Journal of Computational Mathematics, Vol.18, No.2, 2000, 173–182.

MOMENT GENERATING FUNCTIONS OF RANDOM VARIABLES AND ASYMPTOTIC BEHAVIOUR FOR GENERALIZED FELLER OPERATORS[*1)]

Ji-hua Xu Jing-hui Zhao
(*Department of Mathematics, Hubei University, Wuhan 430062, China*)

Abstract

After giving the representation of moment generating function for the S-λ type random variable by solving a differential equation, we prove that this type random variable is of regular n-r order moment. Furthermore we establish the higher order asymptotic formula for generalized Feller operators by making use of the generalized Taylor formula.

Key words: Generalized Feller operator, Moment generating function, Higher order asymptotic formula, Regular n-r order moment, Generalized Taylor formula.

1. Introduction and Notations

The generalized Feller operators which include many famous operators, such as Bernstein, Szasz-Mirakjan, Baskakov, Meyer-König and Zeller operators, can be constructed by making use of the probabilistic method. In the paper [1][2], Xu Jihua provided a general scheme for its construction, and Zhao Jinghui showed that the Feller type operators are of good approximations for unbounded functions.

Our purpose is to present representation of moment generating functions of the sum of S-λ type random variable sequence $\{\xi_i\}$ and to prove that $\{\xi_i\}$ is of the regular n-r order moment. The importance of the regular moments for studying the approximation of the probability type operators is due to the fact that the asymptotic constants are heavily dependent on them. This can be seen from the discussion in [4]. From our results the asymptotic constants of generalized Feller operators will be deduced. Furthermore we establish the higher asymptotic formula for these operators.

We briefly recall some basic concepts and elementary facts concerning probability and sketch the constructive process of the probability type operators for the paper more self-contained and completeness, refer to [1][2] for further details.

Suppose $S(x)$ is a nonnegative function which can be expanded as a power series with nonnegative coefficients p_k and convergent radius R_1:

$$S(x) = \sum_{k=0}^{\infty} p_k x^k. \tag{1}$$

* Received March 11, 1997.
1) Supported partially by the Natural Science Foundation of Hubei Province.

Let $\lambda(x)$ be an increasing monotone indefinitely differentiable function from $[0, R_1)$ into $[0, R)$ with $\lambda(0) = 0$. We call $S(x)$ and $\lambda(x)$ the original function of derivation and extending factor respectively. Here we note R_1 and R may be finite or infinite.

Let $\{\xi_i\}$ be an independent identically distributed random sequence. If its lattice probability

$$P(\xi_i = k) = \frac{p_k(\lambda(x))^k}{S(\lambda(x))}, \qquad k = 0, 1, ...,$$

where $p_k > 0,\ k = 0, 1, ..,$ satisfy (1), we call $\{\xi_i\}$ the S-λ type random variables.

Let $\eta_n = \sum_{i=1}^n \xi_i$. Its distribution function is denoted by $F_{n,x}(t)$. Let $\theta(u) := uS'(u)/S(u)$, $\varphi(x) := \theta(\lambda(x))$, and let $\sigma^2(x) := \lambda(x)\theta'(\lambda(x))$. The inverse of function $\varphi(x)$ is denoted by $\psi(x)$ whenever it exists.

The generalized Feller operators which deduced by original function of derivation $S(x)$ and extending factor $\lambda(x)$ are of the following form

$$L_n(f, x) = E\left(f(\psi(\tfrac{\eta_n}{n}))\right) = \int_0^\infty f(\psi(\tfrac{t}{n}))\, dF_{n,x}(t). \tag{2}$$

These operators are briefly called PPA operators in [1]. They also have the following series representation

$$L_n(f, x) = \frac{1}{(S(\lambda(x)))^n} \sum_{k=0}^{\infty} f(\psi(\tfrac{k}{n})) p_k^{(n)} (\lambda(x))^k, \quad x \in (0, R). \tag{3}$$

Here R may be finite or infinite, $p_k^{(n)}$ satisfy the recurrence formula

$$p_k^{(n)} = \sum_{j=0}^{k} p_j^{(n-1)} p_{k-j}^{(1)}, \qquad \text{where } p_{k-j}^{(1)} = p_{k-j}.$$

The paper is organized as follows. In next section we represent moment generating function for S-λ type random variable by solving a differential equation. It is important to describe the regular n-r order moment for the random variable, which is done in section 3. Finally in section 4 applying the generalized Taylor formula, we give the higher order asymptotic formula for PPA type operators and present two examples.

2. Moment generating functions of S-λ type random variables

Concerning the numerical character for S-λ type random variables $\{\xi_i\}$, we have (see [1])

$$E\xi_i = \varphi(x), \ \ D\xi_i = \sigma^2(x),$$

where E denotes the expectation of ξ_i, and D the variance. For $\eta_n = \sum_{i=1}^n \xi_i$, $E\eta_n = n\varphi(x)$, $D\eta_n = n\sigma^2(x)$. In order to calculate the moment generating functions of the random variable η_n, we establish the following lemma.

Lemma 1. *Let $\{L_n\}$ be the PPA type operators, defined by (2) and (3), deduced by S-λ type random variable. Then*

$$L_n(e^{\alpha\varphi(t)}, x) = \frac{[S(\lambda(x)e^{\alpha/n})]^n}{[S(\lambda(x))]^n},$$

and

$$L_n(\varphi(t)e^{\alpha\varphi(t)},x) = \frac{[S(\lambda(x)e^{\alpha/n})]^n}{[S(\lambda(x))]^n}\theta(\lambda(x)e^{\alpha/n}).$$

Proof. By (3), we have

$$L_n(e^{\alpha\varphi(t)},x) = \frac{1}{[S(\lambda(x))]^n}\sum_{k=0}^{\infty} e^{\alpha k/n}p_k^{(n)}(\lambda(x))^k = \frac{1}{[S(\lambda(x))]^n}\sum_{k=0}^{\infty} p_k^{(n)}(\lambda(x)e^{\alpha/n})^k.$$

We are also easy to see that

$$\sum_{k=0}^{\infty} p_k^{(n)}(\lambda(x)e^{\alpha/n})^k = S^n(\lambda(x)e^{\alpha/n}).$$

Furthermore it leads to by direct calculation

$$\begin{aligned}
L_n(\varphi(t)e^{\alpha\varphi(t)},x) &= \frac{1}{[S(\lambda(x))]^n}\sum_{k=0}^{\infty}\frac{k}{n}e^{\alpha k/n}p_k^{(n)}(\lambda(x))^k \\
&= \frac{1}{[S(\lambda(x))]^n}\frac{\lambda(x)e^{\alpha/n}}{n}\sum_{k=0}^{\infty} kp_k^{(n)}(\lambda(x)e^{\alpha/n})^{k-1} \\
&= \frac{1}{[S(\lambda(x))]^n}\frac{\lambda(x)e^{\alpha/n}}{n}\Big[S^n(u)\Big]'_{u=\lambda(x)e^{\alpha/n}} \\
&= \frac{1}{[S(\lambda(x))]^n}\lambda(x)e^{\alpha/n}\Big[S(\lambda(x)e^{\alpha/n})\Big]^{n-1}S'(\lambda(x)e^{\alpha/n}) \\
&= \frac{[S(\lambda(x)e^{\alpha/n})]^n}{[S(\lambda(x))]^n}\left[\frac{S'(\lambda(x)e^{\alpha/n})}{S(\lambda(x)e^{\alpha/n})}\lambda(x)e^{\alpha/n}\right] \\
&= \frac{[S(\lambda(x)e^{\alpha/n})]^n}{[S(\lambda(x))]^n}\theta(\lambda(x)e^{\alpha/n})).
\end{aligned}$$

Theorem 1. *Denote the moment generating functions* $Ee^{\tau\eta_n}$ *of* S-λ *type random variable* $\eta_n = \sum_{i=1}^n \xi_i$ *by* $G(\tau)$. *Then*

$$G(\tau) = e^{n\,a(\tau;x)}, \tag{4}$$

where $a(\tau;x) = \int_0^\tau \theta(\lambda(x)e^t)\,dt$, *and the parameter* x *fixed.*

Proof. Set $\tau = \alpha/n$. Then

$$G(\tfrac{\alpha}{n}) = Ee^{\alpha\frac{\eta_n}{n}} = L_n(e^{\alpha\varphi(t)},x).$$

Differentiate for α in the above equation,

$$G'(\tfrac{\alpha}{n})\frac{1}{n} = L_n(\varphi(t)e^{\alpha\varphi(t)},x).$$

It follows that

$$\frac{G'(\frac{\alpha}{n})}{G(\frac{\alpha}{n})} = n\frac{L_n(\varphi(t)e^{\alpha\varphi(t)},x)}{L_n(e^{\alpha\varphi(t)},x)}.$$

From this and Lemma 1, we have

$$\frac{G'(\tau)}{G(\tau)} = n\theta(\lambda(x)e^{\tau}). \tag{5}$$

Solving differential equation (5) with the initial condition $G(0) = 1$, we have

$$G(\tau) = e^{n\int_0^{\tau}\theta(\lambda(x)e^t)dt}. \tag{6}$$

Write

$$a(\tau; x) = \int_0^{\tau} \theta(\lambda(x)e^t)dt, \tag{7}$$

giving the proof of Theorem 1.

Differentiate for τ the first and second derivative in equation (7), set $\tau = 0$ and note $\theta(\lambda(x)) = \varphi(x)$, $\theta'(\lambda(x))\lambda(x) = \sigma^2(x)$, we get from Theorem 1

Corollary 1. $a'(0; x) = \varphi(x)$, $a''(0; x) = \sigma^2(x)$.

Put $\bar{\eta}_n = \sum_{i=1}^{n}(\xi_i - E\xi_i) = \sum_{i=1}^{n}(\xi_i - \varphi(x))$ *and denote the moment generating function of* $\bar{\eta}_n$ *by* $g(\tau) = Ee^{\tau\bar{\eta}_n}$ *. In this situation, Theorem 1 gives*

Corollary 2. *Keep the notions above, the moment generating function*

$$g(\tau) = e^{na(\tau;x) - n\varphi(x)\tau}. \tag{8}$$

3. The Regular n-r Order Moment for the Random Variables

We first start from the definition.

Definition 1. *Let* $\{\xi_i\}$ *be an independent identically distributed random sequence, let* $E\xi_i = \varphi(x)$ *and let* $D\xi_i = \sigma^2(x)$*. The quantity*

$$T_{n,r} = T_{n,r}(x) = E(\sum_{i=1}^{n}(\xi_i - \varphi(x)))^r$$

is called the n-r *order moment for* $\{\xi_i\}$*.*

Remark. The n-r order moment in this paper is similar to one in [3], but it differ in factor n^r from one in [4].

Applying representation (2) for the generalized Feller operators we can easily deduce that

$$\begin{aligned} T_{n,r}(x) &= E(\eta_n - n\varphi(x))^r = n^r E(\tfrac{\eta_n}{n} - \varphi(x))^r \\ &= n^r L_n((\varphi(t) - \varphi(x))^r, x). \end{aligned} \tag{9}$$

It is known from (9) that the moment $T_{n,r}(x)$ of the random variable sequence $\{\xi_i\}$ is corresponding to the moment of the generalized Feller operators. This moment is of essential importance for approximation by operators L_n.

Definition 2. *The* n-r *order moments of random variables* $\{\xi_i\}$ *are called regular if their moments satisfy estimates*

$$T_{n,2k}(x) = (2k-1)!!\, n^k \sigma^{2k}(x) + \mathcal{O}(n^{k-1}), \tag{10}$$

$$T_{n,2k+1}(x) = \mathcal{O}(n^k). \tag{11}$$

Not all random variableshave regular n-r order moments. More than there do not exist finite n-r order moments for some random variables. The following theorem will point out that it has regular n-r order moment to S-λ type random variables.

Theorem 2. *The n-r order moments of the S-λ type random variable sequence $\{\xi_i\}$ are regular, i.e. it satisfied estimates (10) and (11)*

Proof. We differentiate $g(\tau)$ r-th for τ in equation (8) and obtain

$$g^{(r)}(\tau) = E(\sum_{i=1}^{n}(\xi_i - \varphi(x)))^r e^{\tau\sum_{i=1}^{n}(\xi_i-\varphi(x))}.$$

Set $\tau = 0$. Then

$$g^{(r)}(0) = E(\sum_{i=1}^{n}(\xi_i - \varphi(x)))^r = T_{n,r}. \tag{12}$$

On the other hand, $g(\tau) = e^{na(\tau;x)-n\varphi(x)\tau}$. Write $h(\tau) = n(a'(\tau;x) - \varphi(x))$, we have

$$g'(\tau) = e^{na(\tau;x)-n\varphi(x)\tau}(na'(\tau;x) - n\varphi(x)) = g(\tau)h(\tau). \tag{13}$$

Applying the Leibniz formula in (13), we have

$$g^{(r+1)}(\tau) = g^{(r)}(\tau)h^{(0)}(\tau) + \binom{r}{1}g^{(r-1)}(\tau)h^{(1)}(\tau) + \cdots + g^{(0)}(\tau)h^{(r)}(\tau).$$

Notice that $g^{(0)}(0) = g(0) = 1,\ h^{(0)}(0) = h(0) = 0,\ h^{(1)}(0) = na''(0;x) = n\sigma^2(x),\ h^{(r)}(0) = na^{(r+1)}(0;x)$, and $g^{(1)}(0) = 0,\ g^{(2)}(0) = na''(0;x) = n\sigma^2(x)$. Furthermore applying mathematical induction we get by calculation

$$g^{(2k)}(0) = (2k-1)!!\, n^k\sigma^{2k}(x) + \mathcal{O}(n^{k-1}), \tag{14}$$

and

$$g^{(2k+1)}(0) = \mathcal{O}(n^k). \tag{15}$$

Combining estimate (14), (15) with (12), we complete the proof.

The n-r order regular moments of the S-λ type random variables determine that the PPA type operators have good behaviour for approximation. As a corollary of Theorem 2, we know that the asymptotic constants (see [4]) of PPA type operators have a unified formula. In order to state the formula we set $W^{r+\alpha}M = \{f\ :\ f^{(r)} \in Lip_M\alpha\}$ and

$$E(M, L_n, r, x) = \sup_{f\in W^{r+\alpha}M}|L_n(f,x) - \sum_{j=0}^{r}\frac{f^{(j)}(x)}{j!}T_{n,j}(x)n^j|.$$

Corollary 3. *For PPA type operators L_n the following relations hold*

$$\lim_{n\to\infty} E(M, L_n, r+\alpha, x)(n^2T_{n,2})^{-\frac{r+\alpha}{2}}(x) = \begin{cases} \dfrac{M2^{\frac{r+3\alpha-2}{2}}\Gamma(\frac{r+\alpha+1}{2})}{(r+\alpha)\cdots(1+\alpha)\sqrt{\pi}}, & \textit{for } r \textit{ odd}, \\[2ex] \dfrac{M2^{\frac{r+\alpha}{2}}\Gamma(\frac{r+\alpha+1}{2})}{(r+\alpha)\cdots(1+\alpha)\sqrt{\pi}}, & \textit{for } r \textit{ even}. \end{cases} \tag{16}$$

Proof. Notice $T_{n,2}(x) = n\sigma^2(x)$, and combine (12), (14) with (15) we have

$$\lim_{n\to\infty} \frac{T_{n,2k}}{T_{n,2}^k} = (2k-1)!!, \tag{17}$$

and

$$\lim_{n\to\infty} \frac{T_{n,2k+1}}{T_{n,2}^{k+1}} = 0. \tag{18}$$

From (17) and (18), similar to the proof of Theorem 1 in [4], we obtained (16).

4. Higher Order Asymptotic Formula for PPA Operators

In his paper [3], R. A. Khan obtained the higher order asymptotic formula for Feller operators $L_n(f,x)$ by making use of probabilistic method. This reads as (we keep the same notions as in [3])

$$\lim_{n\to\infty} n^k \Big(L_n(f,x) - f(b^{(1)}(x) - \sum_{s=1}^{2k-1} \frac{n^{-s}}{s!} T_{n,s} f^{(s)}(b^{(1)}(x)) \Big) = \Big(\frac{b^{(2)}(x)}{2} \Big)^k \frac{f^{(2k)}(b^{(1)}(x))}{k!}. \tag{19}$$

In order to get the result Khan first took the Taylor expansion of $f(\frac{S_n}{n})$ at $b^{(1)}(x)$, and then considered the expectation

$$Ef(\frac{S_n}{n}) = L_n(f,x) = f(b^{(1)}(x)) + \sum_{s=1}^{2k-1} \frac{n^{-s} f^{(s)}(b^{(1)}(x))}{s!} T_{n,s}(x) + R_n.$$

By estimating the remainder R_n as usual arrived at the asymptotic equation (19).

However it is point out in [5] and [1], only when $E\xi_i = x$ $Ef(S_n/n)$ is Feller operators. When $E\xi_i = \varphi(x)$, the corresponding approximation operators are $L_n(f,x) = Ef(\psi(S_n/n))$ which are generalized Feller operators. Therefore (19) is true only for $b^{(1)} = x$.

We use a different method to deal with this topic. In our situations, function f may be unbounded having exponential increasing $f(x) = \mathcal{O}(e^{\alpha\varphi(x)})$, where $\varphi(x)$ tends to infinite as $x \to \infty$.

Above all we generalize the classic Taylor formula into the following form. Let $\varphi(x)$ be an indefinitely differentiable function defined on the interval I (finite or infinite).

Lemma 2. *Let $f^{(n)}(t) \in C(I)$, i.e. f is n times continuous differentiable on I. For $x \in I$ fixed, we have*

$$\begin{aligned} f(t) =& f(x) + f_1(x)(\varphi(t) - \varphi(x)) + \frac{f_2(x)}{2!}(\varphi(t) - \varphi(x))^2 + \cdots + \frac{f_n(x)}{n!}(\varphi(t) - \varphi(x))^n \\ &+ \xi(t,x)(\varphi(t) - \varphi(x))^n, \end{aligned}$$

where $f_1(x) = \frac{f'(x)}{\varphi'(x)}, f_2(x) = \frac{f_1'(x)}{\varphi'(x)}, \cdots f_k(x) = \frac{f_{k-1}'(x)}{\varphi'(x)}$, and $\xi(t,x)$ is bounded in the neighborhood of x. Moreover for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\xi(t,x) < \varepsilon$ if $|t - x| < \delta$.

Proof. Assume first that $f(t)$ is an n-th polynomial with respect to $\varphi(t)$. Therefore we can write $f(t)$ as the form

$$f(t) = c_0 + c_1(\varphi(t) - \varphi(x)) + c_2(\varphi(t) - \varphi(x))^2 + \cdots + c_n(\varphi(t) - \varphi(x))^n.$$

The coefficients $c_j,\ j = 0, 1, .., n$, are determined by equations

$$\begin{aligned} c_0 &= f(x), \\ c_1 &= \frac{f'(x)}{\varphi'(t)} = \frac{f_1(x)}{1!}, \\ &\cdots\ \cdots\ \cdots \\ c_k &= \frac{f'_{k-1}(x)}{k!\varphi'(x)} = \frac{f_k(x)}{k!}, \\ &\cdots\ \cdots\ \cdots \\ c_n &= \frac{f'_{n-1}(x)}{n!\varphi'(x)} = \frac{f_n(x)}{n!}. \end{aligned}$$

For n times continuous differentiable function $f(t)$, we put

$$f(t) - \sum_{k=0}^{n} \frac{f_k(x)}{k!}(\varphi(t) - \varphi(x))^k = R_n(t, x).$$

An analogous approach similar to the calculations for remainder of Taylor expansion gives

$$R_n(t, x) = \xi(t, x)(\varphi(t) - \varphi(x))^n,$$

where $\xi(t, x)$ satisfies the statements of Lemma 2.

Lemma 3. *Let $\{\xi_i\}$ be a sequence of S-λ type random variable and let $\eta_n = \sum_{i=1}^{n} \xi_i$. Then for every $\delta > 0$ there exists $\rho > 0,\ 0 < \rho < 1$, such that*

$$P(|\eta_n - n\varphi(x)| \geq n\delta) \leq 2\rho^n. \tag{20}$$

Proof. Put $\bar{\xi_i} = \xi_i - \varphi(x)$. We have $E\bar{\xi_i} = 0$ for $i = 1, 2, ..., n$. Write $\bar{\eta}_n = \sum_{i=1}^{n} \bar{\xi_i}$. From Lemma 3 in [3], we have for every $\delta > 0$ there exists $\rho > 0,\ 0 < \rho < 1$, such that

$$P(|\bar{\eta}_n| \geq n\delta) \leq 2\rho^n.$$

That is

$$P(|\eta_n - n\varphi(x)| \geq n\delta) \leq 2\rho^n.$$

Theorem 3. *Let $L_n(f, x)$ be PPA operators and let an unbounded function $f(x)$ satisfy $f(x) = \mathcal{O}(e^{\alpha\varphi(x)}),\ \alpha > 0$, with finite $2k$th derivative $f^{(2k)}(x)$. Then*

$$\lim_{n\to\infty} n^k \left(L_n(f, x) - \sum_{j=1}^{2k-1} \frac{n^{-j}}{j!} f_j(x) T_{n,j}(x) \right) = \left(\frac{\sigma^2(x)}{2} \right)^k \frac{f_{2k}(x)}{k!}. \tag{21}$$

Proof. Applying the generalized Taylor formula (see Theorem 3) for $f(t)$ in the interval $(0, R)$, we have

$$f(t) = f(x) + \sum_{j=1}^{2k} \frac{f_j(x)}{j!}(\varphi(t) - \varphi(x))^j + \xi(t, x)(\varphi(t) - \varphi(x))^n, \tag{22}$$

where $f_j(x)$ are defined as in Theorem 3 and $\xi(t,x) = \mathcal{O}(e^{\alpha\varphi(t)})$ $(t \to \infty)$. Furthermore for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\xi(t,x)| < \varepsilon$ if $|t-x| < \delta$.

We first note that the PPA operators are of the basic properties $L_n(1,x) = 1$ and for $j = 1,2,...,n$,

$$L_n((\varphi(t)-\varphi(x))^j, x) = n^{-j}E[(\eta_n - n\varphi(x))] = n^{-j}T_{n,j}(x).$$

Now taking the operators L_n in both sides in (22) gives

$$L_n(f,x) = f(x) + \sum_{j=1}^{2k} \frac{n^{-j}f_j(x)}{j!}T_{n,j}(x) + R_{n,2k}. \tag{23}$$

Here $R_{n,2k} = L_n(\xi(t,x)(\varphi(t)-\varphi(x))^{2k}, x)$ denotes the remainder.

In order to estimate the remainder term we introduce the characteristic function $I_A(\omega)$ of a set A:

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

With the aid of the characteristic function we can write the remainder as

$$\begin{aligned} |R_{n,2k}| &= E[\xi(\psi(\frac{\eta_n}{n}, x) \cdot (\frac{\eta_n}{n} - \varphi(x))^{2k}] \\ &= E[(\frac{\eta_n}{n} - \varphi(x))^{2k} \cdot |\xi(\psi(\frac{\eta_n}{n}, x)| I_{\{|\frac{\eta_n}{n} - \varphi(x))| < \delta\}}] \\ &\quad + E[(\frac{\eta_n}{n} - \varphi(x))^{2k} \cdot |\xi(\psi(\frac{\eta_n}{n}, x)| I_{\{|\frac{\eta_n}{n} - \varphi(x))| \geq \delta\}}] \\ &:= J_1 + J_2. \end{aligned}$$

It is easy to see that

$$J_1 \leq \varepsilon n^{-2k} T_{n,2k}(x). \tag{24}$$

We next estimate J_2. Applying the Cauchy-Schwartz inequality two times

$$\begin{aligned} J_2 &\leq M\, E[(\frac{\eta_n}{n} - \varphi(x))^{2k} \cdot e^{\alpha\eta_n/n} \cdot I_{\{|\frac{\eta_n}{n} - \varphi(x))| \geq n\delta\}}] \\ &\leq M\Big\{ E[(\frac{\eta_n}{n} - \varphi(x))^{4k} \cdot e^{2\alpha\eta_n/n}] \Big\}^{1/2} \cdot [P(|\frac{\eta_n}{n} - \varphi(x))| \geq n\delta)]^{1/2} \\ &\leq M\left\{ E(\frac{\eta_n}{n} - \varphi(x))^{8k} \cdot E(e^{4\alpha\eta_n/n}) \right\}^{1/4} \cdot [P(|\frac{\eta_n}{n} - \varphi(x)| \geq n\delta)]^{1/2} \end{aligned}$$

Combining (9) with (10) leads to

$$E(\frac{\eta_n}{n} - \varphi(x))^{8k} = n^{-8k}T_{n,8k}(x) = \mathcal{O}(n^{-4k}),$$

Lemma 1 to

$$Ee^{4\alpha\eta_n/n} = L_n(e^{4\alpha\varphi(t)}, x) = \frac{[S(\lambda(x)e^{4\alpha/n}]^n}{[S(\lambda(x))]^n} \to e^{4\alpha\varphi(x)} \ (n \to \infty),$$

while Lemma 2 to

$$P(|\eta_n - n\varphi(x)| \geq n\delta) \leq 2\rho^n \quad (0 < \rho < 1).$$

Therefore we finally get

$$J_2 = \mathcal{O}(n^{-k}\rho^{\frac{n}{2}}). \tag{25}$$

We have $R_{n,2k} = \varepsilon_n n^{-k}$ by (24) and (25). Finally (23) and (10) give the proof of Theorem 4.

We present two examples as follows.

Example 1. Baskakov operators

Baskakov operators are defined as

$$B_n(f,x) = \sum_{k=0}^{\infty} f(\tfrac{k}{n})\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, \qquad x \in [0,\infty).$$

Its original function of derivation and extending factor are $S(x) = 1/(1-x)$ and $\lambda(x) = x/(1+x)$, see [1]. The associated S-λ type random variable $\{\xi_i\}$ submits to negative binomial distribution.

$E\xi_i = x,\ D\xi_i = \sigma^2(x) = x(1+x),\ \varphi(x) = x = \psi(x)$, which belongs to Feller type operators.

For a function $f(x)$ defined on $[0,\infty)$ with $2k$-th derivative and $f(x) = \mathcal{O}(e^{\alpha x})$, we have

$$\lim_{n\to\infty} n^k\left(B_n(f,x) - \sum_{j=1}^{2k-1} f_j(x)T_{n,j}(x)\right) = \left(\frac{x(1+x)}{2}\right)^k \frac{f_{2k}(x)}{k!}, \quad x \in [0,\infty),$$

where

$$f_j(x) = f^{(j)}(x), \qquad j = 1,2,...,2k,$$

and

$$T_{n,1}(x) = 0,\ T_{n,2}(x) = nx(1+x),\ T_{n,3}(x) = nx(1+x)(1+2x),\ ...$$

Example 2. Meyer-König and Zeller operators

Meyer-König and Zeller operators are defined as

$$M_n(f,x) = \sum_{k=0}^{\infty} f(\tfrac{k}{n+k})\binom{n+k-1}{k}(1-x)^n x^k, \qquad x \in [0,1).$$

Its original function of derivation and extending factor are $S(x) = 1/(1-x)$ and $\lambda(x) = x$, see [1]. The associated S-λ type random variable $\{\xi_i\}$ submits to geometric distribution.

$E\xi_i = \varphi(x) = \frac{x}{1+x},\ D\xi_i = \sigma^2(x) = \frac{x}{(1-x)^2}$. The inverse function $\psi(x)$ of $\varphi(x)$ is $\frac{x}{1+x}$. This operator belongs to generalized Feller type operators.

For a function $f(x)$ defined on $[0,1)$ with $2k$-th derivative and $f(x) = \mathcal{O}(e^{\alpha\frac{x}{1-x}})$, $(\alpha > 0)$, we have by Theorem 3

$$\lim_{n\to\infty} n^k\left(M_n(f,x) - \sum_{j=1}^{2k-1} \frac{n^{-j}}{j!} f_j(x)T_{n,j}(x)\right) = \left(\frac{x}{2(1-x)}\right)^k \frac{f_{2k}(x)}{k!}, \quad x \in [0,1),$$

where

$$\begin{aligned}
f_1(x) &= (1-x)^2 f'(x),\\
f_2(x) &= (1-x)^2[(1-x)^2 f'(x) - 2(1-x)^3 f''(x)],\\
f_3(x) &= (1-x)^4[6f'(x) - 6(1-x)f''(x) + (1-x)^2 f'''(x)],\\
&\cdots\ \cdots\ \cdots,
\end{aligned}$$

and

$$T_{n,1}(x) = 0,\ T_{n,2}(x) = n\frac{x}{(1-x)^2},\ T_{n,3}(x) = n\frac{x(1+x)}{(1-x)^3}, \ldots$$

References

[1] Xu Jihua, General method for constructing operators of probability type by power series, Northeastern Math. J., **5** (1989), 2: 246–252.

[2] Zhao Jinghui, Quantify behaviors of approximation degree to unbounded functions by generalized Feller operators derived S-λ, *Acta Mathematica Scientia,* **17**:4 (1997), 371-377.

[3] R.A. Khan, Some probabilistic method in the theory of approximation operators, *Acta Math. Hung.,* **35**:1-2 (1980), 193–203.

[4] Zhou Xinlong, Asymptotic constants of positive linear operators, *Acta Math. Sinica,* **29**:3 (1986), 362–367.

[5] B. Levikson, On the behaviour of a certain class of approximation operators for discontinuous functions, *Acta Math. Hung.,* **33**:3-4 (1979), 299–306.