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PROBABILISTIC NUMERICAL APPROACH FOR PDE AND ITS APPLICATION IN THE VALUATION OF EUROPEAN OPTIONS*

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Abstract

This paper suggests a probabilistic numerical approach for a class of PDE. First of all, by simulating Brownian motion and using Monte-Carlo method, we obtain a probabilistic numerical solution for the PDE. Then, we prove that the probabilistic numerical solution converges in probability to its solution. At the end of this paper, as an application, we give a probabilistic numerical approach for the valuation of European Options, where we see volatility σ, interest rate r and divident rate D_0 as functions of stock S, respectively.

Key words: Brownian motion, Probabilistic numerical solution, European options.

1. Introduction

This paper is aimed to give a probabilistic numerical approach for PDE. Probabilistic numerical method can get the solution one by one, which differs from other numerical methods, such as the finite element and finite difference method, and realize total parallel computing easily. Another advantage of this method is that it suits for problems of high-dimension because it is dimension-independent.

Consider the following Cauchy problem of convection-diffusion equations. For simplicity, we will only discuss 1-dimension problem. The method can be easily extended to higher dimensional problems. Find $u = u(x,t)$ such that

$$(I)\begin{cases} u_t = a(x)u_{xx} + b(x)u_x + c(x)u & (1.1)\\ u(x,0) = \phi(x) & (1.2)\end{cases}$$

where $u(x,t)$ is an unknown function defined on $R^1 \times (0,T], \phi(x)$ is an initial function which satisfies some conditions. $a(x) > 0, b(x)$ and $c(x)$ are given functions.

In the first place, let $u(x,t) = v(y,t)$, $y = y(x)$. Then, we have

$$\begin{aligned} u_t &= v_t \\ u_x &= v_y \cdot y_x \\ u_{xx} &= v_{yy} \cdot (y_x)^2 + v_y \cdot y_{xx} \end{aligned}$$

and

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$$v_t = a(x)(y_x)^2 v_{yy} + (a(x)y_{xx} + b(x)y_x)v_y + c(x)v. \tag{1.3}$$

Furthermore, let $a(x)(y_x)^2 = \frac{1}{2}$, namely $y(x) = \int_0^x \frac{1}{\sqrt{2a(z)}}dz$. Note that $y(x)$ is a strictly increasing function. There exists a unique inverse function $x = x^{-1}(y)$. Replace all the x in (1.3) by $x^{-1}(y)$, we can get

$$(I')\begin{cases} v_t = \frac{1}{2}v_{yy} + B(y)v_y + C(y)v & (1.4)\\ v(y,0) = \Phi(y), & (1.5)\end{cases}$$

where

$$\begin{aligned} &B(y) = a(x^{-1}(y))y_{xx}(x^{-1}(y)) + b(x^{-1}(y))y_x(x^{-1}(y))\\ &C(y) = c(x^{-1}(y))\\ &\Phi(y) = \phi(x^{-1}(y)). \end{aligned}$$

From above transformations we see that we only need to give a probabilistic numerical solution for problem (I').

Now we introduce some notations and symbols. Let $\{\xi_t, \Im_t, t \geq 0\}$ be a 1-dim standard Brownian motion(for short, BM) starting at y. P_y is a probabilistic measure about BM and E_y is the expectation related to P_y. Define

$$\begin{aligned} &W_s \equiv \int_0^s B(\xi_r)d\xi_r - \frac{1}{2}\int_0^s |B(\xi_r)|^2 dr\\ &L_s \equiv \int_0^s C(\xi_r)dr\\ &Z_s \equiv W_s + L_s \qquad s \geq 0. \end{aligned}$$

2. Probabilistic Numerical Approach for (I')

2.1. Simulating the Path of Brownian Motion

Suppose that $\eta_i^0, \eta_i^1, \cdots, \eta_i^N (i \geq 1)$ are independent random variables with uniform distribution on $(0,1)$. Let

$$\widehat{\xi}_i^k = \sqrt{-2\ln \eta_i^{k-1}}\cos 2\pi\eta_i^k \qquad i \geq 1, k = 1, 2, \cdots, N$$

then $\widehat{\xi}_i^1, \cdots, \widehat{\xi}_i^N$ $(i \geq 0)$ are independent random variables with normal distribution $N(0,1)$

Let $y \in R^1$ and divide time interval $[0,T]$ $(0 < T < \infty)$ into:

$$0 = t_0 < t_1 < \cdots < t_m < \cdots < t_n = T, \quad t_i - t_{i-1} = h.$$

Take

$$\begin{aligned} &\xi_i^k - \xi_{i-1}^k = \sqrt{h}\widehat{\xi}_i^k \quad i \geq 0, k = 1, 2, \cdots, N\\ &\xi_0^k \equiv y. \end{aligned}$$

Then $\xi_i^k - \xi_{i-1}^k, i = 1, \cdots, N$ have normal distribution $N(0, \sqrt{h})$. In addition, $\xi_i^k - \xi_{i-1}^k$ and $\xi_{i-1}^k - \xi_{i-2}^k$ are independent for $i \geq 2$. So, for each $k = 1, 2, \cdots, N$, $\xi_0^k \equiv y$ and $\xi_1^k, \cdots, \xi_i^k, \cdots$ gives a simulation of one path of BM,which starts at y.

2.2. Probabilistic Numerical Solution of (I')

It is known that under the conditions of Theorem 5[1], (I') has the probabilistic solution:

$$v(y,t) = E_y[\Phi(\xi_t)\exp(Z_t)] \tag{2.1}$$

In this paper we can get probabilistic numerical solution of (I') for any given point $(y, t_m) \in R^1 \times [0, T]$. Draw N samples from population of BM and write k sample as ξ_s^k. The results at time $s = t_0, t_1, \cdots, t_i, \cdots$ are $\xi_0^k \equiv y, \xi_1^k, \cdots, \xi_i^k, \cdots$ (without loss of generality we can assume that t_m is just the mth partition point in time interval $[0, T]$).

Firstly, we use Monte-Carlo method to give the approximation of the expectation of (2.1):

$$\overline{v}(y, t_m) = \frac{1}{N}\sum_{k=1}^{N}\{\exp[\sum_{i=1}^{m}(\int_{t_{i-1}}^{t_i} B(\xi_s^k)d\xi_s^k - \frac{1}{2}\int_{t_{i-1}}^{t_i}|B(\xi_s^k)|^2 ds + \int_{t_{i-1}}^{t_i} C(\xi_s^k)ds)]\Phi(\xi_m^k)\}.$$

Let

$$\begin{aligned}
\widehat{W}_l^k &= \sum_{i=1}^{l} B(\xi_i^k)(\xi_i^k - \xi_{i-1}^k) - \frac{1}{2}\sum_{i=1}^{l}\frac{h}{2}(|B(\xi_i^k)|^2 + |B(\xi_{i-1}^k)|^2) \\
\widehat{L}_l^k &= \sum_{i=1}^{l}\frac{h}{2}(C(\xi_i^k) + C(\xi_{i-1}^k)) \\
\widehat{Z}_l^k &= \widehat{W}_l^k + \widehat{L}_l^k
\end{aligned}$$

Secondly, according to the definition of stochastic integral and trapezoidal integral formula, we have

$$\begin{aligned}
\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i} B(\xi_s^k)d\xi_s^k &= \sum_{i=1}^{m} B(\xi_{i-1}^k)(\xi_i^k - \xi_{i-1}^k) + \overline{\epsilon}_k^m(B) \\
\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i} |B(\xi_s^k)|^2 ds &= \sum_{i=1}^{m}\frac{h}{2}(|B(\xi_i^k)|^2 + |B(\xi_{i-1}^k)|^2) + \epsilon_k^m(|B|^2) \\
\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i} C(\xi_s^k)ds &= \sum_{i=1}^{m}\frac{h}{2}(C(\xi_i^k) + C(\xi_{i-1}^k)) + \epsilon_k^m(C).
\end{aligned}$$

Here we set $\sum_{i=1}^{0} \equiv 0$; $\overline{\epsilon}_k^0(\cdot)$, $\epsilon_k^0(\cdot) \equiv 0$).

Round the remainder terms $\overline{\epsilon}_k^m(B), \epsilon_k^m(|B|^2), \epsilon_k^m(C)$,substitute into (2.1) and finally we gain the probabilistic numerical solution of (I'):

$$\widehat{v}(y, t_m) = \frac{1}{N}\sum_{k=1}^{N}[\exp(\widehat{Z}_m^k)\Phi(\xi_m^k)] \tag{2.2}$$

2.3. Convergence

Suppose $\Phi \in C_b^0(R^1)$, the class of bounded continuous functions, $B, C \in C_b^1(R^1)$, the class of functions that their first derivatives are bounded continuous. $||\cdot||_0, ||\cdot||_1$ represents the norm of $C_b^0(R^1), C_b^1(R^1)$ respectively.

Theorem. *Suppose $v(y,t_m)$ be probabilistic solution of (I'). Then for $\forall\alpha, 0<\alpha<1$, with the confidence level $1-\alpha$, we have*

$$|\widehat{v}(y,t_m) - v(y,t_m)| \leq O(\sqrt{h}) + O(N^{-\frac{1}{4}}).$$

Hence, the probabilistic numerical solution of (I') converges to its probabilistic solution in probability as $h \to 0, N \to \infty$.

Proof. From [4], we know that for any $\alpha, 0<\alpha<1$,

$$|v(y,t_m) - \overline{v}(y,t_m)| \leq C_{\frac{\alpha}{5}} D[\exp(Z_{t_m})\Phi(\xi_{t_m})] \cdot N^{-\frac{1}{2}} \tag{2.3}$$

holds with the confidence level $1-\frac{\alpha}{5}$, where $C_{\frac{\alpha}{5}}$ satisfies $\int_{-\frac{\alpha}{5}}^{\frac{\alpha}{5}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1-\frac{\alpha}{5}$, $D[\cdot]$ represents the variance.

According to the triangle inequality:

$$|\widehat{v}(y,t_m) - v(y,t_m)| \leq |v(y,t_m) - \overline{v}(y,t_m)| + |\overline{v}(y,t_m) - \widehat{v}(y,t_m)|.$$

So, we only need to estimate the term $|\overline{v}(y,t_m) - \widehat{v}(y,t_m)|$. In order to get the estimate, we give some lemmas first. Let

$$\begin{aligned} W_s^k &\equiv \int_0^s B(\xi_r^k) d\xi_r^k - \frac{1}{2}\int_0^s |B(\xi_r^k)|^2 dr \\ L_s^k &\equiv \int_0^s C(\xi_r^k) dr \\ Z_s^k &\equiv W_s^k + L_s^k \qquad s \geq 0. \end{aligned}$$

Lemma 2.1. *With the confidence level $1-\frac{\alpha}{5}$ we have*

$$|\epsilon_k^m(|B|^2)| \leq 6\sqrt{h} C_\delta ||B||_1^2 T \tag{2.4}$$

$$|\epsilon_k^m(C)| \leq 3\sqrt{h} C_\delta ||C||_1 T, \tag{2.5}$$

where δ satisfies $(1-\delta)^n \geq 1-\frac{\alpha}{5}$, $nh = T$.

Proof. Choose $\delta > 0, (1-\delta)^n \geq 1-\frac{\alpha}{5}$. Then with the confidence level $1-\delta$, ξ_t^k will lie in $O_{i-1}^k \equiv [\xi_{i-1}^k - \sqrt{h}C_\delta, \xi_{i-1}^k + \sqrt{h}C_\delta]$ when $t \in [t_{i-1}, t_i]$. Set

$$\begin{aligned} \Delta_i^k(|B|^2) &= \max_{-C_\delta \leq a \leq C_\delta} ||B(\xi_{i-1}^k + \sqrt{h}a)|^2 - |B(\xi_{i-1}^k)|^2| \\ \Delta_i^k(C) &= \max_{-C_\delta \leq a \leq C_\delta} |C(\xi_{i-1}^k + \sqrt{h}a) - C(\xi_{i-1}^k)|. \end{aligned}$$

Then,with the confidence level $1-\frac{\alpha}{5}$ we have

$$\begin{aligned}|\epsilon_k^m(|B|^2)| &= |\int_0^{t_m}|B(\xi_s^k)|^2ds-\sum_{i=1}^m\frac{h}{2}(|B(\xi_i^k)|^2+|B(\xi_{i-1}^k)|^2)|\\ &\le \frac{1}{2}\sum_{i=1}^m|\int_{t_{i-1}}^{t_i}[(|B(\xi_s^k)|^2-|B(\xi_i^k)|^2)+(|B(\xi_s^k)|^2-|B(\xi_{i-1}^k)|^2)]ds|\\ &\le \sum_{i=1}^m\Delta_i^k(|B|^2)h.\end{aligned}$$

Similarly, we can get

$$|\epsilon_k^m(C)|\le\frac{3}{2}\sum_{i=1}^m\Delta_i^k(C)h.$$

From the Taylor expansion, there exists $y_i^{*k}\in O_{i-1}^k$, such that

$$\Delta_i^k(|B|^2)=\max_{-C_\delta\le a\le C_\delta}|2B(y_i^{*k})B_y(y_i^{*k})\sqrt{h}a|\le 4C_\delta\sqrt{h}||B||_1^2.$$

By the same way

$$\Delta_i^k(C)\le 2\sqrt{h}C_\delta||C||_1.$$

Hence, the Lemma is proved.

Lemma 2.2. *With the confidence level* $1-\frac{\alpha}{5}$,*we have*

$$\frac{1}{N}\sum_{k=1}^N[\sup_{0\le s\le t_m}\exp(W_s^k)]\le M+C_{\frac{\alpha}{5}}D[\sup_{0\le s\le t_m}\exp(W_s)]N^{-\frac{1}{2}} \tag{2.6}$$

where $M<\infty$ *is a constant.*

Proof. From §8.4[2],we know

$$G\equiv E[\sup_{0\le s\le t_m}\exp(W_s)]\le M<\infty$$

for some constant M. On the one hand, by Monte-Carlo method we can get the approximation of G:

$$\overline{G}=\frac{1}{N}\sum_{k=1}^N[\sup_{0\le s\le t_m}\exp(W_s^k)].$$

With the confidence level $1-\frac{\alpha}{5}$, we have

$$|G-\overline{G}|\le C_{\frac{\alpha}{5}}D[\sup_{0\le s\le t_m}\exp(W_s)]N^{-\frac{1}{2}}.$$

Hence

$$\overline{G}\le M+C_{\frac{\alpha}{5}}D[\sup_{0\le s\le t_m}\exp(W_s)]N^{-\frac{1}{2}}.$$

Lemma 2.3. *With the confidence level* $1-\frac{2\alpha}{5}$*,we have*

$$\frac{1}{N}\sum_{k=1}^{N}|\overline{\epsilon}_k^m(B)|^2 \le C_{\frac{\alpha}{5}}D[\int_0^{t_m}B(\xi_s)d\xi_s-\sum_{i=1}^{m}B(\xi_i)(\xi_i-\xi_{i-1})]^2N^{-\frac{1}{2}}$$
$$+C_{\frac{\alpha}{5}}D[\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}|B(\xi_s)-B(\xi_i)|^2ds]N^{-\frac{1}{2}}$$
$$+4C_\delta^2||B||_1^2hT. \tag{2.7}$$

Proof. Let

$$B'(\xi_s)=B(\xi_{i-1}) \qquad t_{i-1}<s\le t_i$$

then $B'(\xi_s)$ is a step function of L^2. According to the definition of stochastic integral we know

$$\int_0^{t_m}B'(\xi_s)d\xi_s=\sum_{i=1}^{m}B(\xi_{i-1})(\xi_i-\xi_{i-1}).$$

From the relationship between stochastic integral and path integral[3]:

$$E|\int_0^{t_m}B(\xi_s)d\xi_s-\int_0^{t_m}B'(\xi_s)d\xi_s|^2=E\int_0^{t_m}|B(\xi_s)-B'(\xi_s)|^2ds \tag{2.8}$$

and Monte-Carlo method the approximation of $E|\int_0^{t_m}B(\xi_s)d\xi_s-\int_0^{t_m}B'(\xi_s)d\xi_s|^2$ is:

$$\frac{1}{N}\sum_{k=1}^{N}|\overline{\epsilon}_k^m(B)|^2.$$

Also with the confidence level $1-\frac{\alpha}{5}$ we have

$$|E|\int_0^{t_m}B(\xi_s)d\xi_s-\int_0^{t_m}B'(\xi_s)d\xi_s|^2-\frac{1}{N}\sum_{k=1}^{N}|\overline{\epsilon}_k^m(B)|^2|$$
$$\le C_{\frac{\alpha}{5}}D[\int_0^{t_m}B(\xi_s)d\xi_s-\int_0^{t_m}B'(\xi_s)d\xi_s]^2N^{-\frac{1}{2}}. \tag{2.9}$$

Similarly, the approximation of $E\int_0^{t_m}|B(\xi_s)-B'(\xi_s)|^2ds$ is:

$$\frac{1}{N}\sum_{k=1}^{N}\int_0^{t_m}|B(\xi_s^k)-B'(\xi_s^k)|^2ds,$$

and with the confidence level $1-\frac{\alpha}{5}$

$$|E\int_0^{t_m}|B(\xi_s)-B'(\xi_s)|^2ds-\frac{1}{N}\sum_{k=1}^{N}\int_0^{t_m}|B(\xi_s^k)-B'(\xi_s^k)|^2ds|$$
$$\le C_{\frac{\alpha}{5}}D[\int_0^{t_m}|B(\xi_s)-B'(\xi_s)|^2ds]N^{-\frac{1}{2}}. \tag{2.10}$$

Using the same approach as in Lemma2.1, we obtain:

$$\Delta_i^k(B) \leq 2\sqrt{h}C_\delta ||B||_1.$$

Thus,

$$\begin{aligned}
&\frac{1}{N}\sum_{k=1}^{N}\int_0^{t_m}|B(\xi_s^k)-B'(\xi_s^k)|^2 ds \\
&=\frac{1}{N}\sum_{k=1}^{N}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}|B(\xi_s^k)-B'(\xi_s^k)|^2 ds \\
&\leq \frac{1}{N}\sum_{k=1}^{N}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}4C_\delta^2||B||_1^2 h ds \\
&\leq 4C_\delta^2||B||_1^2 Th.
\end{aligned} \tag{2.11}$$

Combining (2.9), (2.10) and (2.11) yields (2.7).

Lemma 2.4. *With the confidence level* $1-\frac{\alpha}{5}$*,we have*

$$\frac{1}{N}\sum_{k=1}^{N}\exp(2W_s^k) \leq \exp(T||B||_1) + C_{\frac{\alpha}{5}}D[\exp(2W_s)]N^{-\frac{1}{2}}. \tag{2.12}$$

Proof. From §8.4[2], the following inequality:

$$E\exp(2W_s) \leq \exp(s||B||_1) \leq \exp(T||B||_1)$$

holds, proceeding in the same way as Lemma 2.2, we can prove this Lemma.

Now, we can estimate the term $|\overline{v}(y,t_m)-\widehat{v}(y,t_m)|$. Because

$$\begin{aligned}
&|\overline{v}(y,t_m)-\widehat{v}(y,t_m)| \\
&=|\frac{1}{N}\sum_{k=1}^{N}\exp(Z_{t_m}^k)\Phi(\xi_m^k)-\frac{1}{N}\sum_{k=1}^{N}\exp(\widehat{Z}_m^k)\Phi(\xi_m^k)| \\
&\leq ||\Phi||_0|\frac{1}{N}\sum_{k=1}^{N}\exp(Z_{t_m}^k)[1-\exp(-\overline{\epsilon}_k^m(B)+\frac{1}{2}\epsilon_k^m(|B|^2)-\epsilon_k^m(C))]| \\
&\leq ||\Phi||_0\frac{1}{2}\exp(T||C||_1)\sum_{k=1}^{N}\exp(W_{t_m}^k)|1-\exp(-\overline{\epsilon}_k^m(B)+\frac{1}{2}\epsilon_k^m(|B|^2)-\epsilon_k^m(C)|,
\end{aligned}$$

then, according to the exponential inequality, Holder inequality

$$\begin{aligned}
&|e^y-1|\leq 3|y| \qquad \forall |y|\leq 1 \\
&\sum|fg|\leq|\sum f^2|^{1/2}|\sum g^2|^{1/2}
\end{aligned}$$

and Lemma2.1–Lemma2.4,with the confidence level $1-\frac{4\alpha}{5}$ we have:

$$
\begin{aligned}
&|\overline{v}(y,t_m)-\widehat{v}(y,t_m)| \\
&\le 3||\Phi||_0 \exp(T||C||_1)\frac{1}{N}\sum_{k=1}^{N}\exp(W_{t_m}^k)[|\overline{\epsilon}_k^m(B)|+|\frac{1}{2}\epsilon_k^m(|B|^2)|+|\epsilon_k^m(C)|] \\
&\le 3||\Phi||_0 \exp(T||C||_1)[\frac{1}{N}\sum_{k=1}^{N}\exp(2W_{t_m}^k)]^{1/2}[\frac{1}{N}\sum_{k=1}^{N}|\overline{\epsilon}_k^m(B)|^2]^{1/2} \\
&+9||\Phi||_0 \exp(T||C||_1)\sqrt{h}C_\delta||B||_1^2 T\cdot\frac{1}{N}\sum_{k=1}^{N}\exp(W_{t_m}^k) \\
&+9||\Phi||_0 \exp(T||C||_1)\sqrt{h}C_\delta||C||_1 T\cdot\frac{1}{N}\sum_{k=1}^{N}\exp(W_{t_m}^k) \\
&\le 3||\Phi||_0 \exp(||C||_1)[\exp(T||B||_1+O(N^{-\frac{1}{2}})]^{1/2}[O(h)+O(N^{-\frac{1}{2}})]^{1/2} \\
&+9||\Phi||_0 \exp(T||C||_1)\sqrt{h}C_\delta||B||_1^2 T[M+O(N^{-\frac{1}{2}})] \\
&+9||\Phi||_0 \exp(T||C||_1)\sqrt{h}C_\delta||C||_1 T[M+O(N^{-\frac{1}{2}})] \\
&\le O(\sqrt{h})+O(N^{-\frac{1}{4}}).
\end{aligned}
\tag{2.13}
$$

From (2.3), (2.13) we see that, with confidence level $1-\alpha$

$$|\widehat{v}(y,t_m)-v(y,t_m)|\le O(\sqrt{h})+O(N^{-\frac{1}{4}}) \tag{2.14}$$

Summarize the above argument, we know that for $\forall\alpha, 0<\alpha<1, \exists C_{\frac{\alpha}{5}}$ and C_δ, the right hand side of (2.14) approximates to 0 as $N\to\infty, h\to 0$. That is, the probabilistic numerical solution of (I') converges to its solution in probability.

3. Application in the Valuation of European Options

As an application of the probabilistic numerical method, in this section, we will deal with the valuation of European Options, namely, solve the following PDE–Black-Scholes model.

$$
(II)\begin{cases} c_t=\frac{1}{2}\sigma^2 S^2 c_{SS}+(r(S)-D_0(S))Sc_S+r(S)c & (3.1)\\ c(S,0)=\max(S-E,0) & (3.2)\end{cases}
$$

where[5,6]

t^* : current date,

T : expiration date of the option,

t : time to expiration,

S : price of stock at t^*,

$c(S,t)$: the value of an European option at t^*,

σ : volatility,

$r(S)$: the interest rate,

$D_0(S)$: the divident yield,

E : the exercise price.

Remark 3.1. In this paper, the interest rate $r(S)$ and the divident rate $D_0(S)$ are functions of stock S. It is more reasonable to see the interest and divident as the functions of the stock than to see them as constants. We can also deal with the case when volatility σ is also a function of stock S by using probabilistic numerical approach, but for general function $\sigma(S)$,we can not give an explicit transformation like seeing σ as a constant.

According to the above method,we transform the variable first. Let

$$c(S,t) = u(x,t), \quad x = \frac{1}{\sigma}\ln S, \quad S = e^{\sigma x}$$

we can get

$$\begin{aligned} c_t &= u_t \\ c_S &= u_x \cdot \frac{1}{\sigma S} \\ c_{SS} &= u_{xx} \cdot (\frac{1}{\sigma S})^2 + u_x \cdot (-\frac{1}{\sigma S^2}) \end{aligned}$$

And (II) changes into (II'):

$$(II') \begin{cases} u_t = \frac{1}{2}u_{xx} + (\frac{r(e^{\sigma x}) - D_0(e^{\sigma x})}{\sigma} - \sigma)u_x + r(e^{\sigma x})u & (3.3) \\ u(x,0) = \max(e^{\sigma x} - E, 0) & (3.4) \end{cases}$$

So the probabilistic solution and probabilistic numerical solution of (II') are:

$$u(x,t) = E_x[\max(e^{\sigma\xi_t} - E, 0)\exp(M_t)] \tag{3.5}$$

$$\widehat{u}(x,t_m) = \frac{1}{N}\sum_{k=1}^{N}[\max(e^{\sigma\xi_m^k} - E, 0)\exp(\widehat{M}_m^k)] \tag{3.6}$$

respectively, where the definitions of M_t and $\widehat{M}_m^k$ are similar to the definitions of Z_t and $\widehat{Z}_m^k$ in §2.

Now we can obtain the probabilistic numerical solution for the valuation of European Options by simulating Brownian motion starting at x.

$$\widehat{c}(S,t_m) = \widehat{u}(x,t_m)$$

Remark 3.2. we can see (3.6) as an average for a kind of transformation of Brownian motion–stock price. And, the European Options price heavily relies on the probability of event $\{\omega : e^{\sigma\xi_m^k} > E\}$ and stock price at t_m^*. We can see that probabilistic numerical solution is able to reflect the reality directly.

Using the probabilistic numerical approach, we can get the solution one by one and partition-free. Moreover, it suits for total parallel computing. Because this method needs to simulate Brownian motion, its convergence accuracy is not high.

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