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GLOBAL FINITE ELEMENT NONLINEAR GALERKIN METHOD FOR THE PENALIZED NAVIER-STOKES EQUATIONS*1)

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Abstract

A global finite element nonlinear Galerkin method for the penalized Navier-Stokes equations is presented. This method is based on two finite element spaces X_H and X_h, defined respectively on one coarse grid with grid size H and one fine grid with grid size $h << H$. Comparison is also made with the finite element Galerkin method. If we choose $H = O(\varepsilon^{-1/4}h^{1/2})$, $\varepsilon > 0$ being the penalty parameter, then two methods are of the same order of approximation. However, the global finite element nonlinear Galerkin method is much cheaper than the standard finite element Galerkin method. In fact, in the finite element Galerkin method the nonlinearity is treated on the fine grid finite element space X_h and while in the global finite element nonlinear Galerkin method the similar nonlinearity is treated on the coarse grid finite element space X_H and only the linearity needs to be treated on the fine grid increment finite element space W_h. Finally, we provide numerical test which shows above results stated.

Key words: Nonlinear Galerkin method, Finite element, Penalized Navier-Stokes equations.

1. Introduction

In the numerical simulation of the Navier-Stokes equations one encounters three serious difficulties in the case of large Reynolds numbers: the treatment of the incompressibility condition div$u = 0$, the treatment of the nonlinear terms and the large time integration. For the treatment of the incompressibility condition, one use the penalty method in the case of finite elements [1-2] and for the treatment of the nonlinear terms and the large time integration, one use the nonlinear Galerkin method in the framework of finite elements [3]. However, in this work the finite element nonlinear Galerkin method is only used in the time interval $[t_0, \infty)$ and the finite element Galerkin method is used in the finite time interval $[0, t_0]$, $t_0 > 0$ is finite.

Our purpose here is to present a new global finite element nonlinear Galerkin method for the penalized Navier-Stokes equations in the framework of finite elements. This numerical simulation is done in the time interval $[0, \infty)$. Moreover, we analyze the convergence rates of the finite element Galerkin method and the global finite element nonlinear Galerkin method. If $H = O(\varepsilon^{-1/4}h^{1/2})$ is chosen then the global finite element nonlinear Galerkin method provides the same order of approximation as the finite element Galerkin method, where $\varepsilon > 0$ is the penalty parameter. However, in the global finite element nonlinear Galerkin method, the nonlinearity is treated on the coarse grid finite element space X_H and only the linearity is treated on the fine grid increment finite element space W_h; while in the finite element Galerkin method the nonlinearity needs to be treated on the fine grid finite element space X_h. Hence, under the convergence rate of same order, the global finite element nonlinear Galerkin method is much

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cheaper to implement computationally than the finite element Galerkin method. Finally, we provide numerical test which shows the above results stated.

2. The Penalized Navier-Stokes Equations

Let $\Omega \subset R^2$ be a bounded open set with Lipschitz boundary $\Gamma = \partial\Omega$. The Navier-Stokes equations of incompressible flows reads

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \mathrm{grad}p = f, \quad \forall(x,t)\in\Omega\times R^+, \tag{2.1}$$

$$\mathrm{div}u = 0, \quad \forall(x,t)\in\Omega\times R^+, \tag{2.2}$$

where $u = u(x,t)$ is the velocity vector, $p = p(x,t)$ is the pressure, $\nu > 0$ is the kinematic viscosity and f represents the volume driving forces, for simplicity, the constant density ρ was taken equal to 1.

For the penalized equations we suppress the pressure p and the incompressibility equation (2.2) and introduce in (2.1) a penalty term, $\frac{\nu}{\varepsilon}$grad div$u$, $\varepsilon > 0$ the penalty parameter. Hence, we obtain the penalized Navier-Stokes equations:

$$\begin{aligned}&\frac{\partial u_\varepsilon}{\partial t} - \nu\Delta u_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon + \frac{1}{2}(\mathrm{div}u_\varepsilon)u_\varepsilon \\ &\qquad - \frac{\nu}{\varepsilon}\mathrm{grad}\,\mathrm{div}u_\varepsilon = f, \quad \forall(x,t)\in\Omega\times R^+.\end{aligned} \tag{2.3}$$

We have also introduce the supplementary nonlinear term $\frac{1}{2}(\mathrm{div}u_\varepsilon)u_\varepsilon$ which make (2.2) well set.

The equation (2.3) is supplemented by boundary and initial conditions:

$$u_\varepsilon = 0, \qquad \text{on } \Gamma\times R^+, \tag{2.4}$$

$$u_\varepsilon(x,0) = u_0(x), \qquad \forall x\in\Omega. \tag{2.5}$$

We introduce the basic spaces:

$$Y = L^2(\Omega)^2, \quad X = H_0^1(\Omega)^2$$

provided with the scalar products and norms

$$\begin{aligned}(u,v) &= \int_\Omega u(x)\cdot v(x)dx, \quad |u| = (u,u)^{1/2}, &&\forall u,v\in Y,\\ ((u,v)) &= \int_\Omega \mathrm{grad}u\cdot\mathrm{grad}v dx, \quad \|u\| = ((u,u))^{1/2}, &&\forall u,v\in X.\end{aligned}$$

Moreover, we also introduce the following operators:

$$\begin{aligned}&Au = -\Delta u, \quad B(u,v) = (u\cdot\nabla)v + \frac{1}{2}(\mathrm{div}u)v,\\ &Du = \frac{\nu}{\varepsilon}\mathrm{grad}\,\mathrm{div}u.\end{aligned}$$

It is well-known [1-2] that A is a linear unbounded, self-adjoint positive closed operator with the domain

$$D(A) = X\cap H^2(\Omega)^2,$$

and the inverse A^{-1} of A is a compact self-adjoint operator in Y. Then, we obtain the abstract equation

$$\frac{du_\varepsilon}{dt} + \nu Au_\varepsilon + Du_\varepsilon + B(u_\varepsilon,u_\varepsilon) = f. \tag{2.6}$$

Also, we introduce the bilinear forms:

$$a(u,v) = \nu < Au, v >= \nu((u,v)),$$
$$d(u,v) = \frac{\nu}{\varepsilon}(\mathrm{div}u, \mathrm{div}v),$$

and the trilinear forms:

$$b(u,v,w) =< B(u,v), w >= \frac{1}{2}\int_\Omega (u\cdot\nabla)v\cdot w dx - \frac{1}{2}\int_\Omega (u\cdot\nabla)w\cdot v dx$$

Then there hold the following estimates (see [1-2]):

$$b(u,v,w) = -b(u,w,v), \quad \forall u,v,w \in X, \tag{2.7}$$
$$|b(u,v,w)| \le c_0|u|^{1/2}\|u\|^{1/2}(\|v\||w|^{1/2}\|w\|^{1/2} + |v|^{1/2}\|v\|^{1/2}\|w\|),$$
$$|b(u,v,w)| \le c_0(|u|^{1/2}\|u\|^{1/2}\|v\| + \|u\||v|^{1/2}\|v\|^{1/2})|w|^{1/2}\|w\|^{1/2}, \tag{2.8}$$

for all $u,v,w \in X$ and

$$|B(u,v)| \le c_1(|u|^{1/2}|Au|^{1/2}\|v\| + \|u\||v|^{1/2}|Av|^{1/2}),$$
$$|B(u,v)| \le c_1(|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2} + \|u\||v|^{1/2}|Av|^{1/2}), \tag{2.9}$$

for all $u,v \in D(A)$ and

$$|u| \le \lambda_1^{-1/2}\|u\|, \quad \forall u \in X. \tag{2.10}$$

With the above notations, we obtain the variational formulation of (2.5)-(2.6): Find $u_\varepsilon \in L^\infty(R^+;Y)\cap L^2(0,T;X), \ \ \forall T>0$ such that

$$(\frac{du_\varepsilon}{dt}, v) + a(u_\varepsilon, v) + d(u_\varepsilon, v) + b(u_\varepsilon, u_\varepsilon, v) = (f,v), \quad \forall v \in X, \tag{2.11}$$

$$u_\varepsilon(0) = u_0. \tag{2.12}$$

We recall that for u_0 given in $H = \{v \in Y; \mathrm{div}v = 0 \text{ and } v\cdot n|_\Gamma = 0\}$ and $f \in L^\infty(R^+;Y)$, (2.11)-(2.12) admits a unique solution $u_\varepsilon \in L^\infty(R^+;Y)\cap L^2(0,T;X), \ \ \forall T>0$ (see [1-2]).

Furthermore, we can prove that if $u_0 \in X\cap H$, then the solution of (2.11)-(2.12) satisfies the following regularities:

$$|u_\varepsilon(t)|^2 \le M_0^2, \quad \forall t \ge 0, \tag{2.13}$$

$$\|u_\varepsilon(t)\|^2 \le M_1^2, \quad \nu\int_0^t |Au_\varepsilon(\tau)|^2 d\tau \le \|u_0\|^2 + M_2^2 t, \quad \forall t\ge 0, \tag{2.14}$$

where M_1 and M_2 are positive constants and

$$M_0^2 = |u_0|^2 + 2\frac{f_\infty^2}{\nu^2\lambda_1^2}, \quad f_\infty = \sup_{t\ge 0}|f(t)|.$$

Proof. Taking $v = u_\varepsilon$ in (2.11) and using (2.7), (2.10), we obtain

$$\frac{1}{2}\frac{d}{dt}|u_\varepsilon|^2 + \nu\|u_\varepsilon\|^2 = (f,u_\varepsilon) \le \frac{\nu}{4}\|u_\varepsilon\|^2 + \frac{|f|^2}{\nu\lambda_1},$$

which yields

$$\frac{d}{dt}|u_\varepsilon|^2 + \nu\lambda_1|u_\varepsilon|^2 + \frac{\nu}{2}\|u_\varepsilon\|^2 \le 2|f|^2\nu\lambda_1. \tag{2.15}$$

Integrating (2.15) and using (2.12), we have

$$e^{\nu\lambda_1 t}|u_\varepsilon(t)|^2 \le |u_0|^2 + \frac{2f_\infty^2}{\nu^2\lambda_1^2}(e^{\nu\lambda_1 t} - 1), \tag{2.16}$$

$$\frac{\nu}{2}\int_t^{t+1} \|u_\varepsilon(\tau)\|^2 d\tau \le |u_\varepsilon(\tau)|^2 + \frac{2f_\infty^2}{\nu\lambda_1}, \tag{2.17}$$

$$\frac{\nu}{2}\int_0^t \|u_\varepsilon(\tau)\|^2 d\tau \le |u_0|^2 + \frac{2f_\infty^2}{\nu\lambda_1}t. \tag{2.18}$$

From (2.16), we derive (2.13).

Next, we take $v = Au_\varepsilon$ in (2.11). Then we have

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon\|^2 + \nu|Au_\varepsilon|^2 + \frac{\nu}{\varepsilon}|\nabla \mathrm{div} u_\varepsilon|^2 + b(u_\varepsilon, u_\varepsilon, Au_\varepsilon) = (f, Au_\varepsilon). \tag{2.19}$$

Thanks to (2.9) and the Young inequality, we derive

$$(f, Au_\varepsilon) \le \frac{\nu}{4}|Au_\varepsilon|^2 + \nu^{-1}|f|^2, \tag{2.20}$$

$$\begin{aligned} |b(u_\varepsilon, u_\varepsilon, Au_\varepsilon)| &\le 2c_1|u_\varepsilon|^{1/2}\|u_\varepsilon\||Au_\varepsilon|^{3/2} \\ &\le \frac{\nu}{4}|Au_\varepsilon|^2 + 4^2(\frac{2}{\nu})^3 c_1^4|u_\varepsilon|^2\|u_\varepsilon\|^4. \end{aligned} \tag{2.21}$$

Combining (2.19) with (2.20)-(2.21) we arrive at

$$\frac{d}{dt}\|u_\varepsilon\|^2 + \nu|Au_\varepsilon|^2 \le \frac{2}{\nu}|f|^2 + 4(\frac{4}{\nu})^3 c_1^4|u_\varepsilon|^2\|u_\varepsilon\|^4. \tag{2.22}$$

First we deduce from (2.22) that

$$y' \le h + gy, \quad y = \|u_\varepsilon\|^2, \quad h = \frac{2}{\nu}|f|^2, \quad g = 4(\frac{4}{\nu})^3 c_1^4|u_\varepsilon|^2\|u_\varepsilon\|^4. \tag{2.23}$$

Integrating (2.23) from 0 to t, $0 < t \le 1$ for t, we obtain

$$y(t) \le y(0) + \frac{2}{\nu}f_\infty^2 + \int_0^t g(\tau)y(\tau)d\tau. \tag{2.24}$$

From (2.13), (2.18) and the classical Gronwall lemma, we have

$$y(t) = \|u_\varepsilon(t)\|^2 \le (\|u_0\|^2 + \frac{2}{\nu}f_\infty^2)\exp(2(\frac{4}{\nu})^4 c_1^4 M_0^2(|u_0|^2 + \frac{2f_\infty^2}{\nu\lambda_1})), \quad \forall 0 \le t \le 1. \tag{2.25}$$

In order to estimate $\|u_\varepsilon(t)\|$, $\forall t \ge 1$, let we recall the uniform Gronwall lemma (see [2]):

Lemma 2.1. *Let g, h, y be three positive locally integrable functions on (t_0, ∞) satisfying*

$$\frac{dy}{dt} \le gy + h \quad \textit{for } t \ge t_0, \tag{2.26}$$

$$\int_t^{t+r} g(\tau)d\tau \le a_1, \quad \int_t^{t+r} h(\tau)d\tau \le a_2, \quad \int_t^{t+r} y(\tau)d\tau \le a_3, \quad \forall t \ge t_0, \tag{2.27}$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t+r) \le (\frac{a_3}{r} + a_2)\exp(a_1), \quad \forall t \ge t_0. \tag{2.28}$$

Thanks to (2.13) and (2.17), the conditions (2.26)-(2.27) in Lemma 2.1 are satisfied. Hence, by applying Lemma 2.1 with $t_0 = 0$, $r = 1$ to (2.23), we arrive at

$$\|u_\varepsilon(t)\|^2 \le M_1', \quad \forall t \ge 1, \tag{2.29}$$

which and (2.25) yields

$$\|u_\varepsilon(t)\|^2 \le M_1^2, \quad \forall t \ge 0. \tag{2.30}$$

Finally, by integrating (2.22) and using (2.13), (2.30), we obtain

$$\nu\int_0^t |Au_\varepsilon(\tau)|^2 d\tau \le \|u_0\|^2 + (\frac{2}{\nu}f_\infty^2 + 4(\frac{4}{\nu})^3 c_1^4 M_0^2 M_1^4)t. \tag{2.31}$$

Combining (2.30) with (2.31) yields (2.14).

3. Finite Element Galerkin Method

Let Ω be a convex bounded smooth domain in R^2 and let $\{\tau_h\}$ be a family of triangulations of Ω made of triangles K of size h. For a given triangulation τ_h, a finite element space $X_h \subset X$ is defined by

$$X_h = \{v \in C(\bar{\Omega})^2 \cap X; v_h|_K \in P_1, \forall K \in \tau_h\}.$$

We assume that the family of triangulations $\{\tau_h\}$ be quasi-uniform. Then for the finite element space X_h the following approximate properties hold:

(H_1) For each $v \in D(A)$, there exists an approximation $I_h v \in X_h$ such that

$$|v - I_h v| + h\|v - I_h v\| \leq c_2 h^2 |Av|, \quad \forall v \in D(A), \tag{3.1}$$

(where $I_h : C(\bar{\Omega}) \to X_h$ is the nodal value interpolant) and the following inverse inequalities hold

$$\|v_h\| \leq c_3 h^{-1} |v_h|, \quad \forall v_h \in X_h. \tag{3.2}$$

Next, we define the L^2-orthogonal projection $P_h : Y \to X_h$ as follows

$$(P_h v, v_h) = (v, v_h), \quad \forall v_h \in X_h, v \in Y.$$

Then the following properties which are classical consequences of (H_1) (see [3-4]) will be very useful.

$$\|P_h v\| \leq c_4 \|v\|, \quad \forall v \in X, \tag{3.3}$$

$$|v - P_h v| + h\|v - P_h v\| \leq c_4 h^2 |Av|, \quad \forall v \in D(A)$$

$$|\mathrm{div} v - P_h \mathrm{div} v| \leq c_4 h |Av|, \quad \forall v \in D(A), \tag{3.4}$$

$$|v - P_h v| \leq c_4 h \|v\|, \quad \forall v \in X. \tag{3.5}$$

With the above statements, the standard finite element Galerkin approximation of (2.11)-(2.12) based on X_h consists of defining a function vector u_h from R^+ into X_h such that

$$(\frac{du_h}{dt}, v) + a(u_h, v) + d(u_h, v) + b(u_h, u_h, v) = (f, v), \quad \forall v \in X_h, \tag{3.6}$$

$$u_h(0) = P_h u_0. \tag{3.7}$$

For the finite element Galerkin approximate problem (3.6)-(3.7) there holds the similar existence results to ones of problem (2.11)-(2.12) and the following boundedness:

$$|u_h(t)|^2 \leq M_0^2, \|u_h(t)\|^2 \leq M_1^2, \quad \forall t \geq 0. \tag{3.8}$$

We aim now to derive the error estimates for the finite element Galerkin method in terms of the parameter h.

Setting $E = P_h u_\varepsilon - u_h$, then (2.11)-(2.12) and (3.6)-(3.7) yield the following error equation:

$$\begin{aligned}(\frac{dE}{dt}, v) &+ a(E, v) + d(E, v) + b(u_\varepsilon, u_\varepsilon, v) \\ &- b(u_h, u_h, v) + d(u_\varepsilon - P_h u_\varepsilon, v) = 0, \quad \forall v \in X_h,\end{aligned} \tag{3.9}$$

where $E(0)=0$. Taking $v=E$ in (3.9) and using (2.7)-(2.10), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|E|^2+v\|E\|^2+\frac{\nu}{\varepsilon}|\mathrm{div}E|^2+b(u_\varepsilon-P_hu_\varepsilon,u_\varepsilon,E)+b(E,u_\varepsilon,E)\\ +b(u_h,u_\varepsilon-P_hu_\varepsilon,E)+d(u_\varepsilon-P_hu_\varepsilon,E)=0,\end{aligned}\tag{3.10}$$

$$\begin{aligned}|b(u_\varepsilon-P_hu_\varepsilon,u_\varepsilon,E)|&\le 2c_0\lambda_1^{-1/2}\|u_\varepsilon-P_hu_\varepsilon\|\|u_\varepsilon\|\|E\|\\ &\le\frac{\nu}{8}\|E\|^2+\frac{\nu}{8}c_0^2\lambda_1^{-1}\|u_\varepsilon\|^2\|u_\varepsilon-P_hu_\varepsilon\|^2,\end{aligned}\tag{3.11}$$

$$|b(E,u_\varepsilon,E)|\le\frac{\nu}{4}\|E\|^2+(\frac{2}{\nu}c_0^2\|u_\varepsilon\|^2+(\frac{4}{\nu})^3c_0^4|u_\varepsilon|^2\|u_\varepsilon\|^2)|E|^2\tag{3.12}$$

$$|b(u_h,u_\varepsilon-P_hu_\varepsilon,E)|\le\frac{\nu}{8}\|E\|^2+\frac{8}{\nu}c_0^2\lambda_1^{-1}\|u_h\|^2\|u_\varepsilon-P_hu_\varepsilon\|^2,\tag{3.13}$$

$$\begin{aligned}|d(u_\varepsilon-P_hu_\varepsilon,E)|&\le\frac{\nu}{\varepsilon}|\mathrm{div}(u_\varepsilon-P_hu_\varepsilon)||\mathrm{div}E|\\ &\le\frac{\nu}{2\varepsilon}|\mathrm{div}E|^2+\frac{\nu}{2\varepsilon}|\mathrm{div}u_\varepsilon-P_h\mathrm{div}u_\varepsilon|^2.\end{aligned}\tag{3.14}$$

Combining (3.10) with (3.11)-(3.14) yields

$$\begin{aligned}\frac{d}{dt}|E|^2+\nu\|E\|^2+\frac{\nu}{\varepsilon}|\mathrm{div}E|^2\le g(t)|E|^2\\ +\frac{\nu}{\varepsilon}|\mathrm{div}u_\varepsilon-P_h\mathrm{div}u_\varepsilon|^2+\frac{16}{\nu}c_0^2\lambda_1^{-1}(\|u_\varepsilon\|^2+\|u_h\|^2)\|u_\varepsilon-P_hu_\varepsilon\|^2,\end{aligned}$$

where $g(t)=\frac{4}{\nu}c_0^2\|u_\varepsilon\|^2+2(\frac{4}{\nu})^3c_0^4|u_\varepsilon|^2\|u_\varepsilon\|^2$. Thanks to (3.4), (3.8) and (2.14), the above inequality yields

$$\frac{d}{dt}|E|^2+\nu\|E\|^2+\frac{\nu}{\varepsilon}|\mathrm{div}E|^2\le g(t)|E|^2+(\frac{32}{\nu}c_0^2\lambda_1^{-1}M_1^2+\frac{\nu}{\varepsilon})c_4^2h^2|Au_\varepsilon|^2.\tag{3.15}$$

Integrating (3.15), we obtain

$$\begin{aligned}|E(t)|^2+\nu\int_0^t\|E\|^2d\tau+\frac{\nu}{\varepsilon}\int_0^t|\mathrm{div}E|^2d\tau\\ \le\exp(\int_0^tg(\tau)d\tau)(\frac{\nu}{\varepsilon}+\frac{32}{\nu}c_0^2\lambda_1^{-1}M_1^2)c_4^2h^2\int_0^t|Au_\varepsilon|d\tau.\end{aligned}\tag{3.16}$$

With the triangle inequality, (3.16) yields

$$\begin{aligned}|u_\varepsilon(t)-u_h(t)|^2+\nu\int_0^t\|u_\varepsilon-u_h\|^2d\tau\\ +\frac{\nu}{\varepsilon}\int_0^t|\mathrm{div}u_\varepsilon-\mathrm{div}u_h|^2d\tau\le k(t)(1+\frac{\nu}{\varepsilon})h^2.\end{aligned}\tag{3.17}$$

where

$$k(t)=c_4^2\exp\{\int_0^tg(\tau)d\tau\}\int_0^t|Au_\varepsilon(\tau)|^2d\tau\max\{1,\frac{32}{\nu\lambda_1}c_0^2M_1^2\}.$$

Thanks to (2.13)-(2.14), $k(t)$ is uniform bounded as $t\to 0$ and $k(t)\to\infty$ as $t\to\infty$.

With the above discussions, there holds the following convergence theorem.

Theorem 3.1. *If $u_0\in X$, $\mathrm{div}u_0=0$ and $f\in L^\infty(R^+;Y)$, then $u_h(t),t\ge 0$, satisfies the error estimates (3.17).*

4. Global Finite Element Nonlinear Galerkin Method

In this section, we are given two parameters h and H with $H>h>0$. As in sections, we introduce the associated space X_h and X_H with $X_H\subset X_h$. In the applications X_H corresponds to a space associated to a coarse grid, while X_h corresponds to a space associated a fine grid. We consider the following splitting of X_h:

$$X_h = X_H + W_h, \quad W_h = (I - P_H)X_h.$$

Note that X_H and W_h are orthogonal with respect to the scalar product $(\cdot,\cdot)$. The following property of W_h will be often used.

(H_2) There exist two constant $c_5 > 0, \quad 0 < \gamma < 1$ such that

$$|w| \leq c_5 H\|w\|, \qquad \forall w \in W_h, \tag{4.1}$$

$$\gamma(\|v\|^2 + \|w\|^2) \leq \|v + w\|^2, \qquad \forall v \in X_H, w \in W_h. \tag{4.2}$$

We refer to [3-4] for the proof of this property.

The global finite element nonlinear Galerkin method associated to (X_h, X_H, W_h) consists in looking for an approximate solution u^h of the form

$$u^h = y + z, \quad \text{with } u^h \in X_h, \quad y \in X_H, \quad z \in W_h,$$

such that

$$\begin{aligned}(\frac{dy}{dt}, v) + a(y+z, v) + d(y+z, v)&\\ + b(y+z, y+z, v) = (f, v), \quad \forall v \in X_H,&\end{aligned} \tag{4.3}$$

$$\begin{aligned}(\frac{dz}{dt}, w) + a(y+z, w) + d(y+z, w) + b(y, y, w)&\\ + b(y, z, w) + b(z, y, w) = (f, w), \quad \forall w \in W_h,&\end{aligned} \tag{4.4}$$

$$y(0) = P_H u_0, \quad z(0) = (P_h - P_H)u_0. \tag{4.5}$$

We consider a basis $\{e_1, \cdots, e_p, e_{p+1}, \cdots, e_q\}$ of X_h such that $\{e_1, \cdots, e_p\}$ is a basis of X_H. Then setting

$$u^h(t) = \sum_{j=1}^{q} g_{j,h}(t)e_j, \quad y(t) = \sum_{j=1}^{p} g_{j,h}(t)e_j, \quad z(t) = \sum_{j=p+1}^{q} g_{j,h}(t)e_j.$$

Clearly, (4.3)-(4.5) is equivalent to an ODE for the $g_{j,h}, 1 \leq j \leq q$. The existence and uniqueness of a solution of this problem defined on a maximal interval $[0, T_h)$ follows promptly from the standard theorems on the Cauchy problem for ODEs.

We aim now to show that $T_h = +\infty$ thanks to some a priori estimates.

By taking $v = y$ in (4.3), $w = z$ in (4.4) and adding the corresponding relations, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u^h|^2 + \nu\|u^h\|^2 + \frac{\nu}{\varepsilon}|\text{div}u^h|^2 &= (f, u^h)\\ &\leq \frac{\nu}{2}\|u^h\|^2 + \frac{1}{2\nu\lambda_1}|f|^2,\end{aligned} \tag{4.6}$$

where, the equality (2.7) is used.

Therefore, by using (2.10) we have

$$\frac{d}{dt}|u^h|^2 + \nu\lambda_1|u^h|^2 + \frac{2\nu}{\varepsilon}|\text{div}u^h|^2 \leq \nu^{-1}\lambda_1^{-1}\sup_{t\geq 0}|f(t)|^2. \tag{4.7}$$

Integrating (4.7), we get that

$$|u^h(t)|^2 \leq e^{-\nu\lambda_1 t}|P_h u_0|^2 + (1 - e^{-\nu\lambda_1 t})\nu^{-2}\lambda_1^{-2}\sup_{t\geq 0}|f(t)|^2$$

$$\leq |u_0|^2 + \nu^{-2}\lambda_1^{-1}\sup_{t\geq 0}|f(t)|^2 = M_0^2, \quad \forall t \geq 0. \tag{4.8}$$

This estimate guarantees that the solution of (4.3)-(4.4) can not blow up in finite time. So that $T_h = +\infty$.

Hereafter, we consider the discrete analog $A_h \in L(X_h, X_h)$ of the operator A given by

$$(A_h\phi, \psi) = \nu^{-1}a(\phi, \psi) \quad \forall \phi, \psi \in X_h.$$

Then by the usual method $(cf.[2-3])$, we can prove that if $u_0 \in X, \mathrm{div}u^0 = 0$ and $f \in L^\infty(R^+; Y)$, then $u^h \in L^\infty(R^+; X) \cap L^2(0, T; D(A_h)), \forall T > 0$, and

$$|u^h(t)|^2 \leq M_0^2, \quad \|u^h(t)\|^2 \leq M_1^2, \quad \forall t \geq 0. \tag{4.9}$$

We aim now to derive error estimates for the global finite element nonlinear Galerkin method in terms of the two parameters H and h.

We set $E = u_h - u^h, \quad e = P_H u_h - y, \quad \varepsilon = (I - P_H)u_h - z$, then the finite element Galerkin approximation and the global finite element nonlinear Galerkin approximation yield the following error equation:

$$\begin{aligned}(\frac{dE}{dt}, v) &+ a(E, v) + d(E, v) + b(E, u_h, v)\\ &+ b(u^h, E, v) + b(z, z, (I - P_H)v) = 0, \quad \forall v \in X_h.\end{aligned} \tag{4.10}$$

Taking $v = E$ in (4.10) and using (2.7), then we obtain

$$\frac{1}{2}\frac{d}{dt}|E|^2 + \nu\|E\|^2 + \frac{\nu}{\varepsilon}|\mathrm{div}E|^2 + b(E, u_h, E) + b(z, z, \varepsilon) = 0. \tag{4.11}$$

Using again (2.8) and (H_2), we have

$$|b(E, u_h, E)| \leq \frac{\nu}{4}\|E\|^2 + (\frac{2}{\nu}c_0^2 + (\frac{4}{\nu})^3 c_0^4|u_h|^2)\|u_h\|^2|E|^2, \tag{4.12}$$

$$|b(z, z, \varepsilon)| \leq c_0c_5H\|z\|^2\|\varepsilon\| \leq \frac{\nu}{4}\|E\|^2 + \nu^{-1}\gamma^{-1}c_0^2c_5^2H^2\|z\|^4. \tag{4.13}$$

Due to the definition of A_h, we have

$$\|z\|^2 \leq c_5^2H^2|A_hz|^2 \leq c_5^2H^2|A_hu^h|^2 \tag{4.14}$$

$$\|z\|^2 \leq \gamma^{-1}\|u^h\|^2 \tag{4.15}$$

Combining (4.11) with (4.12)-(4.13) and using (4.9), we have

$$\frac{d}{dt}|E|^2 + \nu\|E\|^2 + \frac{\nu}{\varepsilon}|\mathrm{div}E|^2 \leq g(t)|E|^2 + \frac{2}{\nu\gamma}c_0^2c_5^4H^4M_1^2|A_hu^h|^2, \tag{4.16}$$

where

$$g(t) = (\frac{4}{\nu}c_0^2 + 2(\frac{4}{\nu})^3c_0^4M_0^2)\|u^h\|^2 \in L^1(0, T), \quad \forall T > 0.$$

Integrating (4.16), we get that

$$\begin{aligned}|E(t)|^2 &+ \nu\int_0^t \|E\|^2 d\tau + \frac{\nu}{\varepsilon}\int_0^t |\mathrm{div}E|^2 d\tau\\ &\leq \frac{2}{\nu\gamma}c_0^2c_5^4M_1^2H^4\exp(\int_0^t g(\tau)d\tau)\int_0^t |A_hu^h|^2 d\tau \leq k(t)H^4.\end{aligned} \tag{4.17}$$

Combining (4.17) and the convergence rate of u_h yields the convergence rate of the global finite element nonlinear Galerkin method.

Theorem 4.1. *If $u_0 \in X$, $divu_0 = 0$ and $f \in L^\infty(R^+; Y)$, then*

$$|u_\varepsilon(t) - u^h(t)|^2 + \nu \int_0^t \|u_\varepsilon - u^h\|^2 d\tau \le k(t)(\varepsilon^{-1}h^2 + H^4), \quad \forall t \ge 0 \tag{4.18}$$

Furthermore, it is well known [5,6] that the convergence rate of the penalized method is as follows:

$$|u(t) - u_\varepsilon(t)|^2 + \nu \int_0^t \|u(\tau) - u_\varepsilon(\tau)\|^2 d\tau \le k(t)\varepsilon, \quad \forall t \ge 0. \tag{4.19}$$

Remark. Theorem 3.1, Theorem 4.1 and (4.19) indicate

$$|u(t) - u_h(t)|^2 + \nu \int_0^t \|u(\tau) - u_h(\tau)\|^2 d\tau \le k(t)(\varepsilon + \varepsilon^{-1}h^2), \quad \forall t \ge 0, \tag{4.20}$$

$$|u(t) - u^h(t)|^2 + \nu \int_0^t \|u(\tau) - u^h(\tau)\|^2 d\tau \le k(t)(\varepsilon + \varepsilon^{-1}h^2 + H^4), \quad \forall t \ge 0, \tag{4.21}$$

namely, the global finite element nonlinear Galerkin method provides the convergence rate of same order as the standard finite element Galerkin method if we choose $H = O(\varepsilon^{-1/4}h^{1/2})$.

However, in this method, the nonlinearity is treated on the coarse grid finite element space X_H and only the linear subproblem needs to be solved on the fine grid finite element incremental space W_h, while in the finite element Galerkin method the nonlinearity needs to be treated on the fine grid finite element space X_h. Due to $h << H, \dim X_H << \dim X_h$ the method presented in this paper can save much more computational time than the standard finite element Galerkin method.

5. Numerical Test

We describe here the results of numerical test performed with the global finite element nonlinear Galerkin method (NG method). Comparison is also made with the standard finite element Galerkin method (SG method). In fact, to proceed to numerical test we need the time discretization of the semi-discrete schemes (3.6)-(3.7) and (4.3)-(4.5). Here we use the Euler implicit difference scheme for the time discretization, where $\Delta t > 0$ is time step size.

We set $\Omega = (0,1) \times (0,1)$, $\nu = 0.005$, $T = 2, \varepsilon = 0.001$ and choose

$$h = \frac{1}{20}, \quad H = \frac{1}{10}, \quad \Delta t = \frac{1}{25}.$$

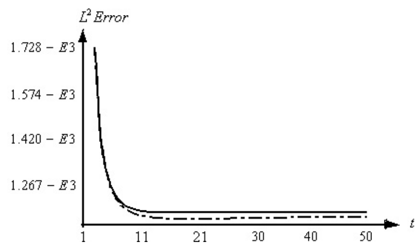
The flow confined in this domain is driven only by constant boundary velocity on the top edge $\Gamma = \{(x,y) | x \in (0,1), y = 1\}$. Particularly, we take

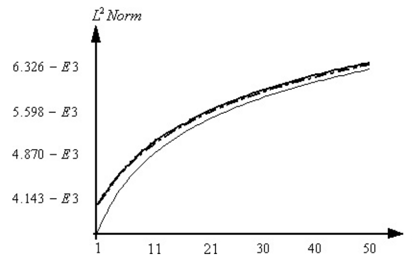
$$u|_\Gamma = 1, \qquad u|_{\partial\Omega\backslash\Gamma} = 0.$$

There are a little difference between our example and the problem we discussed but they are not intrinsic and will not cause any significant difference.

To compare the accuracy of the SG approximation u_h with NG approximation u^h, we need know the exact solution of our example. Here we treat the standard Galerkin approximation u_{h^*} obtained on a fine grid (grid size is $h^* = \frac{1}{40}$) as the exact solution.

The following two figures give the L^2-error comparison of SG and NG approximations and the energy comparison (L^2-norm) of them. In the figures, k stands for time step, thin curve stands for the true solution and the thick curve and dashed curve stand for the SG and NG approximations respectively. It is shown that u_h and u^h has almost the same accuracy.





Next, by the numerical test results, we find that the CPU time of computing u_h corresponding to the SG method is almost as three times long as that of u^h corresponding to the NG method. The saving of CPU time of NG method comes from two reasons. One is that NG method need less computation on each iterative step than SG method. The other reason is the iteration on each time step for solving the nonlinear equation of NG method is easier to converge than that of SG method because the computing scale of NG method is smaller than that of SG method.

Hence, the numerical test and numerical analysis of section 4 show that the global finite element nonlinear Galerkin method is superior to the standard finite element Galerkin method in the numerical computations of the penalized Navier-Stokes equations.

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