\phi - \phi)) \\ &\quad \leq -(\varepsilon + \alpha_0\, h^2)\|\pi_h\sigma - \sigma_h\|_0^2 + \alpha_0\, h^2\, \|\pi_h\sigma - \sigma_h\|_0\, \|\pi_h\sigma\|_0 + C\, h^2\, \|\pi_h\sigma - \sigma_h\|_0\, |\phi|_2 \end{aligned} \tag{2.17}$$

where we have applied the relation (2.14)–(2.16) and (2.11) for the third term in the right hand of (2.17). From (2.13), (2.17) and regularity (2.3), we get

$$\begin{aligned} \|\phi' - \phi_h'\|_0^2 \quad &+ (\varepsilon + \alpha_0\, h^2)\|\pi_h\sigma - \sigma_h\|_0^2 \\ &\leq C\, h\, \|g\|_0 \{\|\phi' - \phi_h'\|_0 + \|\phi_h - I_h\phi\|_0\} + C\, h^2\, \|g\|_0\, \|\pi_h\sigma - \sigma_h\|_0 \\ &\leq C\, h\, \|g\|_0\, \|\phi' - \phi_h'\|_0 + C\, h^2\, \|g\|_0^2 + C\, h^2\, \|g\|_0\, \|\pi_h\sigma - \sigma_h\|_0. \end{aligned} \tag{2.18}$$

So by the Cauchy inequality, we obtain

$$\|\phi' - \phi_h'\|_0 + \sqrt{\varepsilon + \alpha_0\, h^2}\, \|\pi_h\sigma - \sigma_h\|_0 \leq C\, h\, \|g\|_0. \tag{2.19}$$

Thus

$$\begin{aligned} \|w' - w_h'\|_0 \quad &\leq \|\phi - \phi_h\|_0 + \|\varepsilon\, \sigma - (\varepsilon + \alpha_0\, h^2)\sigma_h\|_0 \\ &\leq C\, h\, \|g\|_0 + \varepsilon\, h\|\sigma\|_1 + h^2\|\sigma\|_0 + (\varepsilon + \alpha_0\, h^2)\, \|\pi_h\sigma - \sigma_h\|_0 \\ &\leq C\, h\, \|g\|_0. \end{aligned} \tag{2.20}$$

This completes the proof of the error estimate (2.9).

2.4. The Explicit Formulation of the Exact Solution

In this subsection, we derive an explicit formulation for the exact solution of the Timoshenko beam problem (2.1). For $0 < x < 1$, let

$$\xi = \frac{t}{x}, \quad \eta = \frac{t-x}{1-x}. \tag{2.21}$$

Define

$$v_1(t) = \begin{cases} f_1(\varepsilon)\ \xi + (1 - f_1(\varepsilon))(3\xi^2 - 2\xi^3), & 0 \le t \le x, \\ f_2(\varepsilon)(1-\eta) + (1 - f_2(\varepsilon))(1 - 3\eta^2 + 2\eta^3), & x \le t \le 1, \end{cases} \tag{2.22}$$

$$\psi_1(t) = \begin{cases} (1 - f_1(\varepsilon))(6\xi - 6\xi^2)/x, & 0 \le t \le x, \\ (1 - f_2(\varepsilon))(-6\eta + 6\eta^2)/(1-x), & x \le t \le 1, \end{cases} \tag{2.23}$$

and

$$v_2(t) = \begin{cases} x[\frac{1}{2} f_1(\varepsilon)(-\xi + \xi^2) + (1 - f_1(\varepsilon))(-\xi^2 + \xi^3)], & 0 \le t \le x, \\ (1-x)[\frac{1}{2} f_2(\varepsilon)(\eta - \eta^2) + (1 - f_2(\varepsilon))(\eta - 2\eta^2 + \eta^3)], & x \le t \le 1, \end{cases} \tag{2.24}$$

$$\psi_2(t) = \begin{cases} f_1(\varepsilon)\xi + (1 - f_1(\varepsilon))(-2\xi + 3\xi^2), & 0 \le t \le x, \\ f_2(\varepsilon)(1-\eta) + (1 - f_2(\varepsilon))(1 - 4\eta + 3\eta^2), & x \le t \le 1, \end{cases} \tag{2.25}$$

where

$$f_1(\varepsilon) = \frac{12\varepsilon}{x^2 + 12\varepsilon}, \qquad f_2(\varepsilon) = \frac{12\varepsilon}{(1-x)^2 + 12\varepsilon}.$$

are the local weighting functions used in the Petrov-Galerkin method in [5] as the subvision of the domain is only two elements $I_1 = (0, x)$ and $I_2 = (x, 1)$. From the results of [5], we know that for $i = 1, 2$,

$$\psi_i'' + \varepsilon^{-1}(v_i' - \psi) = 0, \quad in \quad I_1 \quad and \quad I_2,$$
$$\varepsilon^{-1}(v_i'' - \psi_i') = 0, \quad in \quad I_1 \quad and \quad I_2.$$

Substituting $(v_i, \psi_i) \in H_0^1(I) \times H_0^1(I)$, $(i = 1, 2)$ into equation (2.1) and integrating by parts, we derive

$$\begin{aligned} c_{11}\phi(x) + c_{12}w(x) = g_1, \\ c_{21}\phi(x) + c_{22}w(x) = g_2, \end{aligned} \tag{2.26}$$

where

$$c_{11} = \frac{6}{(1-x)^2 + 12\varepsilon} - \frac{6}{x^2 + 12\varepsilon}, \quad c_{12} = \frac{12}{x(x^2 + 12\varepsilon)} + \frac{12}{(1-x)((1-x)^2 + 12\varepsilon)}$$

$$c_{21} = \frac{4}{x(1-x)} - 3\varepsilon\, c_{12}, \qquad c_{22} = c_{11},$$

$$g_1 = \textstyle\int_0^1 g(t) v_1(t)\, dt, \qquad g_2 = \int_0^1 g(t) v_2(t)\, dt.$$

Theorem 2.2. *The linear system (2.26) has an unique solution* $(\phi(x),\ w(x))$.

Proof. It is tedious to compute the determinant

$$\delta = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}$$

and show its value is non-zero. Instead, we let

$$\begin{cases} s1(x) = \phi(x) + \dfrac{2}{1-x} w(x), \\ \\ s_2(x) = -\phi(x) + \dfrac{2}{x} w(x), \end{cases} \tag{2.27}$$

and transform (2.26) into a linear system for $s_1(x)$ and $s_2(x)$:

$$\begin{cases} \dfrac{6}{(1-x)^2+12\varepsilon}s_1(x)+\dfrac{6}{x^2+12\varepsilon}s_2(x)=g_1, \\ \\ (\dfrac{1}{x}+\dfrac{3(1-x)}{(1-x)^2+12\varepsilon})s_1(x)+(-\dfrac{1}{1-x}-\dfrac{3x}{x^2+12\varepsilon})s_2(x)=g_2, \end{cases} \tag{2.28}$$

The transformation (2.27) is invertible since the associated determinant

$$\delta_2=\begin{vmatrix} 1 & \dfrac{2}{1-x} \\ \\ -1 & \dfrac{2}{x} \end{vmatrix}=\frac{2}{x(1-x)}\neq 0.$$

The determinant of the system (2.28) is

$$\delta_1=\begin{vmatrix} \dfrac{6}{(1-x)^2+12\varepsilon} & \dfrac{6}{x^2+12\varepsilon} \\ \\ \dfrac{1}{x}+\dfrac{3(1-x)}{(1-x)^2+12\varepsilon} & -\dfrac{1}{1-x}-\dfrac{3x}{x^2+12\varepsilon} \end{vmatrix}=-\frac{6(1+12\varepsilon)}{x(1-x)(x^2+12\varepsilon)((1-x)^2+12\varepsilon)}\neq 0.$$

Thus the linear system (2.28) has an unique solution, which in trun implies the linear system (2.26) has an unique solution.

For the uniform load case: $g(x)=1, 0\le x\le 1$. By (2.26) we know

$$g_1=\frac{1}{2}x+\frac{1}{2}(1-x)=\frac{1}{2},\quad g_2=-\frac{1}{12}x^2+\frac{1}{12}(1-x)^2=\frac{1}{12}(1-2x).$$

From (2.26), we obtain

$$\phi(x)=\frac{1}{12}x(1-x)(1-2x),$$
$$w(x)=\frac{1}{24}x^2(1-x)^2+\frac{\varepsilon}{2}x(1-x).$$

2.5. The Limit Case

We now consider the limiting case $\varepsilon\to 0$. Denote $(\phi_0(x),w_0(x))$ the limit of $(\phi(x),w(x))$ as $\varepsilon\to 0$. We see that $(\phi_0(x),w_0(x))$ satisfy

$$\phi_0=w_0',\quad w_0^{(4)}=g, \tag{2.29}$$
$$w_0(0)=w_0(1)=w_0'(0)=w_0'(1)=0. \tag{2.30}$$

In fact, the variational formulation of (2.29)-(2.30) is to find $w_0\in H_0^2(I)$ such that

$$\int_0^1 w_0''\,v''\,dt=\int_0^1 g\,v\,dt,\quad \forall v\in H_0^2(I). \tag{2.31}$$

For a fixed $x\in(0,1)$, we introduce the shape functions at node x for Hermite interpolant corresponding to the subvision I_1 , I_2,

$$v_1(t)=\begin{cases} 3(\dfrac{t}{x})^2-2(\dfrac{t}{x})^3, & 0\le t\le x \\ \\ 1-3(\dfrac{t-x}{1-x})^2+2(\dfrac{t-x}{1-x})^3, & x\le t\le 1, \end{cases} \tag{2.32}$$

$$v_2(t) = \begin{cases} x(-(\frac{t}{x})^2 + (\frac{t}{x})^3), & 0 \le t \le x \\ (1-x)(\frac{t-x}{1-x} - 2(\frac{t-x}{1-x})^2 + (\frac{t-x}{1-x})^3, & x \le t \le 1, \end{cases} \tag{2.33}$$

Substituting $v_1, v_2 \in H_0^2(I)$ into equation (2.31) we obtain the linear system

$$\begin{aligned} c_{11}^0 w_0'(x) + c_{12}^0 w_0(x) &= g_1^0, \\ c_{21}^0 w_0'(x) + c_{22}^0 w_0(x) &= g_2^0, \end{aligned} \tag{2.34}$$

where

$$\begin{aligned} c_{11}^0 &= \frac{6}{(1-x)^2} - \frac{6}{x^2}, \quad c_{22}^0 = c_{11}^0 \\ c_{12}^0 &= \frac{12}{x^3} + \frac{12}{(1-x)^3}, \quad c_{21}^0 = \frac{4}{x(1-x)}, \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} g_1^0 &= \int_0^1 g(t) v_1'(t)\, dt, \\ g_2^0 &= \int_0^1 g(t) v_2'(t)\, dt \end{aligned} \tag{2.36}$$

This is the same as is obtained from (2.26) by formally setting $\varepsilon = 0$.

3. The Shallow Arch Problem

3.1. The Model

We follow [6] and consider the following variational formulation for a shallow arch problem: find $(\phi, w, u) \in H_0^1(I) \times H_0^1(I) \times H_0^1(I)$ such that

$$\begin{aligned} (\phi', \psi') + \quad & \tfrac{1}{\varepsilon}(\phi - w', \psi - v') + \tfrac{1}{\varepsilon}(u' + \mu w', z' + \mu v') \\ &= (f, z) + (g, v), \quad \forall (\psi, v, z) \in H_0^1(I) \times H_0^1(I) \times H_0^1(I), \end{aligned} \tag{3.1}$$

where ϕ, w, u denote the rotation, vertical displacement and horizontal displacement, respectively, $\mu = \omega', \omega \in C^1(I) \cap H_0^1(I)$ is the given function and denotes the initial shape of the center-line of the arch, $0 < \varepsilon \ll 1$.

Denote the shear term

$$\gamma = \varepsilon^{-1}(\phi - w') \tag{3.2}$$

and the axial term

$$\lambda = \varepsilon^{-1}(u' + \mu w'). \tag{3.3}$$

It is proved in [6] that there exists a unique solution of (3.1) and there is a constant C independent of f, g such that

$$\|\phi\|_2 + \|\gamma\|_1 + \|\lambda\|_1 \le C(\|f\|_0 + \|g\|_0). \tag{3.4}$$

3.2. Linear Element Scheme

We introduce a family of linear element schemes for solving the shallow arch problem (3.1) Let V_h, Q_h and π_h be defined as the section 2.2. Then we consider the approximation problem to (3.1): find $(\phi_h, w_h, u_h) \in V_h \times V_h \times V_h$ such that

$$\begin{aligned}(\phi_h', \psi_h') + & \tfrac{1}{\varepsilon+\beta_1 h^2}(\pi_h(\phi_h - w_h'), \pi_h(\psi_h - v_h')) + \tfrac{1}{\varepsilon+\beta_2 h^2}(\pi_h(u_h' + \mu w_h'), \pi_h(z_h' + \mu v_h')) \\ & = (f, z_h) + (g, v_h), \quad \forall (\psi_h, v_h, z_h) \in V_h \times V_h \times V_h,\end{aligned} \tag{3.5}$$

where $\beta_1, \beta_2 > 0$ are constants independent of ε and h.

We observe that when $\beta_1 = \beta_2 = 0$, our scheme reduces to the one discussed in [6]. Notice that the discussion in [6] is based on the equivalence of a mixed finite element method. Here we can prove error estimates directly without the saddle point theory.

3.3. Error estimates

Denote

$$\gamma_h = \frac{1}{\varepsilon + \beta_1 h^2} \pi_h(\phi_h - w_h'), \tag{3.6}$$

$$\lambda_h = \frac{1}{\varepsilon + \beta_2 h^2} \pi_h(u_h' + \mu w_h'). \tag{3.7}$$

Theorem 3.1. *Let (ϕ, w, u) and (ϕ_h, w_h, u_h) be the solution of the problem (3.1) and (3.5), respectively, then*

$$\|\phi' - \phi_h'\|_0 + \|w' - w_h'\|_0 + \|u' - u_h'\|_0 \le C\,h\,(\|f\|_0 + \|g\|_0), \tag{3.8}$$

for some constant C independent of h and ε but dependent of β_1 and β_2.

Proof. From (3.1) and (3.5), we have relations

$$(\phi' - \phi_h', \psi_h') + (\gamma - \gamma_h, \psi_h) = 0, \quad \forall \psi_h \in V_h, \tag{3.9}$$

$$(\gamma - \gamma_h, -v_h') + (\lambda - \lambda_h, \mu v_h') = 0, \quad \forall v_h \in V_h, \tag{3.10}$$

$$(\lambda - \lambda_h, z_h') = 0, \quad \forall z_h \in V_h. \tag{3.11}$$

Then

$$\begin{aligned}\|\phi' - \phi_h'\|_0^2 &= (\phi' - \phi_h', \phi' - \phi_I') + (\phi' - \phi_h', \phi_I' - \phi_h') \\ &\le C\,h\,\|\phi''\|_0\,\|\phi' - \phi_h'\|_0 + (\gamma - \gamma_h, \phi_h - \phi_I).\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}(\gamma - \gamma_h, \phi_h - \phi_I) &= (\gamma - \pi_h\gamma, \phi_h - \phi_I) + (\pi_h\gamma - \gamma_h, \phi_h - \phi_I) \\ &\le C\,h\,\|\gamma\|_1\,\|\phi_h - \phi_I\|_0 + (\pi_h\gamma - \gamma_h, \phi_h - \phi_I) \\ &\le C\,h\,\|\gamma\|_1\,\|\phi_h' - \phi_I'\|_0 + (\pi_h\gamma - \gamma_h, \phi_h - \phi_I)\end{aligned} \tag{3.13}$$

where we have used Poincaré inequality in $H_0^1(I)$. From (3.2) and (3.6),

$$\pi_h\phi_I = \varepsilon\pi_h\gamma + \pi_h\,w' - \pi_h(\phi - \phi_I),$$

$$\pi_h\phi_h = (\varepsilon + \beta_1\,h^2)\gamma_h + \pi_h w_h'.$$

Thus

$$\begin{aligned}(\pi_h\gamma - \gamma_h,\, \phi_h - \phi_I) &= (\pi_h\gamma - \gamma_h,\, \pi_h(\phi_h - \phi_I)) \\ &= (\pi_h\gamma - \gamma_h,\, (\varepsilon + \beta_1\,h^2)\gamma_h - \varepsilon\pi_h\gamma) \\ &\quad + (\pi_h\gamma - \gamma_h,\, \pi_h w_h' - \pi_h w' + \pi_h(\phi - \phi_I)) \\ &\le -(\varepsilon + \beta_1\,h^2)\|\pi_h\gamma - \gamma_h\|_0^2 + \beta_1 h^2(\pi_h\gamma - \gamma_h, \pi_h\gamma) \\ &\quad + (\pi_h\gamma - \gamma_h,\, \pi_h(w_h' - w')) + C\,h^2\,\|\phi''\|_0\,\|\pi_h\gamma - \gamma_h\|_0.\end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned}(\pi_h\gamma - \gamma_h, \pi_h(w_h' - w')) &= (\pi_h\gamma - \gamma_h, \pi_h(w_h' - w_I')) \\ &= (\gamma - \gamma_h, w_h' - w_I') \\ &= (\lambda - \lambda_h, \mu(w_h' - w_I')) \\ &= (\lambda - \pi_h\lambda, \mu(w_h' - w_I')) + (\pi_h\lambda - \lambda_h, \mu(w_h' - w_I'))\end{aligned} \tag{3.15}$$

and from (3.3) and (3.7), we have

$$\pi_h(\mu(w_h' - w_I')) = (\varepsilon + \beta_2 h^2)\lambda_h - \varepsilon\pi_h\lambda + \pi_h(\mu w' - \mu w_I') - \pi_h(u_h' - u'),$$

$$\begin{aligned}(\pi_h\lambda - \lambda_h, \mu(w_h' - w_I')) &= (\pi_h\lambda - \lambda_h, \pi_h(\mu(w_h' - w_I'))) \\ &= (\varepsilon + \beta_2 h^2)(\pi_h\lambda - \lambda_h, \lambda_h - \pi_h\lambda) + (\pi_h\lambda - \lambda_h,\, \beta_2 h^2\pi_h\lambda) \\ &\quad +(\pi_h\lambda - \lambda_h, \pi_h(\mu w' - \mu w_I')) - (\pi_h\lambda - \lambda_h, \pi_h(u_h' - u'))\end{aligned} \tag{3.16}$$

In the third term of the above equation, we get

$$\begin{aligned}\pi_h(\mu w' - \mu w_I') &= \pi_h((\pi_h\mu)w' - \mu w_I') + \pi_h((\mu - \pi_h\mu)w') \\ &= (\pi_h\mu)\pi_h w' - (\pi_h\mu)w_I' + \pi_h((\mu - \pi_h\mu)w') \\ &= \pi_h((\mu - \pi_h\mu)w') \\ &= \pi_h((\mu - \pi_h\mu)(w' - w_I')).\end{aligned} \tag{3.17}$$

Then we obtain the third term of (3.16),

$$\begin{aligned}(\pi_h\lambda - \lambda_h,\, \pi_h(\mu w' - \mu w_I')) &\le \|\pi_h\lambda - \lambda_h\|_0\, \|\mu - \pi_h\mu\|_{0,\infty}\, \|w' - w_I'\|_0 \\ &\le C\, h^2\, \|\pi_h\lambda - \lambda_h\|_0\, |\mu|_{1,\infty}\, \|w''\|_0,\end{aligned} \tag{3.18}$$

and the last term of (3.16) is

$$(\pi_h\lambda - \lambda_h, \pi_h(u_h' - u')) = (\lambda - \lambda_h, u_h' - u_I') = 0.$$

Then we can derive the error estimate

$$\begin{aligned}\|\phi' - \phi_h'\|_0^2 &+ (\varepsilon + \beta_1 h^2)\|\pi_h\gamma - \gamma_h\|_0^2 + (\varepsilon + \beta_2 h^2)\|\pi_h\lambda - \lambda_h\|_0^2 \\ &\le C\, h^2(\|f\|_0 + \|g\|_0)^2.\end{aligned} \tag{3.19}$$

We can obtain $\|w' - w_h'\|_0$ and $\|u' - u_h'\|_0$ easily by (3.19) and definitions (3.2), (3.3), (3.6) and (3.7). It completes the proof.

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