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MONOTONIC ITERATIVE ALGORITHMS FOR A QUASICOMPLEMENTARITY PROBLEM*

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Abstract

We present two iterative algorithms, so called SCP and SA respectively, for solving quasicomplementarity problem (QCP). Algorithm SCP is to approximate QCP by a sequence of ordinary complementarity problems (CP). SA is a Schwarz algorithm which can be implemented parallelly. We prove the algorithms above are monotonically convergent.

Key words: Quasicomplementarity problem, Iterative algorithm, Monotonic convergence, Schwarz algorithm.

1. Introduction

Consider the following QCP: find $u \in R^n$ such that

$$\min\{Au - f, \quad u - Bu\} = 0, \tag{1}$$

where A, B: $R^n \to R^n$ are operators, $f \in R^n$. The quasivariational inequality which is equivalent to (1) sounds: to find $u \in R^n$ such that $u \geq Bu$ and

$$(Au, \ \ v - u) \geq (f, \ \ v - u), \quad \forall v \geq Bu. \tag{2}$$

If $Bu \equiv c \in R^n$ for any $v \in R^n$ then $QCP(1)$ reduces into CP. (1) appears in mathematical programming (see, for example, [2] and the references therein), also comes from the discretization of QCP in mathematical physics and control theory (see [1]). For various generalization, see [3] and the references therein.

We assume in this paper that A is a strictly T-monotonic operator, that is:

$$(Au - Av, \ \ (u - v)^+) \geq 0, \quad \forall u, v \in R^n,$$

where the equality holds only if $(u - v)^+ = 0$, $v^+ = \max\{v, 0\}$. The examples in [6] show that strictly T-monotonic operator is widely applicable.

We propose two algorithms for solving (1). The first (SCP) is to solve iteratively a sequence of CP, which produces a sequence of approximate solutions convergent monotonically to a solution of (1). The second (SA) is a Schwarz algorithm, which is parallel algorithm and produce super(sub)solution sequence of (1), convergent monotornically to a solution of (1).

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Schwarz algorithms have been developed rapidly (for example, see [4], [6] and the references therein), But Algorithm SA is the first Schwarz algorithm for quasivariational inequality.

2. Sequential CP Algorithm

Algorithm SCP

1^0. Take $u^0 \in R^n$, $k := 0$;

2^0. Find $u^{k+1} \in R^n$ such that

$$\min\left\{Au^{k+1} - f,\ \ u^{k+1} - Bu^k\right\} = 0; \tag{3}$$

3^0. $k := k+1$, go to 2^0.

At first we study an operator F related to (3) and defined as follows: for any $x \in R^n$ define $y = Fx$ as the solution of the following CP:

$$\min\{Ay - f,\ \ y - Bx\} = 0. \tag{4}$$

It has a unique solution if operator A is continuous, strictly T-monotonic and coercive in some sense (see, for example, [5]). Then F is well-defined.

We call an operator B order-preserved if $v \leq w$ implies $Bv \leq Bw$.

Lemma 1. *Assume B is continuous, order-preserved and there exists $b \in R^n$ such that*

$$Bv \leq b, \qquad \forall v \in R^n. \tag{5}$$

Assume A is continuous, strictly T-monotoinic and

$$\frac{(Av,\ \ v-b)}{\| v \|} \to +\infty \quad (\| v \| \to \infty). \tag{6}$$

Then the range of F, denoted by $R(F)$, is bounded. That is, there exist $p, q \in R^n$ such that

$$p \leq Fx \leq q, \qquad \forall x \in R^n.$$

Proof. Since (6) implies the coerciveness condition of A in [5], (4) has a unique solution for any $x \in R^n$ and F is well-defined. Hence we have

$$\min\{AFx - f, \quad Fx - Bx\} = 0, \qquad \forall x \in R^n.$$

Then for any $x \in R^n$ we have

$$(AFx,\ \ v - Fx) \geq (f,\ \ v - Fx), \qquad \forall v \geq Bx.$$

Letting $v = b$ in it we obtain

$$(AFx, Fx - b) \leq (f, Fx - b), \quad \forall x \in R^n$$

which combining with (6) yields that F is bounded.

Remark 1. If there exists a constant α such that

$$(Av - Aw, v - w) \geq \alpha \| v - w \|^2, \quad \forall v, w \in R^n$$

then operator A is called strong monotonic. We can prove similarly that if A is continuous, strictly T-monotonic and strong monotonic, B is continuous and order-preserved, and (5) holds, then $R(F)$ is bounded and $-\beta e \leq Fx \leq \beta e$, $\forall x \in R^n$, where $\beta = \| b \| + \alpha^{-1} \| f - Ab \|, e = (1, \cdots, 1)$.

Now we give a lemma which displays that the solution of CP is monotonically dependent on the obstacle vector if A is strictly T-monotonic. The lemma is important for our proof of the convergence theorem in this section.

Lemma 2. *Assume A is strictly T-monotonic, $\varphi^m \in R^n, m = 1, 2,$ and*

$$\min\{Aw^m - f, w^m - \varphi^m\} = 0, \quad m = 1, 2.$$

If $\varphi^1 \geq \varphi^2$ then $w^1 \geq w^2$.

Proof. Let $N = \{1, \cdots, n\}$, $I = \{i \in N : (Aw^2 - f)_j = 0\}$. Then for $j \in N \setminus I$ we have $w_j^2 = \varphi_j^2$, $w_j^2 - w_j^1 = \varphi_j^2 - w_j^1 \leq \varphi_j^2 - \varphi_j^1 \leq 0$ and $(w_j^2 - w_j^1)^+ = 0$. Therefore,

$$\begin{aligned} & (Aw^2 - Aw^1, \quad (w^2 - w^1)^+) \\ = \ & \sum_{j \in I} (f - Aw^1)_j (w^2 - w^1)_j^+ + \sum_{j \in N \setminus I} (Aw^2 - Aw^1)_j (w_j^2 - w_j^1)^+ \\ \leq \ & 0. \end{aligned}$$

Hence $(w^2 - w^1)^+ = 0$ and $w^2 \leq w^1$.

Theorem 1. *Assume the conditions of Lemma 1 (or Remark 1) hold. Then (1) has solution and the sequence produced by Algorithm SCP is monotonically decreasing and convergent to a solution of (1) provided $u^0 = q$(or $u^0 = \beta e$).*

Proof. By the process of Algorithm SCP we know that $u^1 \in R(F)$ and then $u^0 \geq u^1$. Since B is order-preserved we have $Bu^0 \geq Bu^1$. It follows from (3) and Lemma 2 that $u^1 \geq u^2$. Then by induction we derive that

$$u^k \geq u^{k+1}, \quad Bu^k \geq Bu^{k+1}, \quad k = 0, 1, \cdots. \tag{7}$$

On the other hand, the boundedness of $R(F)$ implies that $\{u^k\}$ is bounded. Hence there exists $u^* \in R^n$ such that $u^k \to u^* (k \to \infty)$. Letting $k \to \infty$ in (3) we obtain

$$\min\{Au^* - f, \quad u^* - Bu^*\} = 0,$$

i.e. u^* is a solution of (1). The existence of solution is obvious now and the proof is complete.

Remark 2. If $u^0 = p$(*or* $u^0 = -\beta e$) and the other conditions of Theorem 1 hold then $\{u^k\}$ is monotonically increasing and convergent to a solution of (1).

3. Supersolution and Subsolution of (1)

In this section we extend the discussion in [6] on the supersolution and subsolution of CP to QCP.

Let $v \in R^n$. If $\min\{Av - f, v - Bv\} \geq 0$ then call v a supersolution of (1), if $\min\{Av - f, v - Bv\} \leq 0$ then call v a subsolution. Denote by $S_1(S_2)$ the set of supersolutions (subsolutions) of (1). The following lemma indicates the closedness in $S_1(S_2)$ of minimun(maximun) operator.

Lemma 3. *Assume A is continuous and strictly T-monotonic, B is order-preserved, $v^i \in S_1, i = 1, \cdots, m$, and $v = \min\{v_1, \cdots, v^m\}$. Then $v \in S_1$.*

Proof. For any $j \in N$ there exists $j \in \{1, \cdots, m\}$ such that $v_j = v_j^i$. On the other hand, $v_l \le v_l^i$ for any $l \in N$. It follows from lemma3 in [4] that

$$(Av - f)_j \ge (Av^i - f)_j \ge 0.$$

Since B is order-preserved we have $Bv \le Bv^i$ and then

$$Bv \le \min\{Bv^1, \cdots, Bv^m\} \le \min\{v^1, \cdots, v^m\} = v.$$

Hence we obtain

$$\min\{Av - f, v - Bv\} \ge 0,$$

$i.e$ $.v \in S_1$.

Similarly we can prove

Lemma 4. *Assume A, B satisfy the conditions of Lemma 3, $v^i \in S_2, i = 1, \cdots, m$, and $v = \max\{v^1, \cdots, v^m\}$. Then $v \in S_2$.*

Lemma 5. *Assume the conditions of Theorem 1 hold. Then S_2 has maximun element.*

Proof. Consider the following CP :

$$\min\{Aw - f, w - b\} = 0.$$

By a well-known theorem (for examples see [5]), it has a unique solution w. We prove w is a upper bound of S_2. Given $v \in S_2$. Denote $Q = \{i \in N : (Av - f)_i \le 0\}$.

Then for $i \in N \setminus Q$ we have

$$v_i - w_i \le v_i - b_i \le v_i - (Bv)_i \le 0.$$

Hence $(v - w)_i^+ = 0$ for $i \in N \setminus Q$ and then

$$(Av - Aw, (v - w)^+) = \sum_{i \in Q} + \sum_{i \in N \setminus Q}$$
$$\le \sum_{i \in Q} (f - Aw)_i (v - w)_i^+ \le 0.$$

So $v \le w$ for any $v \in S_2$. Let

$$u^* = \sup_{v \in S_2} \{v\}.$$

Then $u^* \le w$. It is not difficuct to show by lemma 4 that there exists a sequence $\{v^i\} \subset S_2$, monotonically increasing and convergent to u^*. Since $v^i \in S_2$ we have

$$\min\{Av^i - f, v^i - bv^i\} \le 0.$$

Hence $$\min\{Au^* - f, u^* - bu^*\} \le 0,$$
i.e. $u^* \in S^2$. The proof is complete.

Remark 3. There is a similar conclusion on the minimun element of S_1.

4. A Schwarz Algorithm

Algorithm SA.

1^0. Decompose $N = N_1 \bigcup \cdots \bigcup N_m$, take $u^0 \in S_2, k := 0$;

2^0. For $i = 1, \cdots, m$ let

$$u_j^{k,i} = u_j^k \quad \text{if} \quad j \in N \setminus N_i$$

and solve parallelly subproblems:

$$\min\{Au^{k,i} - f, u^{k,i} - Bu^k\}_{N_i} = 0; \tag{8}$$

3^0. $u^{k+1} := \max\{u^{k,1}, \cdots, u^{k,m}\}$;

4^0. $k := k + 1$, go to 2^0.

By the way, subproblem (8) is not QCP but CP.

Theorem 2. *Assume that the conditions of Theorem 1 hold. Then the sequence produced by Algorithm SA is contained in S_2, monotonically increasing and convergent to a solution of (1).*

Proof. Let $Q_{0,i} = \{j \in N_i : (Au^0 - f)_j \le 0\}$. Then for $j \in N_i \setminus Q_{0,i}$ we have $u_j^0 \le (Bu^0)_j \le u_j^{0,i}$ and $(u_j^0 - u_j^{0,i})^+ = 0$. Hence

$$\begin{aligned}
\big(Au^0 - Au^{0,i}, (u^0 - u^{0,i})^+\big) &= \sum_{j \in Q_{0,i}} + \sum_{j \in N_i \setminus Q_{0,i}} + \sum_{j \in N \setminus N_i} \\
\le \quad & \sum_{j \in Q_{0,i}} (f - Au^{0,i})_j (u^0 - u^{0,i})_j^+ + \sum_{j \in N_i \setminus Q_{0,i}} (Au^0 - Au^{0,i})_j (u^0 - u^{0,i})_j^+ \\
\le \quad & 0,
\end{aligned}$$

which implies $u^{0,i} \ge u^0, i = 1, \cdots, m$ and $u^1 \ge u^0$.

Let $I = \{j \in N : (u^1 - Bu^1)_j \le 0\}$. Then

$$\min\{Au^1 - f, u^1 - Bu^1\}_j \le 0, \quad \forall j \in I. \tag{9}$$

Given $j \in N \setminus I$. Then $u_j^1 > (Bu^1)_j$. Without loss of generality we may assume $u_j^1 = u_j^{0,1}$ and $j \in N_1$. Therefore, we have

$$u_j^{0,1} - (Bu^0)_j = u_j^1 - (Bu^0)_j \ge u_j^1 - (Bu^1)_j > 0,$$

which combining with (8) yields $(Au^{0,1} - f)_j = 0$. Noting $u^1 \ge u^{0,1}$ and $u_j^1 = u_j^{0,1}$ we deduce by lemma 3 in [6] that

$$(Au^1)_j \le (Au^{0,1})_j = f_j.$$

It means

$$\min\{Au^1 - f, u^1 - Bu^1\}_j \le 0, \quad \forall j \in N \setminus I. \tag{10}$$

From (9) and (10) we conclude $u^1 \in S^2$. Then by induction we derive that

$$u^k \in S_2, \quad u^{k+1} \ge u^{k,i} \ge u^k, \quad k = 0, 1, \cdots. \tag{11}$$

By lemma 5 we know any u^k is not bigger than the maximun elment of S_2. Hence $\{u^k\}$ has a limit, denoted by u^*. It follows from (11) that u^* is also the limit of $u^{k,i}$. Letting $k \to \infty$ in (8) we obtain

$$\min\{Au^* - f, u^* - Bu^*\} = 0.$$

So u^* is a solution of (1). The proof is complete.

Remark 4. In algorithm SA, if take " $u^0 \in S_1$ " instead of " $u^0 \in S_2$" , then $u^k \in S_1$, monotonically decreasing and convergent to a solution of (1) under the conditions of Theorem 2.

Remark 5. Similarly to that in [6], we can deal with quasivariational inequalities in the case $u \le Bu$ or the case $B_1 u \le u \le B_2 u$.

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