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THE SOLVABILITY CONDITIONS FOR THE INVERSE PROBLEM OF BISYMMETRIC NONNEGATIVE DEFINITE MATRICES*1)

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Abstract

$A = (a_{ij}) \in R^{n\times n}$ is termed bisymmetric matrix if

$$a_{ij} = a_{ji} = a_{n-j+1,n-i+1}, \quad i,j = 1,2\cdots n.$$

We denote the set of all $n \times n$ bisymmetric matrices by $BSR^{n\times n}$.

This paper is mainly concerned with solving the following two problems:

Problem I. Given $X, B \in R^{n\times m}$, find $A \in P_n$ such that $AX = B$,
where $P_n = \{A \in BSR^{n\times n} | \quad x^T Ax \geq 0, \quad \forall x \in R^n\}$.

Problem II. Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$ such that

$$\|A^* - \hat{A}\|_F = \min_{A\in S_E} \|A^* - A\|_F,$$

where $\|\cdot\|_F$ is Frobenius norm, and S_E denotes the solution set of problem I.

The necessary and sufficient conditions for the solvability of problem I have been studied. The general form of S_E has been given. For problem II the expression of the solution has been provided.

Key words: Frobenius norm, Bisymmetric matrix, The optimal solution.

1. Introduction

Inverse eigenvalue problem has widely been used in engineering. For example inverse eigenvalue method is a useful means in vibration design and vibration control of flyer. In resent years a serial of good conclussions have been made for inverse eigenvalue problem [4]. Bisymmetric matrices have practical application in civil engineering and vibration engineering. However, inverse problems of bisymmetric matrix have not be concerned yet. In this paper we will discuss this problem.

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We denote the real $n \times m$ matrices space by $R^{n\times m}$, and $R^n = R^{n\times 1}$, the set of all matrices in $R^{n\times m}$ with rank r by $R_r^{n\times m}$, the set of all $n \times n$ orthogonal matrices by $OR^{n\times n}$, the set of all $n \times n$ symmetric matrices by $SR^{n\times n}$, the column space, the null space and the Moore–Penrose generalized inverse of a matrix A by $R(A), N(A), A^+$ respectively, the identity matrix of order n by I_n, the Frobenius norm of A by $\|A\|_F$. We define inner product in space $R^{n\times m}$, $(A, B) = \mathrm{tr}(B^T A) = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} a_{ij}b_{ij}, \quad \forall A, B \in R^{n\times m}$. Then $R^{n\times m}$ is a Hilbert inner product space. The norm of a matrix produced by the inner product is Frobenius norm.

Definition 1. $A = (a_{ij}) \in R^{n\times n}$, *if*

$$a_{ij} = a_{ji} = a_{n-j+1,n-i+1}, \qquad i, j = 1, 2, \cdots, n.$$

Then we term A as a bisymmetric matrix. The set of all bisymmetric matrices denoted by $BSR^{n\times n}$.

Let

$$k = [\frac{n}{2}], \qquad \text{[x] is the maximum integer number that is not greater than x.} \tag{1.1}$$

When $n = 2k$,

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix}; \tag{1.2}$$

and when $n = 2k + 1$,

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{pmatrix}, S_k = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}_{k\times k}. \tag{1.3}$$

It is easy verified that above D are orthogonal.

Definition 2. *$A \in BSR^{n\times n}$ is termed bisymmetric nonnegative definite [positive definite] if $x^T Ax \geq 0 (> 0)$ for every nonzero x in R^n.*

Let

$$P_n = \{A \in BSR^{n\times n} | \quad x^T Ax \geq 0, \quad \forall x \in R^n\}.$$

Now we consider the following problems:

Problem I. Given $X, B \in R^{n\times m}$, find $A \in P_n$ such that

$$AX = B.$$

Problem II. Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$ such that

$$\|A^* - \hat{A}\|_F = \min_{A\in S_E} \|A^* - A\|_F,$$

where S_E is the solution set of problem I.

At first, in this paper, we will discuss the geometric constrution of $BSR^{n\times n}$. Then we will give the necessary and sufficient conditions for the solvability of problem I and the expression of the general solution of problem I, and prove that S_E is a closed convex set. At last, we will prove that there exists an unique solution of problem II and give expression of the solution for problem II.

2. The Solvability Conditions and General Form of Solution for Problem I

At first we discuss the construction of $BSR^{n\times n}$.

Lemma 1.[3] *$A \in BSR^{n\times n}$ if and only if*

$$A = S_n A S_n, \qquad A = A^T$$

Therom 1.

$$BSR^{2k\times 2k} = \left\{ \begin{pmatrix} M & HS_k \\ S_k H & S_k M S_k \end{pmatrix} \middle| \quad M, H \in SR^{k\times k} \right\}. \tag{2.1}$$

$$BSR^{(2k+1)\times(2k+1)} = \left\{ \begin{pmatrix} N & C & HS_k \\ C^T & \rho & C^T S_k \\ S_k H & S_k C & S_k N S_k \end{pmatrix} \middle| \quad \begin{matrix} N, H \in SR^{k\times k}, \\ C \in R^k, \rho \in R^1 \end{matrix} \right\}. \tag{2.2}$$

whether n is odd or even number, the general form of elements in $BSR^{n\times n}$ is

$$D \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} D^T, \qquad A_{11} \in SR^{(n-k)\times(n-k)}, \quad A_{22} \in SR^{k\times k}. \tag{2.3}$$

Proof. We only prove(2.1). If

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in BSR^{2k\times 2k}.$$

Then

$$A = A^T, \quad \begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{2.4}$$

(2.4) is equivalent to

$$\begin{matrix} A_{11} = A_{11}^T, \quad A_{12} = A_{21}^T, & A_{22} = A_{22}^T, \\ A_{22} = S_k A_{11} S_k & (A_{12} S_k)^T = A_{12} S_k \end{matrix}. \tag{2.5}$$

Let $\qquad A_{12} S_k = H, \qquad A_{11} = M.$

Then $\quad H = H^T, \quad M = M^T, \quad A_{12} = HS_k, \quad A_{21} = S_k H, \quad A_{22} = S_k M S_k.$

It implies that

$$A = \begin{pmatrix} M & HS_k \\ S_k H & S_k M S_k \end{pmatrix}.$$

Hence $BSR^{2k\times 2k} \subseteq \left\{ \begin{pmatrix} M & HS_k \\ S_k H & S_k M S_k \end{pmatrix} \middle| M, H \in SR^{k\times k} \right\}.$

Conversely, it is obvious that for every $M, H \in SR^{k\times k}$

$$\begin{pmatrix} M & HS_k \\ S_k H & S_k M S_k \end{pmatrix} = \begin{pmatrix} M & HS_k \\ S_k H & S_k M S_k \end{pmatrix}^T$$

and

$$\begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix} \begin{pmatrix} M & HS_k \\ S_k H & S_k M S_k \end{pmatrix} \begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix} = \begin{pmatrix} M & HS_k \\ S_k H & S_k M S_k \end{pmatrix}$$

From Lemma 1, it follows that $\begin{pmatrix} M & HS_k \\ S_kH & S_kMS_k \end{pmatrix} \in BSR^{2k\times 2k}$. Thus (2.1) holds. The form (2.2) can be obtained by the similar method. Futhermore, for $n = 2k$

$$D^T \begin{pmatrix} M & HS_k \\ S_kH & S_kMS_k \end{pmatrix} D = \begin{pmatrix} M+H & 0 \\ 0 & M-H \end{pmatrix}$$

we have the form (2.3). For $n = 2k+1$, D is the form (1.3)

$$\begin{aligned} & D^T \begin{pmatrix} N & C & HS_k \\ C^T & \rho & C^TS_k \\ S_kH & S_kC & S_kNS_k \end{pmatrix} D \\ &= \frac{1}{2} \begin{pmatrix} I_k & 0 & S_k \\ 0 & \sqrt{2} & 0 \\ I_k & 0 & -S_k \end{pmatrix}^T \begin{pmatrix} N & C & HS_k \\ C^T & \rho & C^TS_k \\ S_kH & S_kC & S_kNS_k \end{pmatrix} \begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{pmatrix} \\ &= \begin{pmatrix} N+H & \sqrt{2}C & 0 \\ \sqrt{2}C^T & \rho & 0 \\ 0 & 0 & N-H \end{pmatrix} \end{aligned}$$

Hence

$$\begin{pmatrix} N & C & HS_k \\ C^T & \rho & C^TS_k \\ S_kH & S_kC & S_kNS_k \end{pmatrix} = D \begin{pmatrix} N+H & \sqrt{2}C & 0 \\ \sqrt{2}C^T & \rho & 0 \\ 0 & 0 & N-H \end{pmatrix} D^T$$

It implies that the elements in $BSR^{n\times n}$ have the form (2.3) when $n = 2k+1$.

On the other hand, it can be directly verify that matrices in form (2.3) belong to $BSR^{n\times n}$ from Lemma 1

Let

$$SR_0^{n\times n} = \{A \in SR^{n\times n} | \quad x^TAx \geq 0, \quad \forall x \in R^n\},$$

which is the set of all symmetric nonnegative definite matrices.

Lemma 2.[2] *Given $Y \in R^{n\times m}$, $Z \in R_r^{n\times m}$, then there is a matrix $A \in SR_0^{n\times n}$ such that*

$$AZ = Y \tag{2.6}$$

if and only if

$$Z^TY = Y^TZ \in SR_0^{m\times m}, \quad rank(Z^TY) = rank(Y) \tag{2.7}$$

and the general solution of (2.6) can be represented as

$$A = YZ^+ + (YZ^+)^T(I_n - ZZ^+) + (I_n - ZZ^+)Y(Z^TY)^+Y^T(I_n - ZZ^+) + U_2GU_2^T,$$

where $U_2 \in R^{n\times(n-r)}$ is an unit column-orthogonal matrix and $R(U_2) = N(X^T)$, $G \in SR_0^{(n-r)\times(n-r)}$ is arbitrary.

Therom 2. *Given $X, B \in R^{n\times m}$, let*

$$D^TX = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \qquad X_2 \in R^{k\times m} \tag{2.8}$$

$$D^TB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \qquad B_2 \in R^{k\times m} \tag{2.9}$$

where D is the same as (1.2) or (1.3). Then Problem I has a solution if and only if

$$X_i^T B_i \in SR_0^{m\times m}, \qquad rank(X_i^T B_i) = rank(B_i), \qquad i = 1, 2. \tag{2.10}$$

And the general solution of problem I can be represented as

$$A = DA_0D^T + D\begin{pmatrix} U_2G_1U_2^T & 0 \\ 0 & P_2G_2P_2^T \end{pmatrix} D^T,$$
$$\forall G_1 \in SR_0^{(n-k-r_1)\times(n-k-r_1)}, \forall G_2 \in SR_0^{(k-r_2)\times(k-r_2)}. \tag{2.11}$$

where

$$A_0 = \begin{pmatrix} A_{11}^0 & 0 \\ 0 & A_{22}^0 \end{pmatrix}, \tag{2.12}$$

$$\begin{aligned} A_{11}^0 &= B_1X_1^+ + (B_1X_1^+)^T(I_{n-k} - X_1X_1^+) \\ &\quad +(I_{n-k} - X_1X_1^+)B_1(X_1^TB_1)^+B_1^T(I_{n-k} - X_1X_1^+), \\ A_{22}^0 &= B_2X_2^+ + (B_2X_2^+)^T(I_k - X_2X_2^+) \\ &\quad +(I_k - X_2X_2^+)B_2(X_2^TB_2)^+B_2^T(I_k - X_2X_2^+). \end{aligned} \tag{2.13}$$

where $r_1 = rank(X_1)$, $r_2 = rank(X_2)$, $U_2 \in R^{(n-k)\times(n-k-r_1)}$, $P_2 \in R^{k\times(k-r_2)}$ are unit column-orthogonal matrices and $R(U_2) = N(X_1^T)$, $R(P_2) = N(X_2^T)$.

Proof. By Therom 1 $AX = B$, $A \in P_n$ is equivalent to that exists $A_{11} \in SR_0^{(n-k)\times(n-k)}$, $A_{22} \in SR_0^{k\times k}$ such that

$$D\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} D^TX = B$$

i.e

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Hence, $AX = B$, $A \in P_n$ has a solution if and only if

$$A_{11}X_1 = B_1, \qquad A_{22}X_2 = B_2, \qquad A_{11} \in SR_0^{(n-k)\times(n-k)}, \quad A_{22} \in SR_0^{k\times k}. \tag{2.14}$$

has a solution. From Lemma 2, (2.14) has solutions if and only if

$$X_i^TB_i \in SR_0^{m\times m}, \quad \mathrm{rank}(X_i^TB_i) = \mathrm{rank}(B_i), \quad i = 1, 2.$$

therefore (2.10) holds, and

$$\begin{aligned} A_{11} &= B_1X_1^+ + (B_1X_1^+)^T(I_{n-k} - X_1X_1^+) \\ &\quad +(I_{n-k} - X_1X_1^+)B_1(X_1^TB_1)^+B_1^T(I_{n-k} - X_1X_1^+) + U_2G_1U_2^T, \\ A_{22} &= B_2X_2^+ + (B_2X_2^+)^T(I_k - X_2X_2^+) \\ &\quad +(I_k - X_2X_2^+)B_2(X_2^TB_2)^+B_2^T(I_k - X_2X_2^+) + P_2G_2P_2^T, \\ G_1 &\in SR_0^{(n-k-r_1)\times(n-k-r_1)}, G_2 \in SR_0^{(k-r_2)\times(k-r_2)}, r_1 = rank(X_1), r_2 = rank(X_2). \end{aligned}$$

Let

$$\begin{aligned} A_{11}^0 &= B_1X_1^+ + (B_1X_1^+)^T(I_{n-k} - X_1X_1^+) \\ &\quad +(I_{n-k} - X_1X_1^+)B_1(X_1^TB_1)^+B_1^T(I_{n-k} - X_1X_1^+), \\ A_{22}^0 &= B_2X_2^+ + (B_2X_2^+)^T(I_k - X_2X_2^+) + (I_k - X_2X_2^+)B_2(X_2^TB_2)^+B_2^T(I_k - X_2X_2^+). \end{aligned}$$

$$A_0 = \begin{pmatrix} A_{11}^0 & 0 \\ 0 & A_{22}^0 \end{pmatrix}.$$

Then

$$A_{11} = A_{11}^0 + U_2 G_1 U_2^T, \qquad A_{22} = A_{22}^0 + P_2 G_2 P_2^T.$$

According to Therom 1 the general solution for problem I has the expression (2.11).

Definition 3. *Suppose $x \in R^n$ satisfies $S_n x = x$ we term x as symmetric vector; if x satisfies $S_n x = -x$ we term x as anti-symmetric vector.*

Lemma 3.[2] *Suppose $X \in R^{n\times m}, \Lambda = diag(\lambda_1, \lambda_2 \cdots, \lambda_m)$. Then $AX = X\Lambda, A \in SR_0^{n\times n}$ has a solution if and only if*

$$\lambda_i \geq 0, \quad (\lambda_i - \lambda_j) x_i^T x_j = 0, \quad i, j = 1, 2, \cdots, m,$$

where x_i is the i-th cloumn of X

From [6] we known that there are symmetric vector and anti-symmetric vector for bisymmetric matrix A .

Corollary 1. *When $n = 2k$ suppose $X \in R^{2k\times m}$, and X has following form*

$$X = \begin{pmatrix} X_1 & Y_1 \\ S_k X_1 & -S_k Y_1 \end{pmatrix} \tag{2.15}$$

which of them $X_1 \in R^{k\times l}$, $Y_1 \in R^{k\times(m-l)}$.

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \tag{2.16}$$

where

$$\Lambda_1 = diag(\lambda_1, \cdots, \lambda_l), \qquad \Lambda_2 = diag(\lambda_{l+1}, \cdots, \lambda_m)$$

Let $B = X\Lambda$, then problem I has a solution if and only if

$$\lambda_i \geq 0, \quad (\lambda_i - \lambda_j) x_i^T x_j = 0, \quad i, j = 1, 2, \cdots, l, \tag{2.17}$$

$$\lambda_i \geq 0, \quad (\lambda_i - \lambda_j) y_i^T y_j = 0, \quad i, j = l+1, 2, \cdots, m, \tag{2.18}$$

where x_i is the i-th cloumn of X_1; y_j is the j-th cloumn of Y_1.

Proof. From (2.3) it follows that $AX = X\Lambda$ is equivalent to

$$D \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} D^T \begin{pmatrix} X_1 & Y_1 \\ S_k X_1 & -S_k Y_1 \end{pmatrix} = \begin{pmatrix} X_1 & Y_1 \\ S_k X_1 & -S_k Y_1 \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$$

i.e

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \sqrt{2} X_1 & 0 \\ 0 & \sqrt{2} Y_1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} X_1 \Lambda_1 & 0 \\ 0 & \sqrt{2} Y_1 \Lambda_2 \end{pmatrix}$$

Hence $AX = X\Lambda$ has a solution if and only if $A_{11}X_1 = X_1\Lambda_1$, $A_{22}Y_1 = Y_1\Lambda_1$, $\quad A_{11}, A_{22} \in SR_0^{k\times k}$ has a solution. From Lemma 3 it follows that (2.17) and (2.18) hold.

Corollary 2. *When $n = 2k+1$ suppose $X \in R^{2k+1\times m}$, and X has following form*

$$X = \begin{pmatrix} X_1 & Y_1 \\ \alpha^T & 0^T \\ S_k X_1 & -S_k Y_1 \end{pmatrix}, \tag{2.19}$$

where $X_1 \in R^{k\times l}$, $Y_1 \in R^{k\times(m-l)}$, $\alpha^T = (\alpha_1, \alpha_2, \cdots, \alpha_l) \in R^{1\times l}$, $0^T = (0, 0, \cdots, 0) \in R^{1\times(m-l)}$.

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \tag{2.20}$$

where

$$\Lambda_1 = diag(\lambda_1, \cdots, \lambda_l), \qquad \Lambda_2 = diag(\lambda_{l+1}, \cdots, \lambda_m)$$

Let $B = X\Lambda$, *then problem I has a solution if and only if*

$$\lambda_i \geq 0, \quad (\lambda_i - \lambda_j)x_i^T x_j = 0, \quad i, j = 1, 2, \cdots, l, \tag{2.21}$$

$$\lambda_i \geq 0, \quad (\lambda_i - \lambda_j)y_i^T y_j = 0, \quad i, j = l+1, 2, \cdots, m, \tag{2.22}$$

where x_i *is the i-th cloumn of* $\begin{pmatrix} \sqrt{2}X_1 \\ \alpha^T \end{pmatrix}$; y_j *is the j-th cloumn of* Y_1.

Proof. From (2.3) it follows that $AX = X\Lambda$ is equivalent to

$$D\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} D^T \begin{pmatrix} X_1 & Y_1 \\ \alpha^T & 0^T \\ S_kX_1 & -S_kY_1 \end{pmatrix} = \begin{pmatrix} X_1 & Y_1 \\ \alpha^T & 0^T \\ S_kX_1 & -S_kY_1 \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix},$$

$$A_{11} \in SR_0^{(k+1)\times(k+1)}$$

i.e

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \sqrt{2}X_1 \\ \alpha^T \end{pmatrix} & 0 \\ 0 & Y_1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \sqrt{2}X_1 \\ \alpha^T \end{pmatrix}\Lambda_1 & 0 \\ 0 & \sqrt{2}Y_1\Lambda_2 \end{pmatrix}.$$

Hence $AX = X\Lambda$ has a solution if and only if

$$A_{11}\begin{pmatrix} \sqrt{2}X_1 \\ \alpha^T \end{pmatrix} = \begin{pmatrix} \sqrt{2}X_1 \\ \alpha^T \end{pmatrix}\Lambda_1, \quad A_{22}Y_1 = Y_1\Lambda_2, \quad A_{11} \in SR_0^{k\times k}, A_{22} \in SR_0^{k\times k}$$

has a solution. From Lemma 3 it follows that (2.21) and (2.22) hold.

3. The Expression of the Solution for Problem II

Lemma 4.[1] *Suppose* V *is a real Hilbert space,* $(\cdot, \cdot)$ *denotes inner product ,* $\|\cdot\|$ *respresents norm produced by inner product,* $K \subset V$ *is a closed convex cone whose vertex is located at the zero point.* K^* *is the dual cone of* K *in* $(K^\perp)^\perp$. *Then, for every* $u \in V$, *there is an unique* $u_0 \in K^\perp, u_+ \in K, -u_- \in K^*$ *such that*

$$(u_+, u_-) = 0, \qquad u = u_0 + u_+ + u_-$$

and

$$\|u - u_+\| \leq \|u - v\|, \quad \forall v \in K,$$

where $K^\perp$ *is the set of all elements which are orthogonal to set* K.

Lemma 5.[1] *Suppose the set of all* $n \times n$ *anti-symmetric matrices denoted by* $ASR^{n\times n}$. *the set of all matrices which are orthogonal to* $SR_0^{n\times n}$ *by* $(SR_0^{n\times n})^\perp$, *the dual cone of* $SR_0^{n\times n}$ *in* $((SR_0^{n\times n})^\perp)^\perp$ *by* $(SR_0^{n\times n})^*$, *then*

$$(SR^{n\times n})^\perp = ASR^{n\times n}, \quad (ASR^{n\times n})^\perp = SR^{n\times n},$$

$$(SR_0^{n\times n})^{\perp} = ASR^{n\times n} \quad [(SR_0^{n\times n})^{\perp}]^{\perp} = SR^{n\times n}, \quad SR_0^{n\times n} = (SR_0^{n\times n})^{*}. \tag{3.1}$$

In Lemma 4 taking $K = SR_0^{n\times n}$ and by Lemma 4, Lemma 5 we obtain

Lemma 6. *For every given $F \in R^{n\times n}$, then there exists an unique $F_0 \in ASR^{n\times n}$, $F_+ \in SR_0^{n\times n}$, $-F_- \in SR_0^{n\times n}$ such that*

$$\begin{aligned} &(F_+, F_-) = 0, \\ &F = F_0 + F_+ + F_- \end{aligned} \tag{3.2}$$

and

$$\|F - F_+\| = \min_{M \in SR_0^{n\times n}} \|F - M\|.$$

From Lemma 6 we know that every matrix F can be decomposed as sum of an anti-symmetric matrix F_0, a symmetric nonnegative definite matrix F_+ and a symmetric nonpositive definite matrix F_-. If $(F_+, F_-) = 0$ the decomposition is unique. We denote the symmetric nonnegative definite matrix and the symmetric nonpositive definite matrix above unique decomposition of F by $[F]_+$ and $[F]_-$ respectively.

Lemma 7. *When solution set S_E of problem I is non-empty, then S_E is a closed convex cone with vertex DA_0D^T, where A_0 is determined by (2.12), (2.13).*

Proof. Taking any two matrices of S_E

$$A_1 = DA_0D^T + D\begin{pmatrix} U_2G_1U_2^T & 0 \\ 0 & P_2G_2P_2^T \end{pmatrix} D^T,$$
$$\forall G_1 \in SR_0^{(n-k-r_1)\times(n-k-r_1)}, \forall G_2 \in SR_0^{(k-r_2)\times(k-r_2)},$$

$$A_2 = DA_0D^T + D\begin{pmatrix} U_2\bar{G}_1U_2^T & 0 \\ 0 & P_2\bar{G}_2P_2^T \end{pmatrix} D^T,$$
$$\forall \bar{G}_1 \in SR_0^{(n-k-r_1)\times(n-k-r_1)}, \forall \bar{G}_2 \in SR_0^{(k-r_2)\times(k-r_2)}.$$

$$A_1X = B, \qquad A_2X = B$$

let

$$F = DA_0D^T + \alpha(A_1 - DA_0D^T) + \beta(A_2 - DA_0D^T), \quad \forall \alpha, \beta \geq 0.$$

then

$$F = DA_0D^T + D\begin{pmatrix} U_2(\alpha G_1 + \beta\bar{G}_1)U_2^T & 0 \\ 0 & P_2(\alpha G_2 + \beta\bar{G}_2)P_2^T \end{pmatrix} D^T \in P_n,$$

$$\begin{aligned} FX &= DA_0D^TX + D\begin{pmatrix} U_2(\alpha G_1 + \beta\bar{G}_1)U_2^T & 0 \\ 0 & P_2(\alpha G_2 + \beta\bar{G}_2)P_2^T \end{pmatrix} D^TX \\ &= DA_0\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + D\begin{pmatrix} U_2(\alpha G_1 + \beta\bar{G}_1)U_2^T & 0 \\ 0 & P_2(\alpha G_2 + \beta\bar{G}_2)P_2^T \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ &= D\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = B. \end{aligned}$$

and

$$\forall t \geq 0, \forall \mu \geq 0, t + \mu = 1, (tA_1 + \mu A_2)X = tA_1X + \mu A_2X = tB + \mu B = B.$$

From the definition, we know that S_E is a convex cone with vertex DA_0D^T, it is clearly closed.

Therom 3. *Given $B, X \in R^{n\times m}$, $A^* \in R^{n\times n}$ and X, B satifies conditions of the Therom 2. A_0 is the same as (2.12).*

Let

$$D^TA^*D - A_0 = \begin{pmatrix} A^*_{11} & A^*_{12} \\ A^*_{21} & A^*_{22} \end{pmatrix}$$

Then Problem II has an unique optimal approximate solution which can be represented as

$$\begin{aligned} \hat{A} &= DA_0D^T + D\begin{pmatrix} U_2\hat{G}_1U_2^T & 0 \\ 0 & P_2\hat{G}_2P_2^T \end{pmatrix} D^T, \\ \hat{G}_1 &= [U_2^TA^*_{11}U_2]_+, \quad \hat{G}_2 = [P_2^TA^*_{22}P_2]_+, \end{aligned} \tag{3.3}$$

where $r_1 = rank(X_1)$, $r_2 = rank(X_2)$, $U_2 \in R^{(n-k)\times(n-k-r_1)}$, $P_2 \in R^{k\times(k-r_2)}$ are unit column-orthogonal matrices and $R(U_2) = N(X_1^T)$, $R(P_2) = N(X_2^T)$.

Proof. Because X and B satisfy the conditions of the Therom 2 the solution set S_E of problem I is nonempty. From Lemma 7 we know S_E is a colsed convex cone. Hence the coresponding problem II has an unique optimal approximate solution. Choose U_1, P_1 such that $U = (U_1, U_2) \in OR^{(n-k)\times(n-k)}$, $P = (P_1, P_2) \in OR^{k\times k}$. Attention to U, P, D are orthogonal matrices. From (2.11) we have

$$\begin{aligned}
&\|A - A^*\|^2 = \left\|DA_0D^T + D\begin{pmatrix} U_2G_1U_2^T & 0 \\ 0 & P_2G_2P_2^T \end{pmatrix} D^T - A^*\right\|^2 \\
&= \left\|A_0 + \begin{pmatrix} U_2G_1U_2^T & 0 \\ 0 & P_2G_2P_2^T \end{pmatrix} - D^TA^*D\right\|^2 \\
&= \left\|\begin{pmatrix} U_2G_1U_2^T & 0 \\ 0 & P_2G_2P_2^T \end{pmatrix} - \begin{pmatrix} A^*_{11} & A^*_{12} \\ A^*_{21} & A^*_{22} \end{pmatrix}\right\|^2 \\
&= \|U_2G_1U_2^T - A^*_{11}\|^2 + \|A^*_{12}\|^2 + \|A^*_{21}\|^2 + \|P_2G_2P_2^T - A^*_{22}\|^2 \\
&= \left\|\begin{pmatrix} -U_1^TA^*_{11}U_1 & -U_1^TA^*_{11}U_2 \\ -U_2^TA^*_{11}U_1 & G_1 - U_2^TA^*_{11}U_2 \end{pmatrix}\right\|^2 + \|A^*_{12}\|^2 + \|A^*_{21}\|^2 \\
&+ \left\|\begin{pmatrix} -P_1^TA^*_{22}P_1 & -P_1^TA^*_{22}P_2 \\ -P_2^TA^*_{22}P_1 & G_2 - P_2^TA^*_{22}P_2 \end{pmatrix}\right\|^2 \\
&= \|-U_1^TA^*_{11}U_1\|^2 + \|-U_1^TA^*_{11}U_2\|^2 + \|-U_2^TA^*_{11}U_1\|^2 + \|G_1 - U_2^TA^*_{11}U_2\|^2 + \|A^*_{12}\|^2 \\
&+ \|A^*_{21}\|^2 + \|-P_1^TA^*_{22}P_1\|^2 + \|-P_1^TA^*_{22}P_2\|^2 + \|-P_2^TA^*_{22}P_1\|^2 + \|G_2 - P_2^TA^*_{22}P_2\|^2.
\end{aligned}$$

Hence $A \in S_E$ such that $\|A - A^*\| = \min$ is equivalent to

$$\begin{aligned} \|G_1 - U_2^TA^*_{11}U_2\| &= \min, \quad G_1 \in SR_0^{(n-k-r_1)\times(n-k-r_1)}, \\ \|G_2 - P_2^TA^*_{22}P_2\| &= \min, \quad G_2 \in SR_0^{(n-r_2)\times(n-r_2)}. \end{aligned} \tag{3.4}$$

From Lemma 6 the solution (3.4) have the following expressions

$$\hat{G}_1 = [U_2^TA^*_{11}U_2]_+, \qquad \hat{G}_2 = [P_2^TA^*_{22}P_2]_+. \tag{3.5}$$

Taking (3.5) into (2.11) we obtain (3.3)

4. The Algorithm Description and Numerical Example

According to Theorem 3 we now give an algorithm of the optimal approximate solution of problem II as the following steps:

(1) according to (2.8) and (2.9) calulate $X_i, B_i, i = 1, 2$.

(2) Calculate Cholesky decomposition $X_i^T B_i, i = 1.2$. If $X_i^T B_i$ is a symmetric nonnegative definite matrix go to (3), otherwise go to (13).

(3) Calculate $rank(X_i^T B_i)$, $rank(B_i)$. If $rank(X_i^T B_i) = rank(B_i)$ go to (4), otherwise go to (13).

(4) Find unit orthogonal basis of linear equations $X_1^T z = 0$ and $Y_1^T t = 0$. We can get unit column-orthogonal matrix U_2 and P_2.

(5) According to (2.13) calculate A_{11}^0, A_{22}^0, $A_0 = \begin{pmatrix} A_{11}^0 & 0 \\ 0 & A_{22}^0 \end{pmatrix}$.

(6) Calculate $D^T A^* D - A_0 = \begin{pmatrix} A_{11}^* & A_{12}^* \\ A_{21}^* & A_{22}^* \end{pmatrix}$.

(7) Calculate $k = [\frac{n}{2}]$.

(8) Calculate eigenvalues of $\dfrac{U_2^T A_{11}^* U_2 + U_2^T (A_{11}^*)^T U_2}{2}$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-k}$, and corresponding unit eigenvectors $u_1, u_2, \ldots u_{n-k}$.

(9) Calculate eigenvalues of $\dfrac{P_2^T A_{22}^* P_2 + P_2^T (A_{22}^*)^T P_2}{2}$, $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_k$, and corresponding unit eigenvectors $\omega_1, \omega_2, \ldots \omega_k$.

(10) Find the minimum positve eigenvalue and write it as $\mu_{i_0} (i_0 \leq n - k)$. Calculate $\hat{G}_1 = \sum_1^{i_0} \mu_i u_i u_i^T$.

(11) Find the minimum positve eigenvalue and write it as $\theta_{j_0} (j_0 \leq k)$. Calculate $\hat{G}_2 = \sum_1^{j_0} \theta_i \omega_i \omega_i^T$.

(12) Acoording to (3.3) calculate $\hat{A}$.

(13) Stop.

In above steps we can calculate A_0 by using stable singular values decomposition. In [1] stability has been analysed for step (10) and step (11). Hence, this algorithm is stable.

Example 1. Taking

$$X = \begin{pmatrix} -0.6 & -1.2 & 0.1 & 0.4 \\ 1.2 & 2.4 & -0.2 & -0.8 \\ 1.2 & 2.4 & 0.2 & 0.8 \\ -0.6 & -1.2 & -0.1 & -0.4 \end{pmatrix}, \quad B = \begin{pmatrix} -1.2 & -2.4 & 0 & 0 \\ 2.4 & 4.8 & 0 & 0 \\ 2.4 & 4.8 & 0 & 0 \\ -1.2 & -2.4 & 0 & 0 \end{pmatrix},$$

$$A^* = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0.5 & -4 & 2 & 0 \\ 0 & 2.5 & 1 & -1 \\ 2 & 4 & 0 & 1 \end{pmatrix}.$$

We obtain

$$D^TX = \begin{pmatrix} -0.8487 & -1.6973 & 0 & 0 \\ 1.6973 & 3.3946 & 0 & 0 \\ 0 & 0 & 0.1414 & 0.5658 \\ 0 & 0 & -0.2829 & -1.1315 \end{pmatrix},$$

$$D^TB = \begin{pmatrix} -1.6973 & -3.3946 & 0 & 0 \\ 3.3946 & 6.7893 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} -0.8487 & -1.6973 & 0 & 0 \\ 1.6973 & 3.3945 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0.1414 & 0.5658 \\ 0 & 0 & -0.2829 & -1.1315 \end{pmatrix}.$$

$$B_1 = \begin{pmatrix} -1.6973 & -3.3946 & 0 & 0 \\ 3.3946 & 6.7893 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_1^TB_1 = \begin{pmatrix} 7.2022 & 14.4045 & 0 & 0 \\ 14.4041 & 28.8086 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_2^TB_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can verify $X_i^TB_i \in SR_0^{4\times4}, i = 1, 2$, and $rank(X_i^TB_i) = rank(B_i)$. Hence X, B satisfy the conditions of Theorem 2. The singular values decomposition of X_1 and X_2 are

$$X_1 = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}V^T, \qquad X_2 = P\begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix}Q^T$$

where

$$U = \begin{pmatrix} -0.4472 & 0.8944 \\ 0.8944 & 0.4472 \end{pmatrix}, \quad \Sigma = 4.2433, \quad V = \begin{pmatrix} 0.4472 & -0.8944 & 0 & 0 \\ 0.8944 & 0.4472 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$P = \begin{pmatrix} 0.4472 & 0.8944 \\ -0.8944 & 0.4472 \end{pmatrix}, \quad \Gamma = 1.304, \quad Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -0.6356 & -0.7720 \\ 0.2425 & 0 & 0.7490 & -0.6167 \\ 0.9702 & 0 & -0.1872 & 0.1541 \end{pmatrix}.$$

$$\Sigma^{-1} = 0.2357, \qquad \Gamma^{-1} = 0.7669.$$

$$X_1^+ = \begin{pmatrix} -0.0471 & 0.0943 \\ -0.0943 & 0.1885 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_2^+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.0832 & -0.1663 \\ 0.3327 & -0.6655 \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 0.4 & -0.8 & 0 & 0 \\ -0.8 & 1.6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D^T A^* D - A_0 = \begin{pmatrix} 1.6006 & 2.3005 & 1.0003 & 2.5008 \\ 0.5499 & -0.8498 & 0.7502 & -2.2507 \\ -1.0003 & -2.5008 & 0 & -1.5005 \\ 0.7502 & -2.7508 & -0.2501 & -3.7511 \end{pmatrix}$$

$$\frac{U_2^T A_{11}^* U_2 + U_2^T (A_{11}^*)^T U_2}{2} = 2.2505, \qquad \frac{P_2^T A_{22}^* P_2 + P_2^T (A_{22}^*)^T P_2}{2} = -1.4504.$$

$$\hat{G}_1 = 2.2505, \qquad \hat{G}_2 = 0.$$

We obtain the unique solution of corresponding problem II as

$$\hat{A} = \begin{pmatrix} 1.1005 & 0.0501 & 0.0501 & 1.1005 \\ 0.0501 & 1.0254 & 1.0254 & 0.0501 \\ 0.0501 & 1.0254 & 1.0254 & 0.0501 \\ 1.1005 & 0.0501 & 0.0501 & 1.1005 \end{pmatrix}.$$

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