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ON THE ESTIMATIONS OF BOUNDS FOR DETERMINANT OF HADAMARD PRODUCT OF H-MATRICES[*1)]

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Abstract

In this paper, some estimations of bounds for determinant of Hadamard product of H-matrices are given. The main result is the following: if $A=(a_{ij})$ and $B=(b_{ij})$ are nonsingular H-matrices of order n and $\prod_{i=1}^{n} a_{ii}b_{ii}>0$, and A_k and $B_k, k=1,2,\cdots,n$, are the $k\times k$ leading principal submatrices of A and B, respectively, then

$$\det(A\circ B)\ge |a_{11}b_{11}|\prod_{k=2}^{n}\Big[|b_{kk}|\frac{\det\mathcal{M}(A_k)}{\det\mathcal{M}(A_{k-1})}+\frac{\det\mathcal{M}(B_k)}{\det\mathcal{M}(B_{k-1})}\Big(\sum_{i=1}^{k-1}\Big|\frac{a_{ik}a_{ki}}{a_{ii}}\Big|\Big)\Big],$$

where $\mathcal{M}(A_k)$ denotes the comparison matrix of A_k.

Key words: H-matrix, Determinant, Hadamard product.

1. Introduction

Let $R^{m\times n}$ denote the set of $m\times n$ real matrices, S_n^+ denote the set of $n\times n$ positive definite real symmetric matrices. For $A=(a_{ij})$ and $B=(b_{ij})\in R^{m\times n}$, the Hadamard product of A and B is defined as an $m\times n$ matrix denoted by $A\circ B:(A\circ B)_{ij}=a_{ij}b_{ij}$.

We write $A\ge B$ if $a_{ij}\ge b_{ij}$ for all i,j. A real $n\times n$ matrix A is called a nonsingular M-matrix if $A=sI-B$ satisfied: $s>0, B\ge 0$ and $s>\rho(B)$, the spectral radius of B, let M_n denote the set of all $n\times n$ nonsingular M-matrices. Suppose $A\in R^{n\times n}$, its comparison matrix $\mathcal{M}(A)=(m_{ij})$ is defined by the following:

$$m_{ij}=\begin{cases}|a_{ij}|, & \text{if } i=j\\ -|a_{ij}|, & \text{if } i\ne j\end{cases}\tag{1}$$

A real (or complex) $n\times n$ matrix A is called an H-matrix if its comparison matrix $\mathcal{M}(A)$ is a nonsingular M-matrix, let H_n denote the set of all $n\times n$ nonsingular H-matrices.

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On the estimations of bounds for determinant of Hadamard product of matrices, we have the following well-known result.

Oppenheim's inequality: If $A = (a_{ij})$ and $B = (b_{ij}) \in S_n^+$ then

$$\det(A \circ B) \geq \Big(\prod_{i=1}^{n} a_{ii}\Big) \cdot \det(B) \tag{2}$$

Lynn[2] had proved that inequality (2) holds for M-matrices and Fielder and Ptak[3] given a similar result when A is an M-matrix and B is a weakly diagonally dominant matrix. Jianzhou Liu and Li Zhu[1] improved Oppenheim's inequality recently as following theorem:

Theorem 1[1]**.** *If $A = (a_{ij})$ and $B = (b_{ij})$ are nonsingular M-matrices, A_k and B_k, $k = 1, 2, \cdots, n-1$, are the $k \times k$ leading principal submatrices of A and B, respectively, then*

$$det\ (A \circ B) \geq a_{11}b_{11} \prod_{k=2}^{n} \Big[b_{kk}\frac{det(A_k)}{det(A_{k-1})} + \frac{det(B_k)}{det(B_{k-1})}\Big(\sum_{i=1}^{k-1} \frac{a_{ik}a_{ki}}{a_{ii}}\Big)\Big] \tag{3}$$

In this paper, we shall generalize Jianzhou Liu's results and give an inequality similar to (3) for nonsingular H-matrices.

2. Some Lemmas

In this section, we shall give some lemmas which shall be used in the following.

Lemma 1[4]**.** *If A and $B \in M_n$ then $\mathcal{M}(A \circ B) \in M_n$.*

Lemma 2. *If A and $B \in H_n$ then $A \circ B \in H_n$.*

Proof. By the definition of Hadamard product and the definition of comparison matrix, we can easily obtain the following equality:

$$\mathcal{M}(A \circ B) = \mathcal{M}(\mathcal{M}(A) \circ \mathcal{M}(B)) \tag{4}$$

If A and $B \in H_n$ then $\mathcal{M}(A)$ and $\mathcal{M}(B) \in M_n$ and $\mathcal{M}(\mathcal{M}(A) \circ \mathcal{M}(B)) \in M_n$ by Lemma 1, that is: $\mathcal{M}(A \circ B) \in M_n$. So $A \circ B \in H_n$.

Lemma 3[5]**.** *Let $A = (a_{ij}) \in R^{n\times n}$ with $a_{ij} \leq 0$ for all $i \neq j; i, j = 1, 2, \cdots, n$, then the following conditions are equivalent:*

1. *A is a nonsingular M-matrix.*
2. *A has all positive diagonal elements, and there exists a positive diagonal matrix D such that AD is strictly diagonally dominant.*
3. *All of the leading principal minors of A are positive.*

From the definition of H-matrix and Lemma 3, we can easily prove the following result.

Lemma 4. *A matrix A is nonsingular H-matrix if and only if there exists a positive diagonal matrix D such that AD is strictly diagonally dominant.*

Lemma 5[1]**.** *If A is a strictly diagonally dominant matrix with $a_{ii} > 0, i = 1, 2, \cdots, n$, then*

$$det A \geq det \mathcal{M}(A) > 0$$

Lemma 6[1]**.** *Let $A=(a_{ij})$ and $B=(b_{ij})\in M_n$, A_k and $B_k, k=1,2,\cdots,n-1$, be the $k\times k$ leading principal submatrices of A and B, respectively, then*

$$det[\mathcal{M}(A\circ B)]\geq a_{11}b_{11}\prod_{k=2}^{n}\Big[b_{kk}\frac{det(A_k)}{det(A_{k-1})}+\frac{det(B_k)}{det(B_{k-1})}\Big(\sum_{i=1}^{k-1}\frac{a_{ik}a_{ki}}{a_{ii}}\Big)\Big] \tag{5}$$

3. Estimations of Bounds for Determinant of Hadamard Product of H-Matrices

In this section, for A and $B\in H_n$, we study the estimations of bounds for det $(A\circ B)$.

Theorme 2. *If $A=(a_{ij})$ and $B=(b_{ij})\in H_n$, then det $(A\circ B)\neq 0$ and det$(A\circ B)$ has the same sign to $\prod_{i=1}^{n}=a_{ii}b_{ii}$.*

Proof. Let $A=(a_{ij})$ and $B=(b_{ij})\in H_n$, then $A\circ B\in H_n$ by Lemma 2. From Lemma 4 there exists a positive diagonal matrix $D'=diag(d'_1,d'_2,\cdots,d'_n)$ such that $(A\circ B)D'$ is strictly diagonally dominant. It is evident that we can change $D'=diag(d'_1,d'_2,\cdots,d'_n)$ into $D=diag(d_1,d_2,\cdots,d_n)$ by changing signs of $d'_i,(i=1,2,\cdots,n)$ with $|d_i|=d'_i$, $i=1,2,\cdots,n$, such that every diagonal element of $(A\circ B)D$ is positive and $(A\circ B)D$ is also strictly diagonally dominant. Obviously,

$$\Big(\prod_{i=1}^{n}a_{ii}b_{ii}\Big)\cdot\det D=\prod_{i=1}^{n}(a_{ii}b_{ii}d_i)>0 \tag{6}$$

By Lemma 5 we have

$$\det[(A\circ B)D]\geq\det\mathcal{M}[(A\circ B)D]>0$$

hence

$$\det(A\circ B)\det D=\det[(A\circ B)D]>0 \tag{7}$$

From (6) and(7) we obtain:

$$\det(A\circ B)\cdot\Big(\prod_{i=1}^{n}a_{ii}b_{ii}\Big)>0 \tag{8}$$

that is: det $(A\circ B)\neq 0$ and det $(A\circ B)$ has the same sign as $\prod_{i=1}^{n}(a_{ii}b_{ii})$.

Theorem 3. *Let $A=(a_{ij})$ and $B=(b_{ij})\in H_n$ and $\prod_{i=1}^{n}a_{ii}b_{ii}>0$, then*

$$det(A\circ B)\geq det[\mathcal{M}(A\circ B)]>0 \tag{9}$$

Proof. Let $A=(a_{ij})$ and $B=(b_{ij})\in H_n$ and $\prod_{i=1}^{n}(a_{ii}b_{ii})>0$, from the proof of the Theorem 2, we know that there exists a diagonal matrix

$$D=diag(d_1,d_2,\cdots,d_n)$$

such that:

$$\det[(A\circ B)D] \geq \det\mathcal{M}[(A\circ B)D] > 0 \tag{10}$$

It is easy to prove that:

$$\mathcal{M}[(A\circ B)D] = \mathcal{M}(A\circ B)\mathcal{M}(D)$$

therefore, by (10) we have:

$$\det(A\circ B)\cdot\det D \geq \det[\mathcal{M}(A\circ B)]\cdot\det\mathcal{M}(D) > 0 \tag{11}$$

By the Theorem 2 and the assumed condition $\prod_{i=1}^{n}(a_{ii}b_{ii}) > 0$, we have det $(A\circ B) > 0$.

From (11), it follows that det $D > 0$, so det $D =$ det $[\mathcal{M}(D)]$. Thus, it follows from (11) that

$$\det(A\circ B) \geq \det[\mathcal{M}(A\circ B)] > 0$$

Theorem 4. *Let $A = (a_{ij})$ and $B = (b_{ij}) \in H_n$ and $\prod_{i=1}^{n} a_{ii}b_{ii} > 0$, A_k and B_k, $k = 1, 2, \cdots, n-1$, be the $k\times k$ leading principal submatrices of A and B, respectively, then*

$$\begin{aligned} det[\mathcal{M}(A\circ B)] \geq & |a_{11}b_{11}|\prod_{k=2}^{n}\Big[|b_{kk}|\frac{det\,\mathcal{M}(A_k)}{det\,\mathcal{M}(A_{k-1})} \\ & + \frac{det\,\mathcal{M}(B_k)}{det\,\mathcal{M}(B_{k-1})}\Big(\sum_{i=1}^{k-1}\Big|\frac{a_{ik}a_{ki}}{a_{ii}}\Big|\Big)\Big]. \end{aligned} \tag{12}$$

Proof. From A and $B \in H_n$, we know $A\circ B \in H_n$ by Lemma 2, that is: $\mathcal{M}(A\circ B) \in M_n$. It is easy to prove that

$$\mathcal{M}(A\circ B) = \mathcal{M}(\mathcal{M}(A)\circ\mathcal{M}(B))$$

hence,

$$\det[\mathcal{M}(A\circ B)] = \det[\mathcal{M}(\mathcal{M}(A)\circ\mathcal{M}(B))].$$

Obviously, $\mathcal{M}(A)$ and $\mathcal{M}(B) \in M_n$,

$$\mathcal{M}(A)_k = \mathcal{M}(A_k), \mathcal{M}(B)_k = \mathcal{M}(B_k),\ k = 1, 2, \cdots, n-1,$$

so by Lemma 6 we have

$$\begin{aligned} \det[\mathcal{M}(A\circ B)] =& \det[\mathcal{M}(\mathcal{M}(A)\circ\mathcal{M}(B))] \\ \geq & |a_{11}b_{11}|\prod_{k=2}^{n}\Big[|b_{kk}|\frac{\det\mathcal{M}(A)_k}{\det\mathcal{M}(A)_{k-1}} \\ & + \frac{\det\mathcal{M}(B)_k}{\det\mathcal{M}(B)_{k-1}}\Big(\sum_{i=1}^{k-1}\Big|\frac{a_{ik}a_{ki}}{a_{ii}}\Big|\Big)\Big] \\ =& |a_{11}b_{11}|\prod_{k=2}^{n}\Big[|b_{kk}|\frac{\det\mathcal{M}(A_k)}{\det\mathcal{M}(A_{k-1})} \end{aligned}$$

$$
+ \frac{det\mathcal{M}(B_k)}{\det\mathcal{M}(B_{k-1})}\Big(\sum_{i=1}^{k-1}\Big|\frac{a_{ik}a_{ki}}{a_{ii}}\Big|\Big)\Big] \tag{13}
$$

where $\mathcal{M}(A)_k$ and $\mathcal{M}(B)_k$, $k = 1, 2, \cdots, n-1$, denote the $k \times k$ leading principal submatrices of $\mathcal{M}(A)$ and $\mathcal{M}(B)$, respectively.

Theorem 5. *Let $A = (a_{ij})$ and $B = (b_{ij}) \in H_n$ and $\prod_{i=1}^{n} a_{ii}b_{ii} > 0$, A_k and B_k ($k = 1, 2, \cdots, n-1$) be the $k \times k$ leading principal submatrices of A and B, respectively, then*

$$
\begin{aligned}
det(A \circ B) \geq & |a_{11}b_{11}| \prod_{k=2}^{n}\Big[|b_{kk}|\frac{det\mathcal{M}(A_k)}{det\mathcal{M}(A_{k-1})} \\
& + \frac{det\mathcal{M}(B_k)}{det\mathcal{M}(B_{k-1})}\Big(\sum_{i=1}^{k-1}\Big|\frac{a_{ik}a_{ki}}{a_{ii}}\Big|\Big)\Big].
\end{aligned} \tag{14}
$$

Proof. By Theorem 3 and theorem 4, the conclusion is obvious.

Theorem 6. *Let $A = (a_{ij})$ and $B = (b_{ij}) \in H_n$ and $\prod_{i=1}^{n} a_{ii}b_{ii} > 0$, then*

$$
\begin{aligned}
det(A \circ B) \geq & \prod_{k=1}^{n}|b_{kk}| \cdot det\mathcal{M}(A) \\
& + \prod_{i=1}^{n}|a_{ii}| \cdot det\mathcal{M}(B) \cdot \Big(\prod_{k=2}^{n}\sum_{i=1}^{k-1}\Big|\frac{a_{ik}a_{ki}}{a_{ii}a_{kk}}\Big|\Big).
\end{aligned} \tag{15}
$$

Proof. By Theorem 5, the inequality (14) holds. Since all terms appearing in (14) are nonnegative, we have

$$
\begin{aligned}
\det(A \circ B) \geq & |a_{11}||b_{11}| \prod_{k=2}^{n}\Big[|b_{kk}|\frac{\det\mathcal{M}(A_k)}{\det\mathcal{M}(A_{k-1})} + \frac{\det\mathcal{M}(B_k)}{\det\mathcal{M}(B_{k-1})}\Big(\sum_{i=1}^{k-1}\Big|\frac{a_{ik}a_{ki}}{a_{ii}}\Big|\Big)\Big] \\
\geq & |a_{11}||b_{11}| \prod_{k=2}^{n}\Big(|b_{kk}|\frac{\det\mathcal{M}(A_k)}{\det\mathcal{M}(A_{k-1})}\Big) \\
& + |a_{11}b_{11}| \prod_{k=2}^{n}\Big[\frac{\det\mathcal{M}(B_k)}{\det\mathcal{M}(B_{k-1})}\Big(\sum_{i=1}^{k-1}\Big|\frac{a_{ik}a_{ki}}{a_{ii}}\Big|\Big)\Big] \\
= & \Big(\prod_{k=1}^{n}|b_{kk}|\Big) \cdot \det\mathcal{M}(A) \\
& + \Big(\prod_{i=1}^{n}|a_{ii}|\Big) \cdot \det\mathcal{M}(B) \cdot \Big[\prod_{k=2}^{n}\Big(\sum_{i=1}^{k-1}\Big|\frac{a_{ik}a_{ki}}{a_{ii}a_{kk}}\Big|\Big)\Big].
\end{aligned}
$$

4. Remark

When A and $B \in M_n$, A and $B \in H_n$, and $a_{ii} > 0$, $b_{ii} > 0$, $i = 1, 2, \cdots, n$, and $\mathcal{M}(A) = A$, $\mathcal{M}(B) = B$, $\mathcal{M}(A_k) = A_k, \mathcal{M}(B_k) = B_k$, $k = 1, 2, \cdots, n-1$, so the Theorem 4, Theorem 5 and Theorem 6 are the generalizations of Theorem 2.3, Theorem 2.1 and Corollary 2.2 of [1], respectively.

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