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DISCONTINUOUS FINITE ELEMENT METHOD FOR CONVECTION-DIFFUSION EQUATIONS*1)

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Abstract

A discontinuous finite element method for convection-diffusion equations is proposed and analyzed. This scheme is designed to produce an approximate solution which is completely discontinuous. Optimal order of convergence is obtained for model problem. This is the same convergence rate known for the classical methods.

Key words: Discontinuous finite element method, Convection-diffusion equations.

1. Introduction

The finite element approximation of the convection-diffusin equations has been investigated using several different approaches (see e.g. [3] [4] and the references therein). Previous analysis in primal formulation of these problems was done for two types of approximation schemes : one which produces a continuous piecewise polynomial approximation and one which produces a piecewise polynomial approximation which are continuous for certain number of moments accross interelement edges [2] (nonconforming approximation). All these finite element methods have optimal order of convergence, assuming sufficient regularity.

In this paper, we propose and analyze a new finite element method which produce a completely discontinuous piecewice polynomial approximation of convection-diffision equations. This method have optimal order of convergence as classical one.

An outline of the paper is as follows. In the next sections the method is introduced for model problem; existance and uniqueness to the discrete problem is given, and optimal error estimate is obtained for a model problem.

2. Model Problem and Finite Element Approximation

Let Ω be a simply connected polygonal domain of R^2. We consider the model problem : Find u such that

$$\begin{cases} -\Delta u + \beta.\nabla u + \sigma u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

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where $\beta \in (W^{1,+\infty}(\Omega))^2$, $\sigma \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. In the sequel we make the assumption

$$\sigma - \frac{1}{2}\,\mathrm{div}\beta \geq \gamma_0 > 0.$$

Assume that we have a regular triangulations $\mathcal{T}_h$ of the domain Ω with triangular finite elements whose diameters are less or equal than h.

First, we introduce the following spaces :

$$W_h = \{v \in L^2(\Omega) : v_{|T} \in H^1(T) \text{ and } \frac{\partial u}{\partial n_T} \in L^2(\partial T), \forall T \in \mathcal{T}_h \text{ and } \ v_{|\partial T \cap \Gamma} = 0$$
$$\text{if } meas(\partial T \cap \Gamma) \neq 0\}$$

where $\dfrac{\partial u}{\partial n_T}$ is the outward normal derivative of the restriction of u to T.

$$V_h = \{v \in L^2(\Omega) : v_{|T} \in P_1(T), \forall T \in \mathcal{T}_h \ \text{ and } \ v_{|\partial T \cap \Gamma} = 0, \ \text{if } meas(\partial T \cap \Gamma) \neq 0\}.$$

Let us remark, that we have

$$V_h \subset W_h. \tag{2.2}$$

Finally, we introduce some notation that we will need in the definition and analysis of the finite element approximation of the model problem.

Let E_I be the set of all interior edges and E_T the set of edges of T. For each interior edge l we choose an arbitrary normal direction n and denote the two triangles sharing this edge T_+ and T_- where n points outwards T_+. For a boundary edge l we take n as the outward normal.

We define the jump of $v \in W_h$ on l by

$$[[v]]_l(x) = v_{|T_+}(x) - v_{|T_-}(x), \forall x \in l.$$

For all $T \in \mathcal{T}_h$, we denoted by ∂T^- and ∂T^+ the set defined by :

$$\partial T^- = \{x \in \partial T, \text{ such that } \ \beta.n(x) < 0\}$$

and

$$\partial T^+ = \{x \in \partial T, \text{ such that } \ \beta.n(x) > 0\}$$

where n is the outward normal to T. And we set, for all $v \in W_h$

$$\forall l \in E_I, v^{\mp}(x) = \lim_{\epsilon \longrightarrow 0} v(x \mp \epsilon\beta(x)), \ x \in l,$$

we use the convention $u^{\mp} = 0$ on Γ.

Let $(u, v) \in (W_h)^2 \longrightarrow B_d(u, v)$ the bilinear form defined by

$$\begin{cases} B_d(u,v) = \displaystyle\sum_{T \in \mathcal{T}_h} \int_T \nabla u \nabla v dx + \sum_{l \in E_I} 3\frac{meas(l)}{meas(T_+)} \int_l [[u]]_l [[v]]_l d\sigma \\ \qquad - \displaystyle\sum_{l \in E_I} \int_l \frac{\partial(u_{|T+})}{\partial n}[[v]]_l d\sigma - \alpha \sum_{l \in E_I} \int_l \frac{\partial(v_{|T+})}{\partial n}[[u]]_l d\sigma \end{cases}$$

where $\alpha \in \{-1,0,1\}$, and $(u,v) \in (W_h)^2 \longrightarrow B_c(u,v)$ the bilinear form defined by

$$B_c(u,v) = \sum_{T\in\mathcal{T}_h} \int_T (\beta.\nabla u + \sigma u) v dx + \int_{\partial T^-} |\beta.n|(u^+ - u^-) v^+ d\sigma.$$

The "discontinuous" Galerkin Finite Element Approximation to the solution of (2.1) is defined by :

$$\begin{cases} \text{Find } u_h \in V_h, \text{ such that} \\ B_d(u_h,v_h) + B_c(u_h,v_h) = \sum_{T\in\mathcal{T}_h} \int_T f v dx. \quad \forall v_h \in V_h \end{cases} \tag{2.3}$$

2.1 Existence, uniqueness results and error estimates

In this section, we proceed to verify certain properties of the form B which will be needed to prove existence and uniqueness results and to get error estimate. In the sequel, we denoted by C a positive generic constant not dependent of h and not necesseraily the same in any two place.

First, let us introduce the norms on $W_h \times W_h$ defined by

$$\forall v \in W_h, \quad [[v]]_d^2 = \sum_{T\in\mathcal{T}_h} |v|_{1,T}^2 + \sum_{l\in E_I} \frac{meas(l)}{meas(T_+)} \int_l [[v]]_l^2 d\sigma, \tag{2.4}$$

$$\forall v \in W_h, \quad [[v]]_c^2 = \sum_{T\in\mathcal{T}_h} \|v\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \int_{\partial T} |\beta.n|(v^+ - v^-)^2 d\sigma \tag{2.5}$$

and

$$[[v]]_h^2 = [[v]]_d^2 + [[v]]_c^2. \tag{2.6}$$

Lemma 2.1. *The bilinear form B_d satisfies*

$$\forall v_h \in V_h, \quad B_d(v_h,v_h) \geq \frac{1}{2}[[v_h]]_d^2. \tag{2.7}$$

and

$$B_d(w,v_h) \leq C\{[[w]]_d^2 + \sum_{l\in E_I} \frac{meas(T_+)}{meas(l)} \|\frac{\partial w}{\partial n}\|_{0,l}^2\}^{\frac{1}{2}} [[v_h]]_d, \quad \forall (w,v_h) \in W_h \times V_h. \tag{2.8}$$

Proof. Let $v_h \in V_h$. First, we have

$$B_d(v_h,v_h) = \sum_{T\in\mathcal{T}_h} |v_h|_{1,T}^2 + \sum_{l\in E_I} 3\frac{meas(l)}{meas(T_+)} \int_l [[v_h]]_l^2 d\sigma - (1+\alpha) \sum_{l\in E_I} \int_l \frac{\partial (v_{h|T+})}{\partial n} [[v_h]]_l d\sigma.$$

Using the inverse inequality

$$\|\frac{\partial v_h}{\partial n}\|_{0,l} \leq (\frac{meas(l)}{meas(T_+)})^{\frac{1}{2}} |v_h|_{1,T}$$

we obtain : for all $l \in E_I$

$$-(1+\alpha) \int_l \frac{\partial (v_{h|T+})}{\partial n} [[v_h]]_l d\sigma \geq -\frac{1+\alpha}{4} |v_h|_{1,T}^2 - (1+\alpha) \frac{meas(e)}{meas(T_+)} \int_l [[v_h]]_l^2 d\sigma.$$

Then we have

$$B_d(v_h,v_h) \geq (1-\frac{1+\alpha}{4})\sum_{T\in\mathcal{T}_h}|v_h|^2_{1,T} + (3-(1+\alpha))\sum_{l\in E_I}\frac{meas(l)}{meas(T_+)}\int_l [[v_h]]_l^2 d\sigma,$$

so $\alpha \in \{-1,0,1\}$, we finally obtain the inequality (2.7) :

$$B_d(v_h,v_h) \geq \frac{1}{2}[[v_h]]_d^2.$$

Let $(w,v_h)\in W_h\times V_h$. We have

$$\begin{aligned} B_d(w,v_h) &= \sum_{T\in\mathcal{T}_h}\int_T \nabla w\nabla v_h dx + \sum_{l\in E_I} 3\frac{meas(l)}{meas(T_+)}\int_l [[w]]_l[[v_h]]_l d\sigma \\ &\quad - \sum_{l\in E_I}\int_l \frac{\partial(w_{|T+})}{\partial n}[[v_h]]_l d\sigma - \alpha\sum_{l\in E_I}\int_l \frac{\partial(v_{h|T+})}{\partial n}[[w]]_l d\sigma \\ &\leq C([[w]]_d^2 + \sum_{l\in E_I}\frac{meas(T_+)}{meas(l)}\|\frac{\partial w}{\partial n}\|^2_{0,l}\}^{\frac{1}{2}}([[v_h]]_d^2 + \sum_{l\in E_I}\frac{meas(T_+)}{meas(l)}\|\frac{\partial v_h}{\partial n}\|^2_{0,l}\}^{\frac{1}{2}}. \end{aligned}$$

Using inverse inequality, we deduce that

$$B_d(w,v_h) \leq C\{[[w]]_d^2 + \sum_{l\in E_I}\frac{meas(T_+)}{meas(l)}\|\frac{\partial w}{\partial n}\|^2_{0,l}\}^{\frac{1}{2}}[[v_h]]_d.$$

Lemma 2.2. *The bilinear form B_c satisfies :*
$\forall(w,v_h)\in W_h\times V_h,$

$$B_c(w,w) \geq min(\gamma_0,\frac{1}{2})[[w]]_c^2, \tag{2.9}$$

and

$$B_c(w,v_h) \leq C[[v_h]]_c\{\|w\|^2_{0,\Omega} + \sum_{T\in\mathcal{T}_h}\int_{\partial T}|\beta.n|(w_{|T})^2 d\sigma\}^{\frac{1}{2}} - \sum_{T\in\mathcal{T}_h}\int_T \overline{\beta}\nabla v_h w dx \tag{2.10}$$

where $\overline{\beta} = \frac{1}{meas(T)}\int_T \beta dx.$

Proof. The first inequality is classical (see e.g. [3]).

For the second inequality. Let $(w,v_h)\in W_h\times V_h$. Using Green Formula, we have [3] :

$$\left\{\begin{aligned} B_c(w,v_h) = &\sum_{T\in\mathcal{T}_h}\int_T(\sigma - div\beta)wv_h dx - \int_T(\beta-\overline{\beta})\nabla v_h w dx - \int_T \overline{\beta}\nabla v_h w dx \\ &- \sum_{T\in\mathcal{T}_h}\int_{\partial T^+}|\beta.n|w^-(v_h^- - v_h^+)d\sigma \end{aligned}\right.$$

By classical result in finite elements methods and inverse inequality [1], it is easy to see that

$$\int_T(\beta-\overline{\beta})\nabla v_h w dx \leq Ch_T|\beta|_{1,\infty,T}|v_h|_{1,T}\|w\|_{0,T} \leq C\|v_h\|_{0,T}\|w\|_{0,T}.$$

Then

$$B_c(w,v_h) \leq C[[v_h]]_c\{\|w\|_{0,\Omega}^2 + \sum_{T\in\mathcal{T}_h}\int_{\partial T}|\beta.n|(w_{|T})^2 d\sigma\}^{\frac{1}{2}} - \sum_{T\in\mathcal{T}_h}\int_T \overline{\beta}\nabla v_h w dx$$

Now we are able to proove existance, uniqueness and to give a priori error estimate. More precisely, we have the following

Theorem 2.3. *The discrete problem (2.3) has unique solution $u_h \in V_h$. Moreover, if $u \in W_h$ be the solution of the model problem (2.1), there exists a constant C such that*

$$\left\{\begin{array}{l} [[u-u_h]]_h \leq C \inf\limits_{v_h\in V_h}\{(([[u-v_h]]_h^2 + \sum\limits_{l\in E_I}\frac{meas(T_+)}{meas(l)}\|\frac{\partial(u-v_h)}{\partial n}\|_{0,l}^2 \\ \qquad\qquad + \|u-v_h\|_{0,T}^2 + \int_{\partial T}|\beta.n|(u_{|T}-v_{h|T})^2 d\sigma\}^{\frac{1}{2}} . \\ \qquad\qquad + \sup\limits_{w_h\in V_h}\frac{\sum_{T\in\mathcal{T}_h}\int_T\overline{\beta}\nabla w_h(u-v_h)dx}{[[w_h]]_h}\} \end{array}\right. \qquad (2.11)$$

Proof. The first part of the Theorem follow from (2.7)-(2.9).
For the second part, remark that the solution of the model problem belong to $W_h \cap H_0^1(\Omega)$. Then using Green Formula, we have :

$$\forall v_h \in V_h,\quad B_d(u,v_h) + B_c(u,v_h) = \sum_{T\in\mathcal{T}_h}\int_T f v_h dx = B_d(u_h,v_h) + B_c(u_h,v_h). \qquad (2.12)$$

Using (2.7)-(2.9) and (2.12), we have. For all $v_h \in V_h$

$$\left\{\begin{array}{l} [[u_h-v_h]]_h \leq C \sup\limits_{w_h\in V_h}\frac{B_d(u_h-v_h,w_h)+B_c(u_h-v_h,w_h)}{[[w_h]]_h} \\ \qquad\qquad \leq C \sup\limits_{w_h\in V_h}\frac{B_d(u-v_h,w_h)+B_c(u-v_h,w_h)}{[[w_h]]_h}, \end{array}\right.$$

so by (2.8) and (2.10), we deduce that

$$\left\{\begin{array}{l} [[u_h-u_h]]_h \leq C \inf\limits_{v_h\in V_h}\{(([[u-v_h]]_h^2 + \sum\limits_{l\in E_I}\frac{meas(T_+)}{meas(l)}\|\frac{\partial(u-v_h)}{\partial n}\|_{0,l}^2 \\ \qquad\qquad + \|u-v_h\|_{0,T}^2 + \int_{\partial T}|\beta.n|(u_{|T}-v_{h|T})^2 d\sigma\}^{\frac{1}{2}} \quad . \\ \qquad\qquad + \sup\limits_{w_h\in V_h}\frac{\sum_{T\in\mathcal{T}_h}\int_T\overline{\beta}\nabla w_h(u-v_h)dx}{[[w_h]]_h}\} \end{array}\right.$$

Using triangular inequality, we obtain

$$\left\{\begin{array}{l} [[u-u_h]]_h \leq C \inf\limits_{v_h\in V_h}\{(([[u-v_h]]_h^2 + \sum\limits_{l\in E_I}\frac{meas(T_+)}{meas(l)}\|\frac{\partial(u-v_h)}{\partial n}\|_{0,l}^2 \\ \qquad\qquad + \|u-v_h\|_{0,T}^2 + \int_{\partial T}|\beta.n|(u_{|T}-v_{h|T})^2 d\sigma\}^{\frac{1}{2}} \\ \qquad\qquad + \sup\limits_{w_h\in V_h}\frac{\sum_{T\in\mathcal{T}_h}\int_T\overline{\beta}\nabla w_h(u-v_h)dx}{[[w_h]]_h}\} \end{array}\right.$$

The final Theorem of this section, is now straighforward consequence of estimate (2.11).

Theorem 2.4. *Let $u \in W_h$ be the solution of model problem (2.1). If $u \in H^\sigma(\Omega)$, with $3/2 < \sigma \leq 2$, then there exists a constant C independent of h such that :*

$$[[u - u_h]]_h \leq C h^{\sigma - 1} |u|_{\sigma,\Omega}.$$

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