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A SPLINE METHOD FOR SOLVING TWO-DIMENSIONAL FREDHOLM INTEGRAL EQUATION OF SECOND KIND WITH THE HYPERSINGULAR KERNEL[*1)]

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Abstract

The purpose of this paper is to adopt the quasi-interpolating operators in multivariate spline space $S_2^1(\Delta_{mn}^{2*})$ to solve two-dimensional Fredholm Integral Equations of second kind with the hypersingular kernels. The quasi-interpolating operators are put forward in ([7]). Based on the approximation properties of the operators, we obtain the uniform convergence of the approximate solution sequence on the Second Kind Fredholm intergral equation with the Cauchy singular kernel function.

Key words: Hypersingular integral, Finite-part integral, Quasi-interpolating operator, Nonuniform type-2 triangulation.

1. Introduction

In recent years, the boundary element methods became a reliable and powerful numerical methods for solving the boundary value problems, such as elastoplasticity, etc. In these methods, the original problem is reduced to a boundary integral equation. For the one dimensional boundary, a lot of methods have been put forward recently. But for the two dimensional boundary situation, it is not so easy to be done because the partition can be very complicated. Since P. Zwart obtained an expression of bivariate B-spline [2], R-H Wang and C.K. Chui obtained a quasi-interpolating operators of $S_2^1(\Delta_{mn}^2)$ on uniform type-2 triangulation and its approximation properties ([1]) which have widespread applications in Mechanics and Engineering. Furthermore, R-H Wang and C.K. Chui also obtained the function with minimum support in $S_2^1(\Delta_{mn}^{2*})$ on non-uniform type-2 triangulation and the basis of $S_2^1(\Delta_{mn}^{2*})$ ([4]). In ([7]), we introduced some quasi-interpolating operators of $S_2^1(\Delta_{mn}^{2*})$ on non-uniform type-2 trangulation and show their approximation properties. By using the operators we constructed cubature formulas which can be used to solve hypersingular integrals that arisen from many mechanics and engineering problems.

In present paper, we give a method for solving the two-dimensional Fredholm Integral Equation with hypersingular kernel based on quasi-interpolating operators in $S_2^1(\Delta_{mn}^{2*})$ and prove the uniform convergence of the approximate solution sequence.

2. Quasi-Interpolating Operators of $S(\Delta_{mn}^{2*})$

Let Δ_{mn}^{2*} be a non-uniform type-2 triangulation on the domain $\Omega : [a,b] \otimes [c,d]$, and

$$x_{-2} < x_{-1} < a = x_0 < \cdots < x_m = b < x_{m+1} < x_{m+2},$$
$$y_{-2} < y_{-1} < c = y_0 < \cdots < y_n = d < y_{n+1} < y_{n+2}.$$

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First we consider the linear operators

$$V_{mn} : C(\Omega) \to S_2^1(\Delta_{mn}^{2*}); \tag{2.1}$$

$$V_{mn}(f) = \sum_{ij} f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right) B_{ij}(x,y); \tag{2.2}$$

It is similar to the result in ([1]), we have the following results.

Theorem 2.1. *For $f \in P_1$ and $f = xy$, we have $V_{mn}(f) = f$,* (2.3)

Theorem 2.2. *$W_{mn}(f) = f$ for any $f \in P_2$.*

In terms of theory 2.1, 2.2, and [7], we have the following results.

Let

$$\omega_k(f,\delta) = \sup\{|f(x,y) - f(u,v)| : (x,y),(u,v) \in K, |(x,y) - (u,v)| < \delta\}. \tag{2.4}$$

$$\begin{aligned} \delta_{mn} &= \max[h_i, k_j], \\ \delta_{mn}^* &= \max(\sqrt{9h^2+k^2}, \sqrt{9k^2+h^2}), \\ h &= \max_i(h_i), k = \max_j(k_j); \end{aligned} \tag{2.5}$$

and where K be a compact set and $K \subset \Omega$.

Theorem 2.3. *Let $f \in C(K)$ and $m, n \geq N_0$. we have*

$$\|f - V_{mn}(f)\|_\Omega \leq \omega_k(f, \delta_{mn}^*), \tag{2.6}$$

if $f \in C^1(K)$ then

$$\|f - V_{mn}(f)\|_\Omega \leq \delta_{mn} \max(\omega_\Omega(f_1, \delta_{mn}/2), (\omega_\Omega(f_2, \delta_{mn}/2), \tag{2.7}$$

if $f \in C^2(K)$ then

$$\|f - V_{mn}(f)\|_\Omega \leq \delta_{mn}^2 \|D^2 f\|. \tag{2.8}$$

Theorem 2.4. *Let $f \in C^2(\Omega)$ and $m, n \geq N_0$. If $f \in C^2(K)$, then*

$$\|f - W_{mn}(f)\|_\Omega \leq \frac{1}{2}\delta_{mn}^2 \max[\omega_\Omega(f_{11}, \delta_{mn}/2), 2\omega_\Omega(f_{12}, \delta_{mn}/2), \omega_\Omega(f_{22}, \delta_{mn}/2)] \tag{2.9}$$

if $f \in C^3(K)$ then

$$\|f - W_{mn}(f)\|_\Omega \leq \frac{1}{12}\delta_{mn}^3 \|D^3 f\|. \tag{2.10}$$

Taking note of $\|W_{mn}\| = 3$. it is easy to prove the theorem in term of Taylor expansion.

3. Cubature Formulas

Here we consider integrals of the form

$$\int_\Omega K_p(v_0; v)\Phi(v)dv, v_0 \in \Omega \subset R^2, \tag{3.1}$$

where the kernel K_p admit the expansion

$$K_p(v_0; v) = \sum_{l=0}^{p} \frac{f_{p-l}(v_0;\theta)}{r^{p+2-l}} + K_p^*(v_0, v). \tag{3.2}$$

(ρ, θ) denotes the polar coordinate of v with respect to $v_0 \cdot K_p^*(v_0; v)$ may still become infinite at v_0, but with order less than 2. For simplicity we assume the functions f_{p-1}, Φ and K_p^* smooth in domain Ω. Therefore we can only consider integrals of the form

$$I = \int_\Omega \frac{f_p(v_0, \theta)}{r^p} \Phi(v) dv \tag{3.3}$$

Based on the definition of finite part integrals and the quasi interpolating operators mentioned as above, we obtain the cubature formula for (3.3)

$$I = \int_\Omega \frac{f_p(v_0, \theta)}{r^p} \Phi(v) dv = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{k=i-1}^{i+1} \sum_{q=j-1}^{j+1} \lambda_{kq}(\Phi) b_{ijkq} + R_{mn}(K\Phi); \tag{3.4}$$

where

$$b_{ijkq} = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} B_{kq}(v) \frac{f(v_0, \theta)}{r^p} dv. \tag{3.5}$$

By the theorem 2.3, we can obtain some convergence results on the cubature formula (3.4).

Theorem 3.1. *If in* (3.4), *we assume* $\Phi \in C^{p+2}(\Omega)$ *and* $f_p \in C[0, \omega]$, *then the remainder term in* (3.4)

$$R_{mn}(K\Phi) \le C\omega_\Omega(g, \delta_{mn}/2). \tag{3.6}$$

where C *is a constant.*

Theorem 3.2. *If we assume* $\Phi \in C^{p+3}(\Omega)$ *and* $f_p \in C^2[0, \omega]$, *then the remainder in* (3.4)

$$R_{mn}(K\Phi) \le C\delta_{mn}^2 \max\left[\omega_\Omega(g_{11}, \delta_{mn}/2), 2\omega_\Omega(g_{12}, \delta_{mn}/2), \omega_\Omega(g_{22}, \delta_{mn}/2)\right], \tag{3.7}$$

where C *is a constant.*

Theorem 3.3. *If we assume* $\Phi \in C^{p+4}(\Omega)$ *and* $f_p \in C^3[0, \omega]$, *then the remainder in* (3.4)

$$R_{mn}(K\Phi) \le C\delta_{mn}^3 \|D^3(g)\| \tag{3.8}$$

where C *is a constant.*

4. Approximation Method

Without loss generality, we consider the two-dimensional second kind Ferdholm integral equations of the form

$$u(x, y) = \int_\Omega K(x, y, s, t) u(s, t) dt ds + g(x, y), \quad (x, y) \in \Omega, \tag{4.1}$$

where

$$K(x, y, s, t) = \frac{f(x, y, s, t)}{r^d}, \ \ d = 2, 3, \ \ r = \|(x, y) - (s, t)\|_2,$$

$$f \in E, g(x, y) \in C(\Omega), E\{(x, y, s, t) : a \le x, t \le b, c \le y, t \le d\}. \tag{4.2}$$

We use the quasi-interpolating function $W_{mn}(u)(s, t)$ as substitute for the integrating factor $u(s, t)$, then the integral equation (4.1) turn into following $(2nm + 5m + 5n + 13) \times (2nm + 5m + 5n + 13)$ linear algebraic system of equations:

$$u^{hk}(x_p, y_q) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{k=i-1}^{i+1} \sum_{q=j-1}^{j+1} \lambda_{kq}(u^{hk}) b_{ijkq}(x_p, y_q) + g(x_p, y_q); \tag{4.3}$$

$$u^{hk}(x_{p'},y_{q'})=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=i-1}^{i+1}\sum_{q=j-1}^{j+1}\lambda_{kq}(u^{hk})b_{ijkq}(x_{p'},y_{q'})+g(x_{p'},y_{q'}); \tag{4.4}$$

where

$$b_{ijkq}(*,*)=\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}B_{kq}(v)\frac{f(*,*,s,t)}{r^p}dsdt. \tag{4.5}$$

$$(p=-1,0,1\cdots m+1;q=-1,0,1\cdots,n+1;p'=0,1\cdots m+1;q'=0,1,\cdots n+1,).$$

and $(x_p,y_q),(x_{p'},y_{q'})$ are the knots of non-uniform type-2 triangulation Δ_{mn}^{2*}, (cf.Fig.1).

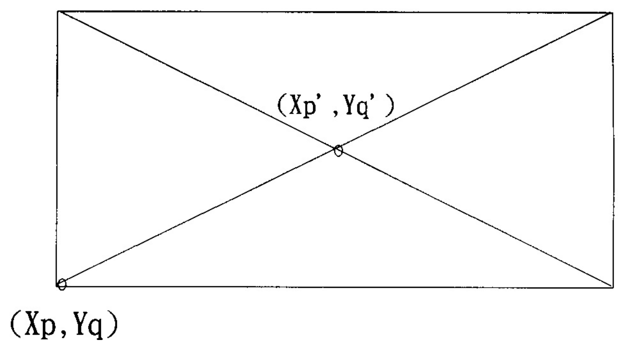

Fig.1

The solution $u(x,y)$ of the equation (4.1) can be evaluated by the values of

$$u^{hk}(x_i,y_j),u^{hk}(x_{i'},y_{j'})$$
$$(i=-1,0,1,\cdots m+1;j=-1,0,1,\cdots n+1;i'=0,1,\cdots m+1;j'=0,1,\cdots n+1).$$

Let

$$u(x)=g(x)+\iint_R K(x;y)u(y)dy \quad x,y\in R,R:[0,1]\otimes[0,1], \tag{4.6}$$

where $g(u)\in C(R),K(u,v)=f(\theta)/r^2$.

Define a linear operator

$$Ku(x)=\iint_R K(x;y)u(y)dy,$$

For the approximation equation of (4.6)

$$u_{mn}(x)=g(x)+\iint_R f(\theta)W_{mn}(u_{mn})(y)/r^2dy,\ \ x\in R,R:[0,1]\otimes[0,1], \tag{4.7}$$

where W_{mn} is a quasi-interpolating operator in $S_2^1(\Delta_{mn}^2)$, we have the following theorem.

Theorem 4.1. *If $g(x)\in L_2$, the sign function $\Phi(\theta)$ of the operator K is bounded and does not rely on the pole, and its modular has a positive infimun on unit sphere B, then the approximation equation (4.7) has a unique solution as $m,n\geq N$, Furthermore, u_{mn} uniformly approaches u as $m,n\to\infty$.*

Proof. If the sign function $\Phi(\theta)$ of operator K is bounded and do not rely on the pole, according to ([8]), the operator K is bounded. Furthermore, if $g(x)\in L_2$ is bounded on unit sphere S and its modular has a positive infimum, then the equation (4.6) has a unique solution in $L_2(R)$. It is obvious that $\Phi^{-1}(\theta)$ is also bounded. We have

$$\|Kh\|_{L_2} \le C\|h\|_{L_2}, \quad h \in L_2(R).$$

Define the linear operator

$$K_{mn}h(x) = \int\int_R K(x;y)W_{mn}h(y)dy$$

Since $K_{mn}h = KW_{mn}h$,

$$\|K_{mn}h\|_{L_2} \le C\|W_{mn}\| \, \|h\|_{L_2} = 3C\|h\|_{L_2}, \quad h \in L_2(R).$$

In the same way, we have also $\|K_{mn}h - Kh\|_{L_2} \le \|K\| \, \|W_{mn}h - h\|_{L_2}$.

As $m,n \to \infty$, $K_{mn}h$ converge uniformly to Kh.

Since W_{mn} is a finite rank operator on $L_2(R) \to S_2^1(\Delta_{mn}^2)$, K_{mn} is a compact operator on $L_2(R) \to L_2(R)$.

Define a set

$$S = \{W_{mn}h; \quad m,n \ge 1, \quad h \in L_2(R)\}.$$

In addition, we have also $W_{mn}L_2(R) \supseteq W_{MN}L_2(R)$ as $n \ge N, m \ge M$. According to the theorem of multivariate spline, S is dense in $L_2(R)$.

Now we can prove that the equation (4.7) has only a zero-solution. In fact, let h be a non-zero solution of $K_{MN}h = h$, and $\|h\| = 1$, $h \in S$ As $n \ge N, m \ge M, K_{mn}h = h$.

In the same time, we have

$$\|K_{mn}h - Kh\|_{L_2} = \|h - Kh\|_{L_2},$$

As $m,n \to \infty$,

$$\|Kh - h\|_{L_2} = 0.$$

This is a contradiction with only having zero-solution of (4.6). So the approximation equation (4.7) has only a zero-solution in S. Because of S dense in $L_2(R)$, the equation $K_{mn}h = h$ has only a zero-solution in $L_2(R)$.

According to the results mentioned as above, there must exist positive N and c, such that for any $n,m \ge N$, approximation equation (4.7) has a unique solution u_{mn}, and

$$\|(I - K_{mn})^{-1}\| \le c$$

holds. Form

$$u_{mn} = g + K_{mn}u_{mn}, \quad u = g + Ku,$$

we obtain

$$u_{mn} - K_{mn}u_{mn} = u - Ku,$$
$$u_{mn} - u = (I - K_{mn})^{-1}(K_{mn} - K)u,$$

and

$$\|u_{mn} - u\|_{L_2} = \|(I - K_{mn})^{-1}\| \, \|(K_{mn} - K)u\|_{L_2} \le C\|(K_{mn} - K)u\|_{L_2},$$

as $m,n \to \infty$, the right side above the inequality approaches zero. Therefore, the theorem 4.1 holds.

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