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LEAST-SQUARES MIXED FINITE ELEMENT METHOD FOR SADDLE-POINT PROBLEM*[1)]

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Abstract

In this paper, a least-squares mixed finite element method for the solution of the primal saddle-point problem is developed. It is proved that the approximate problem is consistent ellipticity in the conforming finite element spaces with only the discrete BB-condition needed for a smaller auxiliary problem. The abstract error estimate is derived.

Key words: least-squares method, mixed finite element approximation, saddle-point problem

1. Introduction

There are many work to investigate the stability of the mixed finite element method for the saddle-point problems, i.e., to construct the finite element spaces, such that the so-called discrete BB-codition is satisfied (c.f. [1],[2],[7],[8] and the references therein). To circumvent the discrete BB-conditon, recently there has been an increased interest in use of least-squares approach for the solution of the mixed finite element approximation of the saddel-point problem (c.f.[3]–[6],[10],[12] and [13]).In this aspect, the saddle-point problem (such as the Stokes problem) is reduced, in general, the first order system by introducing auxiliary variables (such as the stress for the Stokes problem). Thus the bilinear form, in the least-squares mixed finite element approximation for the saddel-point problem, is coercive in the conforming finite element spaces, and the discrete BB-condition is not required.

In this paper, the least-squares mixed approach, a least-squares residual minimization, is introduced for the primal saddel-point problem directly, without use of any auxiliary variables. The ellipticity in the finite element spaces for the least-squares mixed finite element approximation of the primal saddel-point problem is guaranteed, under the assumption of the discrete BB-condition being satisfied for a smaller auxiliary problem, instead for the primal saddel-point problem. And under the same assumption presented previously, the abstract error estimate is derived.

The paper is organized as follows. In section 2, we formulate the least-squares mixed problem and proved the coerciveness of the bilinear form in the case of the BB-condition satisfied for the primal saddle-point problem. In section 3, as a bridge for theoretical

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analysis but not for practical computing, a semi-finite element approximation is presented. Without any discrete BB-condition being required, we derive the stability and the abstract error estimate, which will be used in the next section. In section 4 and 5, the real finite element approximation is presented, the stability and the abstract error estimate are derived with only the discrete BB-conditon being required for a smaller auxiliary problem.

2. The Least-Squares Mixed Formulation for The Saddel-Point Problem

Let V, Q be two real Hilbert spaces with norm $\|\cdot\|_V$ and $\|\cdot\|_Q$ respectively, the norm $\|(\cdot,\cdot)\|_{V\times Q} = (\|\cdot\|_V^2 + \|\cdot\|_Q^2)^{\frac{1}{2}}$, V' and Q' denote the dual spaces of V and Q with norm $\|\cdot\|_{V'}$ and $\|\cdot\|_{Q'}$ respectively, and the dualities of V' and V, Q' and Q be denoted by $<\cdot,\cdot>$. For any given $f \in V', \chi \in Q'$, we consider the following mixed variational equations of the saddel-point problem

$$\begin{cases} \text{to find } \ (u,p) \in V \times Q, & \text{such that} \\ a(u,v) + b(v,p) & =< f, v > \qquad \forall \ v \in V, \\ b(u,q) & =< \chi, q > \qquad \forall \ q \in Q, \end{cases} \tag{2.1}$$

where $a(\cdot,\cdot) : V \times V \to R$ is a symmetric, continuous bilinear form, and V-elliptic, e.i., there exists $\alpha =$ const. > 0, such that

$$a(v,v) \geq \alpha \|v\|_V^2 \qquad \forall \ v \in V, \tag{2.2}$$

$b(\cdot,\cdot) : V \times Q \to R$ is a continuous bilinear form with the following BB-condition: there exists $\beta =$ const. > 0, such that

$$\sup_{v\in V} \frac{b(v,q)}{\|v\|_V} \geq \beta \|q\|_Q \qquad \forall \ q \in Q. \tag{2.3}$$

Then the following theorem is well known (c.f.[7], [8])

Theorem 2.1. *Assume that* (i) *V, Q are the real Hilbert spaces,* (ii) *the bilinear form $a(\cdot,\cdot)$ is symmetric, continuous with V-ellipticity* (2.2)*, and the bilinear form $b(\cdot,\cdot)$ is continuous with BB-condition* (2.3). *Then the abstract problem* (2.1) *has one and only one solution (u,p), and the following inequality holds*

$$\|u\|_V^2 + \|p\|_Q^2 \leq C(\|f\|_{V'}^2 + \|\chi\|_{Q'}^2), \tag{2.4}$$

where $C =$ Const.> 0.

Since the bilinear form $a(\cdot,\cdot)$ is symmetric, continuous and V-elliptic, the space V equipped the inner product $a(\cdot,\cdot)$, also denoted $(\cdot,\cdot) = a(\cdot,\cdot)$, is a Hilbert space, and the corresponding norm is also denoted by $\|\cdot\|_V$. The dual space of V, equipped the inner product $(\cdot,\cdot) = a(\cdot,\cdot)$, is also denoted by V'. It is easily seen that the inequalities (2.2)–(2.4) hold.

To formulate the least-squares mixed form of the problem (2.1), it is needed to introduce the following operators (c.f.[7],[8])

$$\begin{cases} A \in \mathcal{L}(V;V') : \qquad a(u,v) =< Au, v >, \\ B \in \mathcal{L}(V;Q'), \ \ B^T \in \mathcal{L}(Q;V') : \qquad b(v,q) =< Bv, q > =< v, B^T q > . \end{cases} \tag{2.5}$$

Then the problem (2.1) can be formulated in the operators as follows

$$\begin{cases} Au + B^T p & = f \quad \text{in } V' \\ Bu & = \chi \quad \text{in } Q' \end{cases} \tag{2.6}$$

For any given $(v,q) \in V \times Q$, we introduce the least-squares residual in the operator equations (2.6) as follows:

$$I(v,q) = \|Av + B^T q - f\|_{V'}^2 + \|Bv - \chi\|_{Q'}^2. \tag{2.7}$$

Then the least-squares minimizing problem is the following

$$\begin{cases} \text{to find} \quad (u,p) \in V \times Q, \quad \text{such that} \\ I(u,p) = \min I(v,q) \quad \forall \ (v,q) \in V \times Q, \end{cases} \tag{2.8}$$

for which the equivalent weak form is the following

$$\begin{cases} \text{to find} \quad (u,p) \in V \times Q, \quad \text{such that} \\ \mathcal{A}((u,p);(v,q)) = \mathcal{F}(v,q) \quad \forall \ (v,q) \in V \times Q, \end{cases} \tag{2.9}$$

where

$$\mathcal{A}((u,p);(v,q)) = (Au + B^T p, Av + B^T q)_{V'} + (Bu, Bv)_{Q'}, \tag{2.10}$$

$$\mathcal{F}(v,q) = (f, Av + B^T q)_{V'} + (\chi, Bv)_{Q'}. \tag{2.11}$$

Theorem 2.2. *Under the assumptions of the Theorem* 2.1, *the bilinear form* $\mathcal{A}((u,p);(v,q)) : (V \times Q) \times (V \times Q) \to R$ *is continuous and coercive on* $V \times Q$, *i.e., there exists* $\alpha' =$ *Const.* > 0, *such that*

$$\mathcal{A}((v,q);(v,q)) \geq \alpha' \|(v,q)\|_{V\times Q}^2, \qquad \forall \ (v,q) \in V \times Q. \tag{2.12}$$

Proof. For any given $(v,q) \in V \times Q$, let

$$\begin{cases} Av + B^T q & = g \quad \text{in } V', \\ Bv & = \psi \quad \text{in } Q'. \end{cases}$$

Then due to the Theorem 2.1, it can be seen that

$$\begin{aligned} \|v\|_V^2 + \|q\|_Q^2 &\leq C(\|g\|_{V'}^2 + \|\psi\|_{Q'}^2) \\ &= C\{\|Av + B^T q\|_{V'}^2 + \|Bv\|_{Q'}^2 \\ &= C\mathcal{A}((v,q);(v,q)), \end{aligned}$$

where C= Const. > 0, independent of v and q. Thus the proof is completed.

Then by Lax-Milgram Lemma, the following theorem holds

Theorem 2.3. *Under the assumptions of Theorem* 2.1, *the least-equares mixed problem* (2.8) (*or* (2.9)) *has one and only one solution.*

In order to approximate the least-squares mixed problem (2.9) by finite element method, it is needed to express bilinear form $\mathcal{A}((u,p);(v,q))$ (2.10) with the bilinear forms $a(u,v), b(u,q)$ ect.. To do this, by the Riesz representation theorem, we introduce the following Riesz canonical operators σ and s:

$$\begin{cases} \sigma : V' \to V, & \text{such that} \\ < v', v >= (\sigma v', v)_V = a(\sigma v', v) & \forall \;\; v' \in V', v \in V, \end{cases} \tag{2.13}$$

the inner product in V' is defined as follows

$$(v', u')_{V'} = (\sigma v', \sigma u')_V \qquad \forall v', u' \in V', \tag{2.14}$$

and

$$\begin{cases} s : Q' \to Q, & \text{such that} \\ < q', q >= (sq', q)_Q & \forall q' \in Q', q \in Q, \end{cases} \tag{2.15}$$

and the inner product in Q' is defined as follows

$$(q', p')_{Q'} = (sq', sp')_Q \qquad \forall q', p' \in Q'. \tag{2.16}$$

Lemma 2.4.

$$A = \sigma^{-1} \tag{2.17}$$

Proof. By the definitions of the inner product in V and the operator A, we have $\forall \;\; u, v \in V$,

$$(u, v)_V = a(u, v) =< Au, v >= (\sigma Au, v)_V,$$

from which, the relation (2.17) is proved.

Lemma 2.5.

$$\mathcal{A}((u,p);(v,q)) = a(u,v) + b(u,q) + b(v,p) + b(w_q, p) + b(u, r_v), \tag{2.18}$$

where

$$\begin{cases} w_q \in V, & \textit{such that} \\ a(w_q, v) = b(v, q) & \forall \;\; v \in V \end{cases} \tag{2.19}$$

and

$$\begin{cases} r_v \in Q, & \textit{such that} \\ (r_v, q)_Q = b(v, q) & \forall \;\; q \in Q. \end{cases} \tag{2.20}$$

Proof. By the definitions of the operators σ, s, A, B and B^T, and the Lemma 2.4, it can be seen that

$$\begin{aligned} &(i) \;\; (Au, Av)_{V'} = (\sigma Au, \sigma Av)_V = (u, v)_V = a(u, v), \\ &(ii) \;\; (Au, B^T q)_{V'} = (\sigma Au, \sigma B^T q)_V = (u, \sigma B^T q)_V =< u, B^T q >= b(u, q), \\ &(iii) \;\; (Av, B^T p)_{V'} = b(v, p), \end{aligned}$$

and

$$(iv)(B^T p, B^T q)_{V'} = (\sigma B^T p, \sigma B^T q)_V =< \sigma B^T q, B^T p >=< B\sigma B^T q, p >= b(\sigma B^T q, p).$$

Let

$$w_q = \sigma B^T q = A^{-1} B^T q \qquad \text{in} \;\; V,$$

which is equivalent to that

$$Aw_q = B^T q \qquad \text{in} \ \ V'$$

i.e.,

$$< Aw_q, v >=< B^T q, v >=< Bv, q > \qquad \forall \ \ v \in V,$$

then

$$a(w_q, v) = b(v, q) \qquad \forall \ \ v \in V.$$

We now consider that

$$(v) \qquad\qquad (Bu, Bv)_{Q'} = (sBu, sBv)_Q =< Bu, sBv >= b(u, sBv).$$

Let

$$r_v = sBv \qquad \text{in} \ \ Q$$

which is equivalent to that

$$s^{-1} r_v = Bv \qquad \text{in} \ \ Q',$$

i.e.,

$$< s^{-1} r_v, q >=< Bv, q >= b(v, q) \qquad \forall \ \ q \in Q,$$

and

$$< s^{-1} r_v, q >= (r_v, q)_Q,$$

then

$$(r_v, q)_Q = b(v, q) \qquad \forall \ \ q \in Q.$$

Thus

$$(Bu, Bv)_{Q'} = b(u, r_v).$$

Summarizing (i)–(v) and the expression $\mathcal{A}((\cdot,\cdot);(\cdot,\cdot))$(2.10), the proof is completed.

Similarly we have

Lemma 2.6.

$$\mathcal{F}(v, q) =< f, v > +b(\sigma f, q) + b(v, s\chi) \tag{2.21}$$

where $\sigma f \in V$, *such that*

$$a(\sigma f, v) =< f, v > \qquad \forall \ \ v \in V, \tag{2.22}$$

and $s\chi \in Q$, *such that*

$$(s\chi, q)_Q =< \chi, q > \qquad \forall \ \ q \in Q. \tag{2.23}$$

3. The Semi-Finite Element Approximation

In this section, we present the so-called semi-finite element approximation for the least-squares problem (2.9). It will be seen that the semi-finite element approximation is used for the theoretical analysis of the real finite element approximation in the next section, but not for the practical computing.

Let V_h and Q_h be the finite element subspaces of V andQ respectively, then we have the following approximation of the problem (2.9)

$$\begin{cases} \text{to find } \ (\widetilde{u_h}, \widetilde{p_h}) \in V_h \times Q_h, & \text{such that} \\ \mathcal{A}((\widetilde{u_h}, \widetilde{p_h}); (v_h, q_h)) = \mathcal{F}(v_h, q_h) & \forall \ (v_h, q_h) \in V_h \times Q_h, \end{cases} \tag{3.1}$$

where by the Lemma 2.5 and 2.6,

$$\begin{aligned} \mathcal{A}((\widetilde{u_h}, \widetilde{p_h}); (v_h, q_h)) = {} & a(\widetilde{u_h}, v_h) + b(\widetilde{u_h}, q_h) + b(v_h, \widetilde{p_h}) \\ & + b(w_{q_h}, \widetilde{p_h}) + b(\widetilde{u_h}, r_{v_h}), \end{aligned} \tag{3.2}$$

$$\mathcal{F}(v_h, q_h) = < f, v_h > + b(\sigma f, q_h) + b(v_h, s\chi), \tag{3.3}$$

and $w_{q_h}, r_{v_h}, \sigma f$ and $s\chi$ are the "exact" solutions of the problems (2.19), (2.20), (2.22) and (2.23) respectively. Thus the approximate problem (3.1) is called the semi-finite element approximation of (2.9).

Then due to the Theorem 2.2, it can be seen that

$$\mathcal{A}((v_h, q_h); (v_h, q_h)) \geq \alpha' \|(v_h, q_h)\|^2_{V\times Q} \qquad \forall \ (v_h, q_h) \in V_h \times Q_h, \tag{3.4}$$

which means that the continuous bilinear form $\mathcal{A}((\cdot,\cdot);(\cdot,\cdot))$ is $V_h \times Q_h-$ elliptic. Thus by the basic theory of the conforming finite element method for the elliptic problem (c.f.[9]), the following error estimate holds

$$(\|u - \widetilde{u_h}\|^2_V + \|p - \widetilde{p_h}\|^2_Q) \leq C(\inf_{v_h \in V_h} \|u - v_h\|^2_V + \inf_{q_h \in Q_h} \|p - q_h\|^2_Q). \tag{3.5}$$

4. The Finite Element Approximation

In this section, we present the real finite element approximation of the problem (2.9) for the practical computing. To do this, the solutions $w_{q_h}, r_{v_h}, \sigma f$ and $s\chi$ of the problem (2.19), (2.20), (2.22) and (2.23) must be approximated by finite element methods. It turns out that , instead of the problem (3.1), the real finite element approximation of (2.9) should be presented as follows,

$$\begin{cases} \text{to find } \ (u_h, p_h) \in V_h \times Q_h, & \text{such that} \\ \mathcal{A}'_h((u_h, p_h); (v_h, q_h)) = \mathcal{F}_h(v_h, q_h) & \forall \ (v_h, q_h) \in V_h \times Q_h, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} \mathcal{A}'_h((u_h, p_h); (v_h, q_h)) = {} & a(u_h, v_h) + b(u_h, q_h) + b(v_h, p_h) \\ & + b((w_{q_h})_{h'}, p_h) + b(u_h, (r_{v_h})_h), \end{aligned} \tag{4.2}$$

$$\begin{cases} (w_{q_h})_{h'} \in V_{h'}, & \text{such that} \\ a((w_{q_h})_{h'}, v_{h'}) = b(v_{h'}, q_h) & \forall \ v_{h'} \in V_{h'}, \end{cases} \tag{4.3}$$

$$\begin{cases} (r_{v_h})_h \in Q_h, & \text{such that} \\ ((r_{v_h})_h, q_h)_Q = b(v_h, q_h) & \forall \ q_h \in Q_h, \end{cases} \tag{4.4}$$

and

$$\mathcal{F}_h(v_h, q_h) =< f, v_h > +b((\sigma f)_h, q_h) + b(v_h, (s\chi)_h), \tag{4.5}$$

$$\begin{cases} (\sigma f)_h \in V_h, & \text{such that} \\ a((\sigma f)_h, v_h) =< f, v_h > & \forall \ v_h \in V_h, \end{cases} \tag{4.6}$$

$$\begin{cases} (s\chi)_h \in Q_h, & \text{such that} \\ ((s\chi)_h, q_h)_Q =< \chi, q_h > & \forall \ q_h \in Q_h. \end{cases} \tag{4.7}$$

In the problem (4.3), the finite element space $V_{h'} \subset V$ is another space, which is, in general, different from the space V_h, and will be indecated in the following.

In what follows, let C, C' and C" denote the generic positive constants independent of h, and may be of different values in different places.

For the coerciveness of the bilinear form $\mathcal{A}'_h((\cdot,\cdot);(\cdot,\cdot)) : (V_h \times Q_h) \times (v_h \times Q_h) \to R$, we have

Theorem 4.1. *Assume that* (i) *the bilinear form $a(\cdot,\cdot)$ is symmetric, continuous and V-elliptic in the sense of* (2.2), (ii) $V_h \subset V_{h'} \subset V$ *and* (iii) *the bilinear form $b(\cdot,\cdot)$ is continuous and the following discrete BB-condition is satisfied*

$$\sup_{v_{h'} \in V_{h'}} \frac{b(v_{h'}, q_h)}{\|v_{h'}\|_V} \geq \beta' \|q_h\|_Q \qquad \forall \ q_h \in Q_h, \tag{4.8}$$

with $\beta' =$ Const. > 0. Then $\mathcal{A}'_h((\cdot,\cdot);(\cdot,\cdot)) : (V_h \times Q_h) \times (V_h \times Q_h) \to R$ is symmetric, continuous bilinear form and the following coerciveness holds

$$\mathcal{A}'_h((v_h, q_h); (v_h, q_h)) \geq \alpha' (\|v_h\|_V^2 + \|q_h\|_Q^2) \qquad \forall \ (v_h, Q_h) \in V_h \times Q_h. \tag{4.9}$$

where $\alpha' =$ Const. > 0, independent of h.

Proof. The continuity of the bilinear form $\mathcal{A}'_h((\cdot,\cdot);(\cdot,\cdot))$ is obvious. And the symmetry of $\mathcal{A}'_h((\cdot,\cdot);(\cdot,\cdot))$ can be deduced from the following relations

$$b((w_{q_h})_{h'}, p_h) = a((w_{p_h})_{h'}, (w_{q_h})_{h'}) = a((w_{q_h})_{h'}, (w_{p_h})_{h'}) = b((w_{p_h})_{h'}, q_h), \tag{4.10}$$

and

$$b(u_h, (r_{v_h})_h) = ((r_{u_h})_h, (r_{v_h})_h)_Q = ((r_{v_h})_h, (r_{u_h})_h)_Q = b(v_h, (r_{u_h})_h), \tag{4.11}$$

since (4.3) and (4.4).

Finally, we consider the coerciveness of the bilinear form $\mathcal{A}'_h((\cdot,\cdot);(\cdot,\cdot))$. By (4.3), (4.4) and the assumption (ii), from (4.2) we have

$$\begin{aligned} \mathcal{A}'_h((v_h, q_h); (v_h, q_h)) &= a(v_h, v_h) + 2b(v_h, q_h) + b((w_{q_h})_{h'}, q_h) + b(v_h, (r_{v_h})_h) \\ &= a(v_h, v_h) + 2a((w_{q_h})_{h'}, v_h) + a((w_{q_h})_{h'}, (w_{q_h})_{h'}) + ((r_{v_h})_h, (r_{v_h})_h)_Q \\ &= \|v_h + (w_{q_h})_{h'}\|_V^2 + \|(r_{v_h})_h\|_Q^2. \end{aligned} \tag{4.12}$$

With use of the technique in [12], it can be seen that

$$\|(r_{v_h})_h\|_Q^2 = \|(r_{v_h})_h + \gamma q_h\|_Q^2 - 2\gamma((r_{v_h})_h, q_h)_Q - \gamma^2 \|q_h\|_Q^2,$$

and by (4.3), (4.4) and the assumption (ii), we have

$$((r_{v_h})_h, q_h)_Q = b(v_h, q_h) = a((w_{q_h})_{h'}, v_h),$$

from which, the following equality holds

$$\|(r_{v_h})\|_Q^2 = \|(r_{v_h})_h + \gamma q_h\|_Q^2 - 2\gamma a((w_{q_h})_{h'}, v_h) - \gamma^2 \|q_h\|_Q^2, \tag{4.13}$$

where the parameter $\gamma > 0$ will be determined in the following. We now rewrite $\|v_h + (w_{q_h})_{h'}\|_V^2$ into the following form

$$\|v_h + (w_{q_h})_{h'}\|_V^2 = \|v_h\|_V^2 + \|(w_{q_h})_{h'}\|_V^2 + 2a((w_{q_h})_{h'}, v_h). \tag{4.14}$$

From (4.12)–(4.14), we have

$$\begin{aligned} \mathcal{A}'_h((v_h, q_h); (v_h, q_h)) &= \|(r_{v_h})_h + \gamma q_h\|_Q^2 \\ &\quad + \{\|v_h\|_V^2 + \|(w_{q_h})_{h'}\|_V^2 + 2(1-\gamma) a((w_{q_h})_{h'}, v_h)\} - \gamma^2 \|q_h\|_Q^2 \\ &= \|(r_{v_h})_h + \gamma q_h\|_Q^2 + \|v_h + (1-\gamma)(w_{q_h})_{h'}\|_V^2 \\ &\quad + \gamma(2-\gamma)\|(w_{q_h})_{h'}\|_V^2 - \gamma^2 \|q_h\|_Q^2. \end{aligned} \tag{4.15}$$

By the assumption (iii) and (4.3), it can be seen that

$$\|(w_{q_h})_{h'}\|_V \geq \beta' \|q_h\|_Q \qquad \forall \;\; q_h \in Q_h. \tag{4.16}$$

From (4.15) and (4.16), we have

$$\begin{aligned} \mathcal{A}'_h((v_h, q_h); (v_h, q_h)) &\geq \|(r_{v_h})_h + \gamma q_h\|_Q^2 + \|v_h + (1-\gamma)(w_{q_h})_{h'}\|_V^2 \\ &\quad + \gamma(2 - \gamma - \frac{\gamma}{(\beta')^2})\|(w_{q_h})_{h'}\|_V^2 \geq C' \|(w_{q_h})_{h'}\|_V^2, \end{aligned} \tag{4.17}$$

where $C' = \gamma(2 - \gamma - \frac{\gamma}{(\beta')^2}) = \text{Const.} > 0$, and the parameter γ is determined in $(0, \frac{2\beta'^2}{1+\beta'^2})$.

From (4.16) and (4.17), we have

$$\mathcal{A}'_h((v_h, q_h); (v_h, q_h)) \geq C \|q_h\|_Q^2, \tag{4.18}$$

with $C = \text{Const.} > 0$. We now assert that

$$\mathcal{A}'_h((v_h, q_h); (v_h, q_h)) \geq C \|v_h\|_V^2 \qquad \forall \;\; (v_h, q_h) \in V_h \times Q_h. \tag{4.19}$$

In fact, we have

$$\|v_h\|_V^2 \leq 2\|v_h + (w_{q_h})_{h'}\|_V^2 + 2\|(w_{q_h})_{h'}\|_V^2, \tag{4.20}$$

and by (4.3)

$$\begin{aligned} \|(w_{q_h})_{h'}\|_V^2 &= a((w_{q_h})_{h'}, (w_{q_h})_{h'}) = b((w_{q_h})_{h'}, q_h) \\ &\leq \|b\| \cdot \|(w_{q_h})_{h'}\|_V \cdot \|q_h\|_Q, \end{aligned}$$

from which we have

$$\|(w_{q_h})_{h'}\|_V \leq \|b\| \cdot \|q_h\|_Q. \tag{4.21}$$

Then from (4.20) and (4.12), it can be seen that

$$\begin{aligned}\|v_h\|_V^2 &\le 2\|v_h + (w_{q_h})_{h'}\|_V^2 + 2\|b\|^2\|q_h\|_Q^2\\ &\le 2\mathcal{A}'_h((v_h,q_h);(v_h,q_h)) + \frac{2\|b\|^2}{C}\mathcal{A}'_h((v_h,q_h);(v_h,q_h))\\ &\le C"\mathcal{A}'_h((v_h,q_h);(v_h,q_h)) \qquad \forall (v_h,q_h)\in V_h\times Q_h,\end{aligned}$$

from which the assertion (4.19) is proved. Thus by (4.18), (4.19), the proof is completed.

Remark 4.1. *If for any given $q_h \in Q_h$, the exact solution w_{q_h} of the problem* (2.9) *with $q = q_h$ can be obtained, then it is not needed to approximate* (2.19) *by* (4.3), *and we can take $b(w_{q_h}, p_h)$ instead of $b((w_{q_h})_{h'}, p_h)$ in the expression of $\mathcal{A}'_h((u_h,p_h);(v_h,q_h))$* (4.2). *Thus we can obtain the coerciveness of $\mathcal{A}'_h((\cdot,\cdot);(\cdot,\cdot))$ without the discrete BB-condition* (4.8). *For the first order system, the least-squares mixed finite element method is just the case presented previously* (*c.f.* [3]–[6], [10], [12] *and* [13]).

5. The Error Estimate

In this section, we establish the error estimate of the finite element approximation (4.1) as follows:

Theorem 5.1. *If the assumptions* (i)–(iii) *in the Theorem* 4.1 *are satisfied, let (u,p) and (u_h,p_h) be the solutions of the problems* (2.9) *and* (3.2) *respectively, then the following error estimate holds*

$$\begin{aligned}\|(u,p)-(u_h,p_h)\|_{V\times Q} \le C\{&\inf_{(v_h,q_h)\in V_h\times Q_h}\|(u,p)-(v_h,q_h)\|_{V\times Q}\\ &+\inf_{(\phi_{h'},\nu_h)\in V_{h'}\times Q_h}\|(w_p,r_u)-(\phi_{h'},\nu_h)\|_{V\times Q}\\ &+\inf_{(\omega_h,\mu_h)\in V_h\times Q_h}\|(\sigma f, s\chi)-(\omega_h,\mu_h)\|_{V\times Q}\},\end{aligned} \tag{5.1}$$

where w_p and r_u are the solutions of the problems (2.19) *with $q = p$ and (2.20) with $v = u$ respectively, and σf and $s\chi$ are the solutions of the problems* (2.22) *and* (2.23) *respectively.*

Proof. (i) With use of the triangle inequality, we have $\forall\ (v_h,q_h)\in V_h\times Q_h$,

$$\|(u,p)-(u_h,p_h)\|_{V\times Q} \le \|(u,p)-(v_h,q_h)\|_{V\times Q} + \|(u_h,p_h)-(v_h,q_h)\|_{V\times Q}. \tag{5.2}$$

By the Theorem 4.1 and since (u,p) and (u_h,p_h) are the solutions of the problem (2.9) and (4.1) respectively, it can be seen that

$$\begin{aligned}\alpha'\|(u_h,p_h)-(v_h,q_h)\|_{V\times Q}^2 &\le \mathcal{A}'_h((u_h-v_h,p_h-q_h);(u_h-v_h,p_h-q_h))\\ &= \mathcal{A}'_h((u_h,p_h);(u_h-v_h,p_h-q_h)) - \mathcal{A}'_h((v_h,q_h);(u_h-v_h,p_h-q_h))\\ &= \mathcal{F}_h(u_h-v_h,p_h-q_h) - \mathcal{A}((u,p);(u_h-v_h,p_h-q_h))\\ &+ \mathcal{A}((u-v_h,p-q_h);(u_h-v_h,p_h-q_h))\\ &+ \mathcal{A}((v_h,q_h);(u_h-v_h,p_h-q_h)) - \mathcal{A}'_h((v_h,q_h);(u_h-v_h,p_h-q_h))\\ &= (\mathcal{F}_h(u_h-v_h,p_h-q_h) - \mathcal{F}(u_h-v_h,p_h-q_h))\\ &+ (\mathcal{A}((v_h,q_h);(u_h-v_h,p_h-q_h)) - \mathcal{A}'_h((v_h,q_h);(u_h-v_h,p_h-q_h)))\\ &+ \mathcal{A}((u-v_h,p-q_h);(u_h-v_h,p_h-q_h)).\end{aligned} \tag{5.3}$$

From (5.2), (5.3) and by the continuity of $\mathcal{A}((\cdot,\cdot);(\cdot,\cdot))$, it can be seen that

$$\begin{aligned}\|(u,p)-(u_h,p_h)\|_{V\times Q} \le C\{&\inf_{(v_h,q_h)\in V_h\times Q_h}(\|(u,p)-(v_h,q_h)\|_{V\times Q}\\ &+\sup_{(w_h,r_h)\in V_h\times Q_h}\frac{|\mathcal{A}((v_h,q_h);(w_h,r_h))-\mathcal{A}'_h((v_h,q_h);(w_h,r_h))|}{\|(w_h,r_h)\|_{V\times Q}})\\ &+\sup_{(w_h,r_h)\in V_h\times Q_h}\frac{|\mathcal{F}(w_h,r_h)-\mathcal{F}_h(w_h,r_h)|}{\|(w_h,r_h)\|_{V\times Q}}\}.\end{aligned} \tag{5.4}$$

(ii) By (2.21) and (4.5), we have

$$\begin{aligned}\mathcal{F}(w_h,r_h)-\mathcal{F}_h(w_h,r_h) &= b(\sigma f-(\sigma f)_h,r_h)+b(w_h,s\chi-(s\chi)_h)\\ &\le C(\|\sigma f-(\sigma f)_h\|_V\|r_h\|_Q+\|s\chi-(s\chi)_h\|_Q\|w_h\|_V)\\ &\le C\{\|\sigma f-(\sigma f)_h\|_V^2+\|s\chi-(s\chi)_h\|_Q^2\}^{\frac{1}{2}}\|(w_h,r_h)\|_{V\times Q}.\end{aligned} \tag{5.5}$$

Since $(\sigma f)_h$ and $(s\chi)_h$ are the solutions of the problems (4.6) and (4.7) respectively, that means that $(\sigma f)_h$ and $(s\chi)_h$ are the finite element approximations of the solutions of the problems (2.22) and (2.23) respectively, then

$$\|\sigma f-(\sigma f)_h\|_V \le \inf_{\omega_h\in V_h}\|\sigma f-\omega_h\|_V, \tag{5.6}$$

and

$$\|s\chi-(s\chi)_h\|_Q \le \inf_{\mu_h\in Q_h}\|s\chi-\mu_h\|_Q. \tag{5.7}$$

Form (5.5)–(5.7), we have

$$\sup_{(w_h,r_h)\in V_h\times Q_h}\frac{|\mathcal{F}(w_h,r_h)-\mathcal{F}_h(w_h,r_h)|}{\|(w_h,r_h)\|_{V\times Q}} \le C(\inf_{\omega_h\in V_h}\|\sigma f-\omega_h\|_V+\inf_{\mu_h\in Q_h}\|s\chi-\mu_h\|_Q). \tag{5.8}$$

(iii) By (2.18)and (4.2), and taking account of (4.10), (4.11), we have

$$\begin{aligned}&\mathcal{A}((v_h,q_h);(w_h,r_h))-\mathcal{A}'_h((v_h,q_h);(w_h,r_h))\\ &=(b(w_{r_h},q_h)-b((w_{r_h})_{h'},q_h))+(b(v_h,r_{w_h})-b(v_h,(r_{w_h})_h))\\ &=(b(w_{q_h,r_h})-b((w_{q_h})_{h'},r_h))+(b(w_h,r_{v_h})-b(w_h,(r_{v_h})_h))\\ &\le C(\|w_{q_h}-(w_{q_h})_{h'}\|_V\|r_h\|_Q+\|r_{v_h}-(r_{v_h})_h\|_Q\|w_h\|_V)\\ &\le C\{\|(w_{q_h},r_{v_h})-((w_{q_h})_{h'},(r_{v_h})_h)\|_{V\times Q}\|(w_h,r_h)\|_{V\times Q}.\end{aligned} \tag{5.9}$$

Since $(w_{q_h})_{h'}$ and $(r_{v_h})_h$ are the finite element approximations of the problems (2.19) with $q=q_h$ and (2.20) with $v=v_h$ respectively, then

$$\|w_{q_h}-(w_{q_h})_{h'}\|_V \le \inf_{\phi_{h'}\in V_{h'}}\|w_{q_h}-\phi_{h'}\|_V, \tag{5.10}$$

and

$$\|r_{v_h}-(r_{v_h})_h\|_Q \le \inf_{\nu_h\in Q_h}\|r_{v_h}-\nu_h\|_Q. \tag{5.11}$$

From (5.9)–(5.11), we have

$$\sup_{(w_h,r_h)\in V_h\times Q_h} \frac{|\mathcal{A}((v_h,q_h);(w_h,r_h)) - \mathcal{A}'_h((v_h,q_h);(w_h,r_h))|}{\|(w_h,r_h)\|_{V\times Q}} \le C(\inf_{(\phi_{h'},\nu_h)\in V_{h'}\times Q_h} \|(w_{q_h},r_{v_h}) - (\phi_{h'},\nu_h)\|_{V\times Q}). \tag{5.12}$$

Summarizing (5.4), (5.8) and (5.12), the following estimate is established

$$\begin{aligned}&\|(u,p)-(u_h,p_h)\|_{V\times Q}\\ &\le C\{\inf_{(v_h,q_h)\in V_h\times Q_h}(\|(u,p)-(v_h,q_h)\|_{V\times Q} + \inf_{(\phi_{h'},\nu_h)\in V_{h'}\times Q_h}\|(w_{q_h},r_{v_h})-(\phi_{h'},\nu_h)\|_{V\times Q})\\ &+ \inf_{(\omega_h,\mu_h)\in V_h\times Q_h}\|(\sigma f, s\chi)-(\omega_h,\mu_h)\|_{V\times Q}\}\end{aligned} \tag{5.13}$$

(iv) Choosing $v_h = \widetilde{u_h}, q_h = \widetilde{p_h}$ in (5.13), where $(\widetilde{u_h},\widetilde{p_h})$ is the solution of the semi-finite element approximation (3.1) of the problem (2.9), we have

$$\begin{aligned}\|(u,p)-(u_h,p_h)\|_{V\times Q} \le C\{&\|(u,p)-(\widetilde{u_h},\widetilde{p_h})\|_{V\times Q}\\ &+\inf_{(\phi_{h'},\nu_h)\in V_{h'}\times Q_h}\|(w_{\widetilde{p_h}},r_{\widetilde{u_h}})-(\phi_{h'},\nu_h)\|_{V\times Q}\\ &+\inf_{(\omega_h,\mu_h)\in V_h\times Q_h}\|(\sigma f,s\chi)-(\omega_h,\mu_h)\|_{V\times Q}\}.\end{aligned} \tag{5.14}$$

By the error estimate (3.5), we have

$$\|(u,p)-(\widetilde{u_h},\widetilde{p_h})\|_{V\times Q} \le \inf_{(v_h,q_h)\in V_h\times Q_h}\|(u,p)-(v_h,q_h)\|_{V\times Q}. \tag{5.15}$$

We now estimate the second term on the right hand side of (5.14). Firstly we have

$$\inf_{\phi_{h'}\in V_{h'}}\|w_{\widetilde{p_h}}-\phi_{h'}\|_V \le \|w_{\widetilde{p_h}}-(w_p)_{h'}\|_V \le \|w_{\widetilde{p_h}}-w_p\|_V + \|w_p-(w_p)_{h'}\|_V. \tag{5.16}$$

Since $w_{\widetilde{p_h}}$ and w_p are the solutions of (2.19) with $q=\widetilde{p_h}$ and p respectively, then it can be seen that

$$a(w_p - w_{\widetilde{p_h}}, v) = b(v, p-\widetilde{p_h}) \qquad \forall\ \ v\in V,$$

in which let $v = w_p - w_{\widetilde{p_h}} \in V$, and taking into account that $b(\cdot,\cdot)$ is a continuous bilinear form, we have

$$\|w_p - w_{\widetilde{p_h}}\|_V^2 = b(w_p - w_{\widetilde{p_h}}, p-\widetilde{p_h}) \le C\|w_p - w_{\widetilde{p_h}}\|_V\|p-\widetilde{p_h}\|_Q,$$

then

$$\|w_p - w_{\widetilde{p_h}}\|_V \le C\|p-\widetilde{p_h}\|_Q. \tag{5.17}$$

From (5.16) and (5.17), we have

$$\inf_{\phi_{h'}\in V_{h'}}\|w_{\widetilde{p_h}}-\phi_{h'}\|_V \le \{\|p-\widetilde{p_h}\|_Q + \|w_p-(w_p)_{h'}\|_V\}. \tag{5.18}$$

Next, we have

$$\inf_{\nu_h\in Q_h}\|r_{\widetilde{u_h}}-\nu_h\|_Q \le \|r_{\widetilde{u_h}}-(r_u)_h\|_Q \le \|r_{\widetilde{u_h}}-r_u\|_Q + \|r_u-(r_u)_h\|_Q. \tag{5.19}$$

Since $r_{\widetilde{u_h}}$ and r_u are the solutions of (2.20) with $v = \widetilde{u_h}$ and u respectively, then it can be seen that

$$(r_u - r_{\widetilde{u_h}}, q) = b(u - \widetilde{u_h}, q) \qquad \forall \;\; q \in Q,$$

from which, in the similar way as above, we have

$$\|r_u - r_{\widetilde{u_h}}\|_Q \leq C\|u - \widetilde{u_h}\|_V. \tag{5.20}$$

Then we have

$$\inf_{\nu_h \in Q_h} \|r_{\widetilde{u_h}} - \nu_h\|_Q \leq C\{\|u - \widetilde{u_h}\|_V + \|r_u - (r_u)_h\|_Q\}. \tag{5.21}$$

And since $(w_p)_{h'}$ and $(r_u)_h$ are the finite element approximations of w_p and r_u of the problem (2.19) with q= p and problem (2.20) with v= u respectively, then by the standard error estimate of the finite element method(c.f.[9]), it can be seen that

$$\begin{cases} \|w_p - (w_p)_{h'}\|_V \leq C \inf_{\phi_{h'} \in V_{h'}} \|w_p - \phi_{h'}\|_V, \\ \|r_u - (r_u)_h\|_Q \leq \inf_{\nu_h \in Q_h} \|r_u - \nu_h\|_Q. \end{cases} \tag{5.22}$$

Summarizing (5.14), (5.15), (5.18), (5.21) and (5.22), the proof is completed.

The abstract analysis presented previously can be applied to the Stokes problem and plane elasticity.

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