Journal of Computational Mathematics, Vol.17, No.2, 1999, 171–178.

GENERALIZED GAUSSIAN QUADRATURE FORMULAS WITH CHEBYSHEV NODES*[1)]

Ying-guang Shi
(*ICMSEC, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

Explicit expressions of the Cotes numbers of the generalized Gaussian quadrature formulas for the Chebyshev nodes (of the first kind and the second kind) and their asymptotic behavior are given.

Key words: Quadrature formula, Chebyshev polynomials.

1. Introduction

This paper deals with the generalized Gaussian quadrature formulas for Chebyshev nodes (cf. [2]).

Throughout the paper we assume that m and n are positive integers. As usually, $T_n(x)$ and $U_n(x)$ denote the n-th Chebyshev polynomials of the first kind and the second kind, respectively. Among generalized Gaussian quadrature formulas one of the most important cases is the weight

$$w_m(x) := (1-x^2)^{[(m+1)/2]-(m+1)/2}, \tag{1.1}$$

where $[r]$ denotes the largest integer $\le r$. In [5] we pointed out that if we take as nodes of a quadrature formula the zeros of $(1-x^2)U_{n-1}(x)$ (here we replace $n+1$ by n for convenience)

$$x_{kn} = \cos\frac{k\pi}{n}, \quad k = 0, 1, ..., n, \tag{1.2}$$

then the quadrature formula with certain numbers $c_{ikm} := c_{ikmn}$ (called Cotes numbers of higher order)

$$\int_{-1}^{1} f(x)\sigma_m(x)w_m(x)dx = \sum_{k=0}^{n}\sum_{i=0}^{m_k} c_{ikm} f^{(i)}(x_k) \tag{1.3}$$

is exact for all $f \in \boldsymbol{P}_{mn+[(-1)^m-3]/2}$, where

$$\sigma_m(x) := \operatorname{sgn} U_{n-1}(x)^m \tag{1.4}$$

* Received April 2, 1996.
[1)] The Project Supported by National Natural Science Foundation of China.

and

$$m_k := [n_k(m-2)], \quad n_k := \begin{cases} 1, & 1 \le k \le n-1, \\ \frac{1}{2}, & k = 0, n. \end{cases} \tag{1.5}$$

As it turns out, the most interesting property of this quadrature formula is that its nodes do not depend on the index m. In [5] we found the explicit formulas for c_{ikmn} and their asymptotic behavoiur as $n \to \infty$, which provided an answer to an analogue of Problem 26 of Turán [6, p. 47]. To state this results, which will be used later, we put:

$$\Delta_m(x) = (1-x^2)^{[m/2]} U_{n-1}(x)^m, \tag{1.6}$$

$$d_{km} = \Delta_m^{([n_k m])}(x_k) = \begin{cases} m!(1-x_k^2)^{[m/2]} U'_{n-1}(x_k)^m, & 1 \le k \le n-1, \\ (-2)^{[m/2]}\left(\left[\frac{m}{2}\right]\right)! U_{n-1}(1)^m, & k = 0, \\ 2^{[m/2]}\left(\left[\frac{m}{2}\right]\right)! U_{n-1}(-1)^m, & k = n, \end{cases} \tag{1.7}$$

$$L_{km}(x) = \frac{([n_k m])! \Delta_m(x)}{d_{km}(x-x_k)^{[n_k m]}}, \quad k = 0, 1, ..., n, \tag{1.8}$$

$$b_{ikm} = \frac{1}{i!}\left[L_{km}(x)^{-1}\right]^{(i)}_{x=x_k}, \quad i = 0, 1, ...; \quad k = 0, 1, ..., n, \tag{1.9}$$

$$B_{ikm} = \frac{1}{i!}\left\{\sum_{\nu \in \{0,n\} \setminus \{k\}} [2(x_\nu - x) L_{km}(x)]^{-1}\right\}^{(i)}_{x=x_k}, \quad i = 0, 1, ...; \quad k = 0, 1, ..., n, \tag{1.10}$$

$$s_m = \begin{cases} 2, & \text{if } m \text{ is odd}, \\ \pi, & \text{if } m \text{ is even}. \end{cases} \tag{1.11}$$

Then we have (cf. [5]; for $m = 4$ the results of the theorem can be found in [7]).

Theorem A. *Let* (1.2) *be given. Then for each* $k, 0 \le k \le n$, *and for each* i, $0 \le i \le m_k$,

$$\begin{cases} c_{m_k,k,m} = \dfrac{n_k s_m (m-2)!}{d_{k,m-2}[(m-2)!!]^2 n}, \quad m \ge 2, \\ c_{m_k+1,k,m} = 0, \end{cases} \tag{1.12}$$

$$c_{ikm} = c_{i,k,m-2} + \frac{m_k! c_{m_k,k,m}}{i! n_k (m-2)}\{(i + n_k(m-2) - m_k) b_{m_k-i,k,m-2} - \frac{1}{2}[1 + (-1)^{m+1}] B_{m_k-i-1,k,m-2}\}, \quad m \ge 3. \tag{1.13}$$

Moreover,

$$|c_{ikmn}| \le \begin{cases} \dfrac{(1-x_{kn}^2)^{[(m-1)/2]-[(m-1-i)/2]}}{n^{m+1-2[(m-i)/2]}}, & 1 \le k \le n-1, \\ \dfrac{1}{n^{m-2[m/2]+2i+1}}, & k = 0, n, \end{cases} \tag{1.14}$$

$$c_{ikmn} \sim \frac{(1-x_{kn}^2)^{i/2}}{n^{i+1}}, \quad m = even; \quad i = 0, 2, ..., m-2; \quad 1 \le k \le n-1. \tag{1.15}$$

Following Kronrod [3], the object of this paper is to extend the fomula (1.3) to the following formula

$$\int_{-1}^{1} f(x)\sigma_m(x)w_m(x)dx = \sum_{k=0}^{n}\sum_{i=0}^{m_k} C_{ikm} f^{(i)}(x_k) + \sum_{k=1}^{n} D_{km} f(y_k), \tag{1.16}$$

where

$$y_{kn} := \cos\frac{(2k-1)\pi}{2n}, \quad k = 1, 2, ..., n \tag{1.17}$$

are the zeros of $T_n(x)$. We will see that this quadrature formula maintains the above property: its nodes (1.2) and (1.17) do not depend on the index m.

The main result of this paper is

Theorem 1. *Let* (1.2) *and* (1.17) *be given. Then the quadrature formula* (1.16) *is exact for all* $f \in \boldsymbol{P}_{(m+2)n+[(-1)^m-3]/2}$. *Here*

$$C_{ikmn} = \frac{m-1}{m} C_{i,k,m-2,n} + \frac{1}{m} c_{ikmn}, \quad k = 0, 1, ..., n; \quad i = 0, 1, ..., m_k, \tag{1.18}$$

$$D_{kmn} = \frac{(-1)^{m(k-1)} s_m (m!)(1-y_{kn}^2)^{(m/2)-[m/2]}}{(m!!)^2 n}, \quad k = 1, ..., n. \tag{1.19}$$

Moreover,

$$|C_{ikmn}| \le \begin{cases} \dfrac{(1-x_{kn}^2)^{[(m-1)/2]-[(m-1-i)/2]}}{n^{m+1-2[(m-i)/2]}}, & 1 \le k \le n-1, \\ \dfrac{1}{n^{m-2[m/2]+2i+1}}, & k = 0, n, \end{cases} \tag{1.20}$$

$$C_{ikmn} \sim \frac{(1-x_{kn}^2)^{i/2}}{n^{i+1}}, \quad m = even; \quad i = 0, 2, ..., m-2; \quad 1 \le k \le n-1. \tag{1.21}$$

The second main result of this paper gives an extremal property of the quadrature formula (1.16).

Theorem 2. *Let* (1.2) *and* (1.17) *be given. Then*

$$\int_{-1}^{1} \left|(1-x^2)^{(m-1)/2} \prod_{k=1}^{n-1}(x-x_k)^m \prod_{k=1}^{n}(x-y_k)^2\right| dx$$

$$= \min_{\substack{-1<\xi_{n-1}<...<\xi_1<1 \\ -1<\eta_n<...<\eta_1<1}} \int_{-1}^{1} \Big|(1-x^2)^{(m-1)/2}$$

$$\cdot \prod_{k=1}^{n-1}(x-\xi_k)^m \prod_{k=1}^{n}(x-\eta_k)^2\Big| dx. \tag{1.22}$$

Moreover, this solution is unique.

The following lemma plays an crucial role.

Lemma. *Let m and n be positive integers. Then*

$$\int_{-1}^{1} q(x)T_n(x)|U_{n-1}(x)|^{m-1}[sgn\ U_{n-1}(x)](1-x^2)^{(m-1)/2}dx = 0,$$
$$\forall q \in \boldsymbol{P}_{2n-2}. \tag{1.23}$$

The proofs are putted in Section 2 and other results in Section 3. In what follows we agree $c_{ikmn} = C_{ikmn} = 0$ if $i > m_k$.

2. Proofs

2.1. Proof of Lemma

It is easy to see that (1.23) is equivalent to

$$\int_{-1}^{1} U_{k-1}(x)|T_n(x)U_{n-1}(x)^{m-1}|[\text{sgn } (T_n(x)U_{n-1}(x))](1-x^2)^{(m-1)/2}dx = 0,$$
$$k = 1, 2, ..., 2n-1. \tag{2.1}$$

By making the change of variable $x = \cos t$ and integrating over the interval twice, (2.1) becomes

$$\int_{-\pi}^{\pi} \sin kt|(\cos nt)(\sin nt)^{m-1}|\text{sgn } \sin 2nt dt = 0,$$
$$k = 1, 2, ..., 2n-1. \tag{2.2}$$

Since $\sin kt$ is a liear combination of the functions $e^{\pm ikt}$, it will be enough to establish

$$I := \int_{-\pi}^{\pi} e^{ikt}|(\cos nt)(\sin nt)^{m-1}|\text{sgn } \sin 2nt dt = 0,$$
$$k = \pm 1, \pm 2, ..., \pm(2n-1). \tag{2.3}$$

Remembering the periodicity of the functions, by making the change of variable $t = \theta + \pi/n$ we see

$$I = \int_{-\pi}^{\pi} e^{ik(\theta+\pi/n)}|\cos(n\theta+\pi)\sin^{m-1}(n\theta+\pi)|\text{sgn } \sin(2n\theta+2\pi)d\theta$$
$$= e^{ik\pi/n} I.$$

Clearly, $e^{ik\pi/n} \neq 1$, which means $I = 0$. $\square$

2.2. Proof of Theorem 1

According to the Lemma using a well known method for constructing quadrature formulas via interpolation the formula (1.16) with certain constants C_{ikm} and D_{km} must hold for all $f \in \boldsymbol{P}_{(m+2)n+[(-1)^m-3]/2}$.

To compute D_{km} we start with the formula

$$D_{km} = \int_{-1}^{1} \frac{T_n(x)(1-x^2)^{[m/2]}U_{n-1}(x)^{m-1}}{T_n'(y_k)(x-y_k)(1-y_k^2)^{[m/2]}U_{n-1}(y_k)^{m-1}} \sigma_m(x) w_m(x) dx \qquad (2.4)$$

which follows from (1.16). We expand $T_n(x)/(x-y_k)$ as

$$\frac{T_n(x)}{x-y_k} = U_{n-1}(x) + \sum_{j=0}^{n-2} a_j U_j(x) \qquad (2.5)$$

with certain constants a_j. It follows from (1.3) that

$$\int_{-1}^{1} U_j(x)(1-x^2)^{[m/2]}U_{n-1}(x)^{m-1}\sigma_m(x)w_m(x)dx = 0, \quad j \leq n-2.$$

Hence we obtain a simpler formula

$$D_{km} = \frac{1}{T_n'(y_k)(1-y_k^2)^{[m/2]}U_{n-1}(y_k)^{m-1}} \int_{-1}^{1} (1-x^2)^{(m-1)/2}|U_{n-1}(x)|^m dx. \qquad (2.6)$$

It is well known that

$$T_n'(y_k) = \frac{(-1)^{k-1}n}{(1-y_k^2)^{1/2}}, \quad k = 1, 2, ..., n \qquad (2.7)$$

and

$$U_{n-1}(y_k) = \frac{(-1)^{k-1}}{(1-y_k^2)^{1/2}}, \quad k = 1, 2, ..., n. \qquad (2.8)$$

We also know [5, (2.8)]

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^{(m-1)/2}|U_{n-1}(x)|^m dx &= \int_0^{\pi} |\sin n\theta|^m d\theta \\ &= \int_0^{\pi} |\sin\theta|^m d\theta = \frac{s_m(m!)}{(m!!)^2}. \end{aligned} \qquad (2.9)$$

The substitution of these values gives (1.19).

Here from (1.19) we derive an interesting formula

$$D_{jm} = \frac{m-1}{m} D_{j,m-2}, \quad j = 0, 1, ..., n \quad (m \geq 3). \qquad (2.10)$$

To determine C_{ikm} first we assume $m \geq 3$ and consider the function

$$F \in \boldsymbol{P}_{(m-1)n+[(-1)^m-1]/2}$$

which satisfies the interpolatory conditions

$$F^{(\mu)}(x_\nu) = \delta_{i\mu}\delta_{k\nu}, \quad \nu = 0, 1, ..., n; \quad \mu = 0, 1, ..., m_\nu. \tag{2.11}$$

From (1.3) it follows that

$$c_{ikm} = \int_{-1}^{1} F(x)\sigma_m(x)w_m(x)dx. \tag{2.12}$$

On the other hand, first applying (1.16) to F and using (2.12) yields

$$c_{ikm} = C_{ikm} + \sum_{j=1}^{n} D_{jm}F(y_j). \tag{2.13}$$

Next, substituting F into (1.16) in replacing m by $m-2$ and using (2.12) gives

$$c_{ikm} = C_{i,k,m-2} + \sum_{j=1}^{n} D_{j,m-2}F(y_j). \tag{2.14}$$

Substituting (2.10) into (2.13) and cancelling the term

$$\sum_{j=1}^{n} D_{j,m-2}F(y_j)$$

from the two equations (2.13) and (2.14) we obtain the recurrence relations (1.18) with respect to the index m. Although, these relations are derived under the assumption $m \geq 3$, they remain true for $m \leq 2$ (remembering the agreement in the end of Section 1). In fact, we can check directly. First we easily see $C_{0k1} = c_{0k1} = 0$. Next, by (1.12) we have $c_{0k2} = n_k\pi/n$. On the other hand, for

$$\Omega_{n+1}(x) = (1-x^2)U_{n-1}(x)$$

using [1, 22.7.25, p. 782] and [4, (3.6) and (3.7)] we obtain

$$\begin{aligned} C_{0k2} &= \int_{-1}^{1} \frac{T_n(x)\Omega_{n+1}(x)}{T_n(x_k)\Omega'_{n+1}(x_k)(x-x_k)} \frac{dx}{\sqrt{1-x^2}} \\ &= \int_{-1}^{1} \frac{T_n(x)[T_{n+1}(x) - T_{n-1}(x)]}{T_n(x_k)[T'_{n+1}(x_k) - T'_{n-1}(x_k)](x-x_k)} \frac{dx}{\sqrt{1-x^2}} = \frac{n_k\pi}{2n}. \end{aligned} \tag{2.15}$$

This means that (1.18) remains true for $m \leq 2$.

Finnally, (1.20) and (1.21) directly follows from (1.18), (1.14), and (1.15). □

2.3. Proof of Theorem 2

This is an immediate consequence of Theorem 1 by Theorems 4 and 5 in [2]. □

3. Other Results

Theorem 2 admits other generalized Gaussian quadrature formulas. We briefly discuss one of them.

If we rewrite (1.22) as

$$\begin{aligned}\int_{-1}^{1}\left|\prod_{k=1}^{n-1}(x-x_k)^m\prod_{k=1}^{n}(x-y_k)^2\right|(1-x^2)^{(m-1)/2}dx \\ = \min_{\substack{-1<\xi_{n-1}<\ldots<\xi_1<1\\ -1<\eta_n<\ldots<\eta_1<1}}\int_{-1}^{1}\left|\prod_{k=1}^{n-1}(x-\xi_k)^m\prod_{k=1}^{n}(x-\eta_k)^2\right|(1-x^2)^{(m-1)/2}dx,\end{aligned} \tag{3.1}$$

we find that the quadrature formula

$$\int_{-1}^{1} f(x)\sigma_m(x)(1-x^2)^{(m-1)/2}dx = \sum_{k=1}^{n-1}\sum_{i=0}^{m-2} C^*_{ikm}f^{(i)}(x_k) + \sum_{k=1}^{n} D^*_{km}f(y_k) \tag{3.2}$$

is exact for all $f \in \boldsymbol{P}_{(m+2)n-m-1}$.

Determination of D^*_{km} in the same way first leads to a formula in different structure. From (3.2) we obtain

$$D^*_{km} = \int_{-1}^{1}\frac{T_n(x)U_{n-1}(x)^{m-1}}{T'_n(y_k)(x-y_k)U_{n-1}(y_k)^{m-1}}\sigma_m(x)(1-x^2)^{(m-1)/2}dx. \tag{3.3}$$

Comparing this with (2.4) we have

$$D^*_{km} = (1-y_k^2)^{[m/2]}D_{km}.$$

To determine C^*_{ikm} we should consider the function $G \in \boldsymbol{P}_{m(n-1)}$ $(m \geq 2)$ satisfying

$$\begin{cases} G^{(\mu)}(x_\nu) = \delta_{i\mu}\delta_{k\nu}, & \mu = 0,1,\ldots,m-2; \quad \nu = 1,2,\ldots,n-1 \\ G(y_\nu) = 0, & \nu = 1,2,\ldots,n. \end{cases}$$

Then by (3.2) we get

$$\begin{aligned} C^*_{ikm} &= \int_{-1}^{1} G(x)\sigma_m(x)(1-x^2)^{(m-1)/2}dx \\ &= \int_{-1}^{1}(1-x^2)^{[m/2]}G(x)\sigma_m(x)w_m(x)dx. \end{aligned}$$

Using (1.3) and applying the Newton-Leibniz formula the above relation leads to

$$C^*_{ikm} = \sum_{j=i}^{m-2}\binom{j}{i}\left\{(1-x^2)^{[m/2]}\right\}^{(j-i)}_{x=x_k} c_{jkm}. \quad \square$$

References

[1] Abramowitz M., Stegun I.A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (National Bureau of Standards, Applied Mathematics 55, Washington, D. C., 1965).

[2] Bojanov B.D., Braess D., Dyn N., Generalized Gaussian quadrature formulas, *J. Approx. Theory*, **46** (1986), 335–353.

[3] Gautschi W., Gauss-Kronrod quadrature–a survey, Numerical Methods and Approximation Theory, III, (Niš, 1987), (G. V. Milovanović, ed.), Faculty of Electronic Engineering, *Univ. Niš, Niš,* (1988), 39–66.

[4] Li S., Kronrod extension of Turán formula, *Studia Sci. Math. Hungar.*, **29** (1994), 71–83.

[5] Shi Y.G., An analogue of Problem 26 of P. Turán, *Bull. Austral. Math. Soc.*, **53** (1996), 1–12.

[6] Turán P., On some open problems of approximation theory, *J. Approx. Theory*, **29** (1980), 23–85.

[7] Varma A.K., On optimal quadrature fomulae, *Studia Sci. Math. Hungar.*, **19** (1984), 437–446.