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HIGH RESOLUTION SCB SCHEME FOR HYPERBOLIC SYSTEMS OF 2-D CONSERVATION LAWS*1)

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Abstract

In this paper, a new class of high resolution schemes satisfying the "condition A" (SCA) and the "condition B" (SCB) for hyperbolic systems of conservation laws in one and two dimensions are constructed. Moreover, the results of the numerical experiments by using these schemes are given for the system of Euler equations in one and two dimensions.

Key words: SCB scheme, hyperbolic system, conservation law

1. Introduction

Consider numerical solutions of the initial value problem for hyperbolic conservation laws in one dimension

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0 \tag{1.1a}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \tag{1.1b}$$

where $\mathbf{u} = (u_1, u_2, \cdots, u_m)^T$ and $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}), \cdots, f_m(\mathbf{u}))^T$.

And conservation laws in two dimensions

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} + \frac{\partial \mathbf{g}(\mathbf{u})}{\partial y} = 0 \tag{1.2a}$$

$$\mathbf{u}(x,y,0) = \mathbf{u}_0(x,y) \tag{1.2b}$$

where $\mathbf{u} = (u_1, u_2, \cdots, u_m)^T$, $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}), \cdots, f_m(\mathbf{u}))^T$, and $\mathbf{g}(\mathbf{u}) = (g_1(\mathbf{u}), g_2(\mathbf{u}), \cdots, g_m(\mathbf{u}))^T$.

For the scalar conservation laws in one dimension, the TVD concept by A. Harten[3] is widely accepted to design the numerical schemes for theoretical purposes and practical applications. The total variation of a grid function $\{u_j\}$ denoted by $TV(u)$ is defined as

$$TV(u) = \sum_j \mid u_{j+1} - u_j \mid . \tag{1.3}$$

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Let $BV(R)$ be the space of functions with bounded variation. A difference scheme

$$L(u^{n+1}) = R(u^n),$$

is called as TVD scheme, if for any $u^n \in BV(R)$

$$TV(u^{n+1}) \leq TV(u). \tag{1.4}$$

Encouraged by the success of the TVD schemes in $1-D$, one wants to extend the TVD schemes for two dimensions. The total variation of a grid function $\{u_{j,k}\}$ denoted by $TV(u)$ is defined as

$$TV(u) = \sum_{j,k}[\Delta y \mid u_{j+1,k} - u_{j,k} \mid + \Delta x \mid u_{j,k+1} - u_{j,k} \mid]. \tag{1.5}$$

Same as the case in $1 - D$, a difference scheme is called as TVD scheme, if for any $u^n \in BV(R^2)$

$$TV(u^{n+1}) \leq TV(u^n). \tag{1.6}$$

Unfortunately, any conservative TVD scheme for solving scalar conservation laws in two dimensions is at most first order accurate[2]. Hence, it may be worthy of creating new conception beyond TVD in two dimensions. In [7], we have developed a new kind of total variation stability criteria, so-called the "condition A" and the "condition B", and discussed the relationship between them and the TVD conditions. In this paper, we construct a class of high resolution schemes satisfying the "condition A" (SCA) and the "condition B" (SCB) for hyperbolic systems of conservation laws in one dimension and two dimensions. Lastly, some numerical results for the system of Euler equations are given in one and two dimensions.

The organization of this paper is as follows. In section 2, we briefly review the theoretical results in [7]. In section 3 and 4, we construct second order accurate SCA and SCB schemes in one and two dimensions to hyperbolic systems of conservation laws respectively. In section 5, we give the numerical results for the system of Euler equations in one and two dimensions.

2. The "Condition A" and the "Condition B"

In this section, we review the theory developed in [7] for scalar conservation laws ($m = 1$ in (1.1) and (1.2)).

First, let us discuss the conservative schemes in one dimension. Partition the space $[0, T] \times [-\infty, \infty]$. Let Δt and Δx be the temporal and spatial step lengths, respectively. Denote the numerical solution at the point (x_j, t^n) by u_j^n. $x_j = j\Delta x$, $t^n = n\Delta t$ $(j = 0, \pm 1, \pm 2, \cdots, n = 0, 1, 2, \cdots), \lambda = \Delta t/\Delta x$.

A conservative scheme can be written as following:

$$u_j^{n+1} = u_j^n - \lambda(f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}) \tag{2.1}$$

$$f_{j+\frac{1}{2}} = h(u^n_{j-k+1}, \cdots, u^n_{j+k}) \qquad \text{and} \qquad h(u, \cdots, u) = f(u).$$

We say that a scheme satisfies the "condition A" (SCA), if the following inequality holds for every integer n

$$\sum_j \left(\left|u^{n+1}_j - \frac{1}{2}(u^n_{j+1} + u^n_j)\right| + \left|u^{n+1}_j - \frac{1}{2}(u^n_j + u^n_{j-1})\right|\right) \le \sum_j \mid u^n_{j+1} - u^n_j \mid \tag{2.2}$$

For the relationship between the "condition A" and TVD conditions, we have

Theorem 2.1. *A scheme is TVD if it satisfies the "condition A".*

For the schemes of incremental form

$$u^{n+1}_j = u^n_j + C_{j+\frac{1}{2}}(u^n_{j+1} - u^n_j) - D_{j-\frac{1}{2}}(u^n_j - u^n_{j-1}), \tag{2.3}$$

similar to the TVD property[3], we have

Theorem 2.2. *If the schemes* (2.3) *satisfy that*

$$0 \le C_{j+\frac{1}{2}},\ D_{j+\frac{1}{2}} \le \frac{1}{2} \tag{2.4}$$

then, the schemes satisfy the "condition A" (SCA).

Obviously, the condition A is stricter than the TVD condition. The purpose that we introduce the concept of the condition A is to obtain a condition which the second order schemes in two dimensions should be satisfied based on the condition A. The condition B, defined as follows, does be the condition we want to obtain in two dimensions.

Now, let us discuss the conservative schemes in two dimensions.

Partition the space $R^+ \times R^2$. Let Δt be the temporal step length, and Δx, Δy the spatial step lengths in x, y-direction, respectively. Denote the numerical solution at the point (x_j, y_k, t^n) as $u^n_{j,k}$. $x_j = j\Delta x$, $y_k = k\Delta y$, $t^n = n\Delta t$ $(j, k = 0, \pm1, \pm2, \cdots,$ $n = 0, 1, 2, \cdots)$ $\lambda = \Delta t/\Delta x$, $\mu = \Delta t/\Delta y$. So, conservative schemes in two dimension can be written as

$$u^{n+1}_{j,k} = u^n_{j,k} - \lambda(f_{j+\frac{1}{2},k} - f_{j-\frac{1}{2},k}) - \mu(g_{j,k+\frac{1}{2}} - g_{j,k-\frac{1}{2}}) \tag{2.5}$$

where $f_{j+\frac{1}{2},k} = h^f(u^n_{j-l+1,k-s}, \cdots, u^n_{j+l,k+s})$, $g_{j,k+\frac{1}{2}} = h^g(u^n_{j-l,k-s+1}, \cdots, u^n_{j+l,k+s})$ and $h^f(u, \cdots, u) = f(u)$, $h^g(u, \cdots, u) = g(u)$.

Similar to the case in one dimension, we say that a numerical scheme satisfies the "condition A" in two dimensions if

$$\begin{aligned}\frac{1}{2}(\Delta x + \Delta y) \sum_{j,k} \Big[&\left|u^{n+1}_{j,k} - \frac{1}{4}(u^n_{j,k} + u^n_{j+1,k} + u^n_{j,k+1} + u^n_{j+1,k+1})\right| \\ &+ \left|u^{n+1}_{j,k} - \frac{1}{4}(u^n_{j,k} + u^n_{j-1,k} + u^n_{j,k+1} + u^n_{j-1,k+1})\right| \\ &+ \left|u^{n+1}_{j,k} - \frac{1}{4}(u^n_{j,k} + u^n_{j+1,k} + u^n_{j,k-1} + u^n_{j+1,k-1})\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|u_{j,k}^{n+1}-\frac{1}{4}(u_{j,k}^n+u_{j-1,k}^n+u_{j,k-1}^n+u_{j-1,k-1}^n)\right|\Big] \\
\leq &\sum_{j,k}(\Delta y \mid u_{j,k}^n-u_{j-1,k}^n \mid +\Delta x \mid u_{j,k}^n-u_{j,k-1}^n \mid)
\end{aligned} \tag{2.6}
$$

But, what can be obtained is that

$$
\sum_j \Big[\left|u_j^{n+1}-\frac{1}{2}(u_{j+1}^n-u_j^n)\right|+\left|u_j^{n+1}-\frac{1}{2}(u_j^n+u_{j-1}^n)\right|\Big] \leq \frac{1}{2}\sum_j \mid u_{j+1}^n-u_j^n \mid . \tag{2.7}
$$

when the "condition A" in two dimensions degenerates into one dimension and $\Delta x = \Delta y$. Obviously, the condition (2.7) is too constrict to construct high resolution schemes, and conflicts with the "condition A" in one dimension. So, we define another stability criteria, the "condition B", in two dimensions as follows.

We say that a scheme satisfies the "condition B" (SCB), if it satisfies the following inequality:

$$
\begin{aligned}
&\sum_{j,k}\Big[\left|u_{j,k}^{n+1}-\frac{1}{4}(u_{j,k}^n+u_{j+1,k}^n+u_{j,k+1}^n+u_{j+1,k+1}^n)\right| \\
&\quad+\left|u_{j,k}^{n+1}-\frac{1}{4}u_{j,k}^n+u_{j-1,k}^n+u_{j,k+1}^n+u_{j-1,k+1}^n)\right| \\
&\quad+\left|u_{j,k}^{n+1}-\frac{1}{4}(u_{j,k}^n+u_{j,k-1}^n+u_{j+1,k}^n+u_{j+1,k-1}^n)\right| \\
&\quad+\left|u_{j,k}^{n+1}-\frac{1}{4}(u_{j,k}^n+u_{j-1,k}^n+u_{j,k-1}^n+u_{j-1,k-1}^n)\right|\Big] \\
\leq &2\sum_{j,k}[\mid u_{j+1,k}^n-u_{j,k}^n \mid + \mid u_{j,k+1}^n-u_{j,k}^n \mid]
\end{aligned} \tag{2.8}
$$

It is easy to see that a scheme satisfies the "condition B" in two dimensions, then it also satisfies the "condition A" in one dimension, of course is TVD, when it degenerates into one dimension. It is well known that TVD schemes in two dimensions must be the first order accuracy[2]. However, SCB schemes present can be second order accurate. The second order accurate SCB schemes will be obtained in the section 4.

For the schemes of incremental form in two dimensions:

$$
\begin{aligned}
u_{j,k}^{n+1} = &u_{j,k}^n + C_{j+\frac{1}{2},k}(u_{j+1,k}^n-u_{j,k}^n) - D_{j-\frac{1}{2},k}(u_{j,k}^n-u_{j-1,k}^n) \\
&+ C_{j,k+\frac{1}{2}}(u_{j,k+1}^n-u_{j,k}^n) - D_{j,k-\frac{1}{2}}(u_{j,k}^n-u_{j,k-1}^n)
\end{aligned} \tag{2.9}
$$

similar to the Theorem 2.2, we have

Theorem 2.3. *If the coefficients of the schemes* (2.9) *satisfy that*

$$
0 \leq C_{j+\frac{1}{2},k}, D_{j+\frac{1}{2},k}, C_{j,k+\frac{1}{2}}, D_{j,k+\frac{1}{2}} \leq \frac{1}{4} \tag{2.10}
$$

then the schemes satisfy the "condition B" (SCB).

3. Nonlinear Conservation Laws in One Dimension

Firstly, let us consider scalar hyperbolic conservation laws in one dimension:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \ u(x,0) = u_0(x) \tag{3.1}$$

Approximating (3.1), we use the second order accurate conservative scheme

$$u_j^{n+1} = u_j^n - \lambda(f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}) \tag{3.2}$$

$$f_{j+\frac{1}{2}} = \tilde{f}_{j+\frac{1}{2}} + \Phi_{j+\frac{1}{2}} \tag{3.3}$$

where $\tilde{f}_{j+\frac{1}{2}}$ is the numerical flux of the first order accurate scheme. $\Phi_{j+\frac{1}{2}}$ is the antidiffussion term. In general, (the notations are conventional, $f_j = f(u_j^n)$)

$$\tilde{f}_{j+\frac{1}{2}} = \frac{1}{2}(f_j + f_{j+1} - Q(a_{j+\frac{1}{2}})\Delta_{j+\frac{1}{2}}u) \tag{3.4}$$

where

$$Q(x) = \begin{cases} | x |, & | x | > \varepsilon \\ \frac{1}{2\varepsilon}(| x |^2 + \varepsilon^2), & | x | \le \varepsilon. \end{cases}$$

and

$$a_{j+\frac{1}{2}} = \begin{cases} \frac{f_{j+1} - f_j}{u_{j+1} - u_j}, & u_{j+1} \neq u_j \\ f'(u_j), & u_{j+1} = u_j \end{cases}$$

In the smooth region of the solution of (3.1), $\Phi_{j+\frac{1}{2}}$ should satisfies

$$\Phi_{j+\frac{1}{2}} = \frac{1}{2}[Q(a_{j+\frac{1}{2}}) - \lambda a^2_{j+\frac{1}{2}}]\Delta_{j+\frac{1}{2}}u + O(\Delta x^2) \tag{3.5}$$

such that the scheme (3.2) is second order accurate.

As usual, we take

$$\Phi_{j+\frac{1}{2}} = \frac{1}{2}[Q(a_{j+\frac{1}{2}}) - \lambda a^2_{j+\frac{1}{2}}]d_{j+\frac{1}{2}} \tag{3.6}$$

where $d_{j+\frac{1}{2}}$ is

$$d_{j+\frac{1}{2}} = d(\Delta_{j-\frac{1}{2}}u, \Delta_{j+\frac{1}{2}}u, \Delta_{j+\frac{3}{2}}u) \tag{3.7}$$

For the scheme defined above, we have

Theorem 3.1. *If the function d satisfies*

$$d(\alpha_1, \alpha_2, \alpha_3) = r_k\alpha_k, \ (k = 1, 2, 3), \ 0 \le r_k \le \theta \le 2$$

then, the scheme (3.2) *satisfies the "condition A" (SCA) if the inequality*

$$\lambda \max_j Q(a_{j+\frac{1}{2}}) \le \frac{1}{2+\theta} \tag{3.8}$$

holds.

Proof. We rewrite the scheme (3.2) into incremental form

$$u_j^{n+1} = u_j + C_{j+\frac{1}{2}} \Delta_{j+\frac{1}{2}} u - D_{j-\frac{1}{2}} \Delta_{j-\frac{1}{2}} u$$

where

$$C_{j+\frac{1}{2}} = \frac{\lambda}{2} \Big[(Q(a_{j+\frac{1}{2}}) - a_{j+\frac{1}{2}}) + (Q(a_{j-\frac{1}{2}}) - \lambda a^2_{j-\frac{1}{2}}) \frac{d_{j-\frac{1}{2}}}{\Delta_{j+\frac{1}{2}} u} \Big]$$

$$D_{j+\frac{1}{2}} = \frac{\lambda}{2} \Big[(Q(a_{j+\frac{1}{2}}) + a_{j+\frac{1}{2}}) + (Q(a_{j+\frac{3}{2}}) - \lambda a^2_{j+\frac{3}{2}}) \frac{d_{j+\frac{3}{2}}}{\Delta_{j+\frac{1}{2}} u} \Big]$$

According to the definition of the function $Q(x)$, we have

$$C_{j+\frac{1}{2}} \geq 0, \;\; D_{j+\frac{1}{2}} \geq 0.$$

Moreover,

$$C_{j+\frac{1}{2}} \leq \frac{\lambda}{2} [2Q(a_{j+\frac{1}{2}}) + \theta Q(a_{j-\frac{1}{2}})],$$

therefore, $C_{j+\frac{1}{2}} \leq \frac{1}{2}$ if

$$[2+\theta] \frac{\lambda}{2} \max_j Q(a_{j+\frac{1}{2}}) \leq \frac{1}{2}$$

i.e.

$$\lambda \max_j Q(a_{j+\frac{1}{2}}) \leq \frac{1}{2+\theta}.$$

So, $0 \leq C_{j+\frac{1}{2}} \leq \frac{1}{2}$ if the inequality (5.8) holds.

Similarily, we have $0 \leq D_{j+\frac{1}{2}} \leq \frac{1}{2}$ if (3.8) holds. The proof is completed.

Now, we generalize the high resolution SCA schemes to hyperbolic systems of conservation laws.

For the system of equations (1.1), the Jacobian matrix of the functions $\mathbf{f}$

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}. \tag{3.9}$$

$\lambda_1, \lambda_2, \cdots, \lambda_m$ denote the eigenvalues of $\mathbf{A}$. $r_1, r_2, \cdots, r_m$ and $l_1, l_2, \cdots, l_m$ denote right and left eigenvectors of the matrix $\mathbf{A}$, respectively. So,

$$\mathbf{A} = \mathbf{R \Lambda L} \tag{3.10}$$

where

$$\mathbf{R} = (r_1, \cdots, r_m), \;\; \mathbf{\Lambda} = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_m) \text{ and } \mathbf{L} = (l_1, \cdots, l_m)^T. \tag{3.11}$$

In the scheme (3.3), the difference of the function u is replaced to the system (1.1) by

$$(\alpha_k)_{j+\frac{1}{2}} = (l_k)^T_{j+\frac{1}{2}}\Delta_{j+\frac{1}{2}}\mathbf{u} \text{ and } \alpha_{j+\frac{1}{2}} = ((\alpha_1)_{j+\frac{1}{2}}, \cdots, (\alpha_m)_{j+\frac{1}{2}})^T$$

So, the numerical flux can be generalized into

$$\begin{aligned}\mathbf{f}_{j+\frac{1}{2}} =& \frac{1}{2}(\mathbf{f}_j + \mathbf{f}_{j+1}) - \frac{1}{2}\mathbf{R}_{j+\frac{1}{2}}\Big\{Q(\mathbf{\Lambda})_{j+\frac{1}{2}}\alpha_{j+\frac{1}{2}} - \Big[Q(\mathbf{\Lambda})_{j+\frac{1}{2}} - \lambda\mathbf{\Lambda}^2_{j+\frac{1}{2}}\Big] \\ & \Big(d\Big((\alpha_1)_{j-\frac{1}{2}}, (\alpha_1)_{j+\frac{1}{2}}, (\alpha_1)_{j+\frac{3}{2}}\Big), \cdots, d\Big((\alpha_m)_{j-\frac{1}{2}}, (\alpha_m)_{j+\frac{1}{2}}, (\alpha_m)_{j+\frac{3}{2}}\Big)\Big)^T\Big\}\end{aligned} \tag{3.12}$$

where the index $_{j+\frac{1}{2}}$ represents some kind of average such as Roe average, and

$$Q(\mathbf{\Lambda})_{j+\frac{1}{2}} = \text{diag}\left(Q\left((\lambda_1)_{j+\frac{1}{2}}\right), \cdots, Q\left((\lambda_m)_{j+\frac{1}{2}}\right)\right). \tag{3.13}$$

In section 5, we will give numerical results for the system of Euler equations in one dimension by using the scheme (3.12).

4. Nonlinear Conservation Laws in Two Dimensions

In this section , we firstly discuss scalar hyperbolic conservation laws in two dimensions

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} + \frac{\partial g(u)}{\partial y} = 0 \tag{4.1}$$

We use the conservative scheme

$$u^{n+1}_{j,k} = u_{j,k} - \lambda(f_{j+\frac{1}{2},k} - f_{j-\frac{1}{2},k}) - \mu(g_{j,k+\frac{1}{2}} - g_{j,k-\frac{1}{2}}) \tag{4.2}$$

to approximate the equation (4.1).

For the second order accuracy, we take

$$\begin{aligned} f_{j+\frac{1}{2},k} &= \tilde{f}_{j+\frac{1}{2},k} + \Phi_{j+\frac{1}{2},k} + \phi_{j+\frac{1}{2},k} \\ g_{j,k+\frac{1}{2}} &= \tilde{g}_{j,k+\frac{1}{2}} + \Psi_{j,k+\frac{1}{2}} + \psi_{j,k+\frac{1}{2}} \end{aligned}$$

where

$$\begin{aligned} \tilde{f}_{j+\frac{1}{2},k} &= \frac{1}{2}[f_{j,k} + f_{j+1,k} - Q(a_{j+\frac{1}{2},k})\Delta_{j+\frac{1}{2},k}u] \\ \tilde{g}_{j,k+\frac{1}{2}} &= \frac{1}{2}[g_{j,k} + g_{j,k+1} - Q(b_{j,k+\frac{1}{2}})\Delta_{j,k+\frac{1}{2}}u] \\ \Phi_{j+\frac{1}{2},k} &= \frac{1}{2}[Q(a_{j+\frac{1}{2},k}) - \lambda a^2_{j+\frac{1}{2},k}]d_{j+\frac{1}{2},k} \end{aligned}$$

$$\Psi_{j,k+\frac{1}{2}} = \frac{1}{2}[Q(b_{j,k+\frac{1}{2}}) - \mu b^2_{j,k+\frac{1}{2}}]d_{j,k+\frac{1}{2}}$$

and

$$a_{j+\frac{1}{2},k} = \begin{cases} \dfrac{f_{j+1,k} - f_{j,k}}{u_{j+1,k} - u_{j,k}}, & u_{j+1,k} \neq u_{j,k} \\ f'(u_{j,k}), & u_{j+1,k} = u_{j,k} \end{cases}$$
$$b_{j,k+\frac{1}{2}} = \begin{cases} \dfrac{g_{j,k+1} - g_{j,k}}{u_{j,k+1} - u_{j,k}}, & u_{j,k+1} \neq u_{j,k} \\ g'(u_{j,k}), & u_{j,k+1} = u_{j,k} \end{cases}$$

the limiters

$$d_{j+\frac{1}{2},k} = d(\Delta_{j-\frac{1}{2},k}u, \Delta_{j+\frac{1}{2},k}u, \Delta_{j+\frac{3}{2},k}u)$$
$$d_{j,k+\frac{1}{2}} = d(\Delta_{j,k-\frac{1}{2}}u, \Delta_{j,k+\frac{1}{2}}u, \Delta_{j,k+\frac{3}{2}}u)$$
$$\phi_{j+\frac{1}{2},k} = -\mu a_{j+\frac{1}{2},k} m^g_{j+\frac{1}{2},k}, \quad \psi_{j,k+\frac{1}{2}} = -\lambda b_{j,k+\frac{1}{2}} m^f_{j,k+\frac{1}{2}}$$
$$m^g_{j+\frac{1}{2},k} = \text{minmod}\,[\Delta_{j,k+\frac{1}{2}}g, \Delta_{j+1,k+\frac{1}{2}}g, \Delta_{j,k-\frac{1}{2}}g, \Delta_{j+1,k-\frac{1}{2}}g]$$
$$m^f_{j,k+\frac{1}{2}} = \text{minmod}\,[\Delta_{j+\frac{1}{2},k}f, \Delta_{j+\frac{1}{2},k+1}f, \Delta_{j-\frac{1}{2},k}f, \Delta_{j-\frac{1}{2},k+1}f]$$

For making the scheme (4.2) satisties the "condition B", we modify $\psi_{j,k+\frac{1}{2}} - \psi_{j,k-\frac{1}{2}}$ into

$$-\frac{\lambda}{2}[b_{j,k+\frac{1}{2}} m^f_{j,k+\frac{1}{2}} - b_{j,k-\frac{1}{2}} m^f_{j,k-\frac{1}{2}}]\Big(\frac{d_{j-\frac{1}{2},k}}{\Delta_{j+\frac{1}{2},k}u} + \frac{d_{j+\frac{1}{2},k}}{\Delta_{j-\frac{1}{2},k}u}\Big)$$

and $\phi_{j+\frac{1}{2},k} - \phi_{j-\frac{1}{2},k}$ into

$$-\frac{\mu}{2}[a_{j+\frac{1}{2},k} m^g_{j+\frac{1}{2},k} - a_{j-\frac{1}{2},k} m^g_{j-\frac{1}{2},k}]\Big(\frac{d_{j,k-\frac{1}{2}}}{\Delta_{j,k+\frac{1}{2}}u} + \frac{d_{j,k+\frac{1}{2}}}{\Delta_{j,k-\frac{1}{2}}u}\Big)$$

For the scheme defined above, we have

Theorem 4.1. *If the function d satisfies that*

$$d(\alpha_1, \alpha_2, \alpha_3) = r_k \alpha_k, \quad k = 1, 2, 3, \quad 0 \leq r_k \leq \theta \leq 2$$

then the scheme defined above satisfies the "condition B" if the inequalities

$$\lambda \max_{j,k} Q(a_{j+\frac{1}{2},k}) \leq \frac{3}{4(3+2\theta)} \tag{4.3}$$
$$\mu \max_{j,k} Q(b_{j,k+\frac{1}{2}}) \leq \frac{3}{4(3+2\theta)} \tag{4.4}$$

holds.

Proof. Since the proof is similar to that of Theorem 3.1, it is omitted.

Now, we generalize the high resolution SCB schemes to hyperbolic systems of conservation laws.

For the system of equations (1.2), the Jacobian matrices of the functions $\mathbf{f}$ and $\mathbf{g}$

$$\mathbf{A}^x = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \qquad \text{and} \qquad \mathbf{A}^y = \frac{\partial \mathbf{g}}{\partial \mathbf{u}} \tag{4.5}$$

$\lambda_1^x, \lambda_2^x, \cdots, \lambda_m^x$ and $\lambda_1^y, \lambda_2^y, \cdots, \lambda_m^y$ denote the eigenvalues of $\mathbf{A}^x$ and $\mathbf{A}^y$, respectively. $r_1^x, r_2^x, \cdots, r_m^x$, $l_1^x, l_2^x, \cdots, l_m^x$ and $r_1^y, r_2^y, \cdots, r_m^y$, $l_1^y, l_2^y, \cdots, l_m^y$ denote right and left eigenvectors of the matrices $\mathbf{A}^x$ and $\mathbf{A}^y$, respectively. So,

$$\mathbf{A}^x = \mathbf{R}^x \mathbf{\Lambda}^x \mathbf{L}^x \qquad \text{and} \qquad \mathbf{A}^y = \mathbf{R}^y \mathbf{\Lambda}^y \mathbf{L}^y \tag{4.6}$$

where

$$\begin{aligned} &\mathbf{R}^x = (r_1^x, \cdots, r_m^x),\ \mathbf{\Lambda}^x = \operatorname{diag}(\lambda_1^x, \cdots, \lambda_m^x) \text{ and } \mathbf{L}^x = (l_1^x, \cdots, l_m^x)^T \\ &\mathbf{R}^y = (r_1^y, \cdots, r_m^y),\ \mathbf{\Lambda}^y = \operatorname{diag}(\lambda_1^y, \cdots, \lambda_m^y) \text{ and } \mathbf{L}^y = (l_1^y, \cdots, l_m^y)^T. \end{aligned} \tag{4.7}$$

In the scheme (4.2), the difference of the function u is replaced to the system (1.2) in x- or y-direction respectively by

$$(\alpha_s^x)_{j+\frac{1}{2},k} = (l_s^x)^T_{j+\frac{1}{2},k} \Delta^x_{j+\frac{1}{2},k} \mathbf{u} \text{ and } \alpha^x_{j+\frac{1}{2},k} = ((\alpha_1^x)_{j+\frac{1}{2},k}, \cdots, (\alpha_m^x)_{j+\frac{1}{2},k})^T \tag{4.8}$$

$$(\alpha_s^y)_{j,k+\frac{1}{2}} = (l_s^y)^T_{j,k+\frac{1}{2}} \Delta^y_{j,k+\frac{1}{2}} \mathbf{u} \text{ and } \alpha^y_{j,k+\frac{1}{2}} = ((\alpha_1^y)_{j,k+\frac{1}{2}}, \cdots, (\alpha_m^y)_{j,k+\frac{1}{2}})^T \tag{4.9}$$

So, the numerical flux can be generalized into

$$\begin{aligned} \mathbf{f}_{j+\frac{1}{2},k} = {} & \frac{1}{2}(\mathbf{f}_{j,k} + \mathbf{f}_{j+1,k}) - \frac{1}{2}\mathbf{R}^x_{j+\frac{1}{2},k} \Big\{ Q(\mathbf{\Lambda}^x)_{j+\frac{1}{2},k} \alpha^x_{j+\frac{1}{2},k} - \Big[Q(\mathbf{\Lambda}^x)_{j+\frac{1}{2},k} - \lambda (\mathbf{\Lambda}^x)^2_{j+\frac{1}{2},k} \Big] \\ & \bullet \left(d\left((\alpha_1^x)_{j-\frac{1}{2},k}, (\alpha_1^x)_{j+\frac{1}{2},k}, (\alpha_1^x)_{j+\frac{3}{2},k} \right), \cdots, d\left((\alpha_m^x)_{j-\frac{1}{2},k}, (\alpha_m^x)_{j+\frac{1}{2},k}, (\alpha_m^x)_{j+\frac{3}{2},k} \right) \right)^T \\ & + \frac{\mu}{2} (\mathbf{\Lambda}^x)_{j+\frac{1}{2},k} (\mathbf{L}^x)^T_{j+\frac{1}{2}} m^g_{j+\frac{1}{2},k} \Big\} \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \mathbf{g}_{j,k+\frac{1}{2}} = {} & \frac{1}{2}(\mathbf{g}_{j,k} + \mathbf{g}_{j,k+1}) - \frac{1}{2}\mathbf{R}^y_{j,k+\frac{1}{2}} \Big\{ Q(\mathbf{\Lambda}^y)_{j,k+\frac{1}{2}} \alpha^y_{j,k+\frac{1}{2}} - \Big[Q(\mathbf{\Lambda}^y)_{j,k+\frac{1}{2}} - \mu (\mathbf{\Lambda}^x)^2_{j,k+\frac{1}{2}} \Big] \\ & \bullet \left(d\left((\alpha_1^y)_{j,k-\frac{1}{2}}, (\alpha_1^y)_{j,k+\frac{1}{2}}, (\alpha_1^y)_{j,k+\frac{3}{2}} \right), \cdots, d\left((\alpha_m^y)_{j,k-\frac{1}{2}}, (\alpha_m^y)_{j,k+\frac{1}{2}}, (\alpha_m^y)_{j,k+\frac{3}{2}} \right) \right)^T \\ & + \frac{\lambda}{2} (\mathbf{\Lambda}^y)_{j,k+\frac{1}{2}} (\mathbf{L}^x)^T_{j,k+\frac{1}{2}} m^f_{j,k+\frac{1}{2}} \Big\} \end{aligned} \tag{4.11}$$

where the index ${}_{j+\frac{1}{2},k}$ or ${}_{j,k+\frac{1}{2}}$ represents some kind of average such as Roe average, and

$$Q(\mathbf{\Lambda}^x)_{j+\frac{1}{2},k} = \operatorname{diag}\left(Q\left((\lambda_1^x)_{j+\frac{1}{2},k} \right), \cdots, Q\left((\lambda_m^x)_{j+\frac{1}{2},k} \right) \right), \tag{4.12}$$

$$Q(\mathbf{\Lambda}^y)_{j,k+\frac{1}{2}} = \operatorname{diag}\left(Q\left((\lambda_1^y)_{j,k+\frac{1}{2}}\right), \cdots, Q\left((\lambda_m^y)_{j,k+\frac{1}{2}}\right)\right), \tag{4.13}$$

and

$$m^g_{j+\frac{1}{2},k} = \operatorname{minmod}\left[\Delta_{j,k+\frac{1}{2}}\mathbf{g}, \Delta_{j+1,k+\frac{1}{2}}\mathbf{g}, \Delta_{j,k-\frac{1}{2}}\mathbf{g}, \Delta_{j+1,k-\frac{1}{2}}\mathbf{g}\right] \tag{4.14}$$

$$m^f_{j,k+\frac{1}{2}} = \operatorname{minmod}\left[\Delta_{j+\frac{1}{2},k}\mathbf{f}, \Delta_{j+\frac{1}{2},k+1}\mathbf{f}, \Delta_{j-\frac{1}{2},k}\mathbf{f}, \Delta_{j-\frac{1}{2},k+1}\mathbf{f}\right]. \tag{4.15}$$

In section 5, we will give numerical results for the system of Euler equations in two dimensions by using the above scheme.

5. Numerical Experiments

In this section, we give some numerical examples to the schemes constructed in provious sections.

All numerical results presented here were performed on a PC-AST/486 computer.

Firstly, we briefly describe our two test problems in one dimension: Sod and Lax problems.

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0 \tag{5.1}$$

where

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} \rho u \\ \rho u^2 \\ (E+p)u \end{pmatrix} \tag{5.2}$$

where ρ, u, E, and p are respectively density, velocity, total energy, and pressure of a γ-law gas with $\gamma = 1.4$. The pressure satisfies the equation of state $p = 0.4\Big(E - \dfrac{1}{2}\rho u^2\Big)$. The sound speed is $c = \sqrt{1.4p/\rho}$.

The Sod and Lax problems are simple Riemann-problems given respectively by the initial data

$$(\rho(x), u(x), p(x)) = \begin{cases} (1, 0, 1) & \text{if } x < 0.5 \\ (0.125, 0, 0.1) & \text{if } x \geq 0.5 \end{cases} \tag{5.3}$$

and

$$(\rho(x), u(x), p(x)) = \begin{cases} (0.445, 0.698, 3.528) & \text{if } x < 0.5 \\ (0.5, 0, 0.571) & \text{if } x \geq 0.5 \end{cases} \tag{5.4}$$

100 equally space grid points are used over the interval $[0,1]$ for both problems, with $T = 0.18$ for the Roe problem (Fig. 1), and $T = 0.14154$ for the Lax problem (Fig. 2). The CFL restriction=0.6.

Moreover, let us describe two test problems in two dimensions: the regular shock reflection problem and the ramp problem[1].

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} + \frac{\partial \mathbf{g}(\mathbf{u})}{\partial y} = 0 \tag{5.5}$$

where

$$\mathbf{u} = \begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix} \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} m \\ m^2/\rho + p \\ mn/\rho \\ m(E+p)/\rho \end{pmatrix} \quad \mathbf{g}(\mathbf{u}) = \begin{pmatrix} n \\ mn/\rho \\ n^2/\rho + p \\ n(E+p)/\rho \end{pmatrix} \tag{5.6}$$

where ρ, m, n, E, and p are respectively density, momentum in x-direction and y-direction, total energy, and pressure of a γ-law gas with $\gamma = 1.4$. The pressure satisfies the equation of state $p = 0.4\Big(E - \frac{1}{2}(m^2+n^2)/\rho\Big)$. The sound speed is $c = \sqrt{1.4p/\rho}$. $m = \rho u, n = \rho v$, where u, v are velocities in x- and y-directions, respectively.

To begin, let us discuss the treatment of boundary conditions. As the case in [1], the accuracy of our unsplit scheme is also sensitive to the reflecting boundary conditions, which is different from the operator split methods. In our calculation, the treatment of reflecting boundary conditions is as following:

Assume the point x_0 be the boundary grid point, and the next grid points in the calculating domain x_1, x_2. So, To calculate the function value of u on the boundary grid point x_0, u_0^{n+1}, the values of u at $t = t^n$ level on the outside grid points, u_{-1}^n and u_{-2}^n, should be known. For the reflecting boundary conditions, in our calculation, take

$$u_{-1}^n = -u_1^n \qquad \text{and} \qquad u_{-2}^n = -u_2^n. \tag{5.7}$$

For the outflow boundary conditions, take

$$u_{-1}^n = u_0^n \qquad \text{and} \qquad u_{-2}^n = u_0^n. \tag{5.8}$$

For the system of equations (5.5), Jacobian matrices of the functions $\mathbf{f}$ and $\mathbf{g}$ are respectively:

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \qquad \text{and} \qquad \mathbf{B} = \frac{\partial \mathbf{g}}{\partial \mathbf{u}} \tag{5.9}$$

and

$$\mathbf{A} = \mathbf{R}^x \Lambda^x \mathbf{L}^x \qquad \mathbf{B} = \mathbf{R}^y \Lambda^y \mathbf{L}^y \tag{5.10}$$

where the concrete forms of matrices $\mathbf{A}, \mathbf{R}^x, \Lambda^x, \mathbf{L}^x$, $\mathbf{B}, \mathbf{R}^y, \Lambda^y$, and $\mathbf{L}^y$ refer to ref.[4].

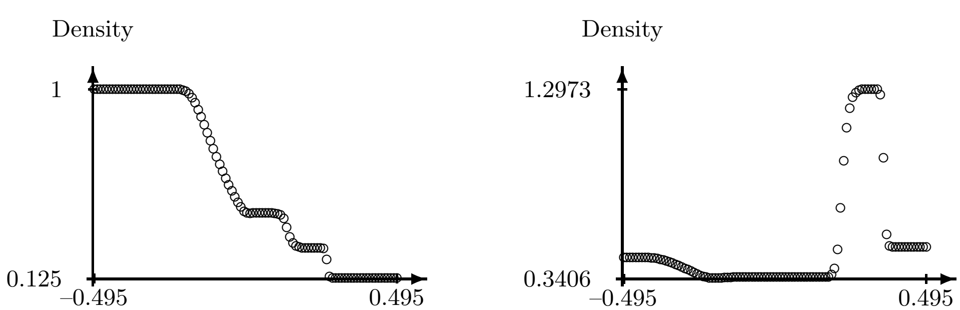

Fig.1a Density

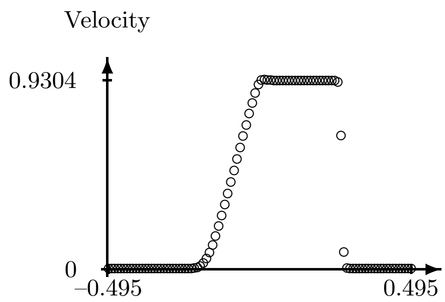

Fig.1b Velocity

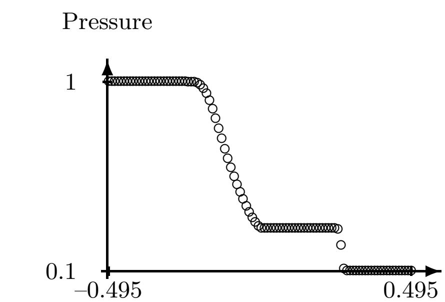

Fig.1c Pressure

Fig.1 The Sod problem

Fig.2a Density

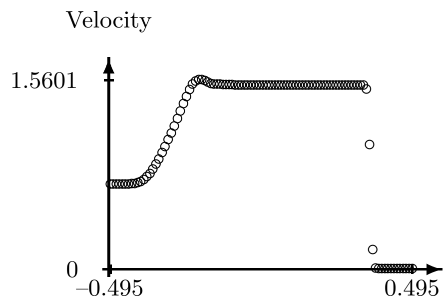

Fig.2b Velocity

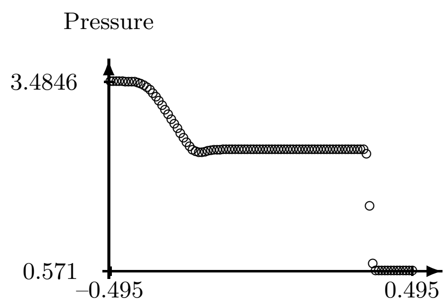

Fig.2c Pressure

Fig.2. The Lax problem

Now, let us consider the regular shock reflection problem [1] (Fig. 3). The computational domain is a rectangle of length 4 and height 1. This domain is divided into a 60×20 rectangular grids, with $\Delta x = 1/15$, $\Delta y = 1/20$. The boundary conditions are that of a reflection condition along the bottom boundary, supersonic outflow along the right boundary, and Dirichlet conditions on the other two sides, given by

$$
\begin{aligned}
&(\rho, u, v, p)^T|_{(0,y,t)} = (1., 2.9, 0., 1/1.4)^T \\
&(\rho, u, v, p)^T|_{(x,1,t)} = (1.69997, 2.61934, -0.50632, 1.52819)^T.
\end{aligned} \tag{5.11}
$$

Initially, set the solution in the entire domain to be that at the left boundary. we iterate for 1000 time steps using CFL=0.45 with $\Delta x = 1/15$ and $\Delta y = 1/20$, at which time the solution reach a steady state. We show the contour plots of the density in the Fig. 4 and the value of density at $y = 0.5$ in the Fig. 5, respectively.

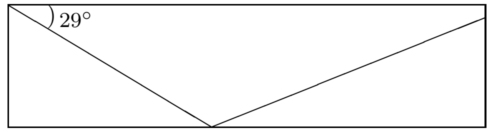

Fig.3 The regular shock reflection problem

Fig.4 Regular shock reflection problem with $\Delta x = 1/15$ and $\Delta y = 1/20$ 1000 time steps, CFL=0.45

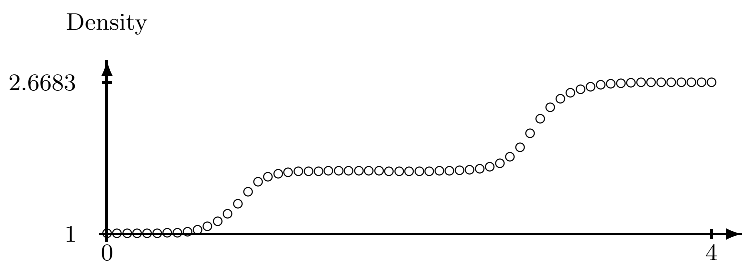

Fig.5. The value of density at $y = 0.5$

The second problem is the ramp problem[4], which is solved on the domain $(x, y) \in (0, 4) \times (0, 1)$ with t running from $t = 0$ to $t = 0.2$. The initial data are

$$\mathbf{u}(x, y, 0) = \begin{cases} \mathbf{u}_L & \text{for } y \geq h(x, 0) \\ \mathbf{u}_R & \text{for } y < h(x, 0) \end{cases} \tag{5.12}$$

where

$$\begin{aligned} &\mathbf{u}_L = (8., 57.1597, -33.0012, 563.544)^T \\ &\mathbf{u}_R = (1.4, 0., 0., 2.5)^T \\ &h(x, t) = \sqrt{3}\Big(x - \frac{1}{6}\Big) - 20t. \end{aligned} \tag{5.13}$$

Data $\mathbf{u_L}$ and $\mathbf{u}_R$ correspond to a Mach 10 planar shock ($\gamma = 1.4$) at an angle of 60^o with the x-axis. The boundary conditions for boundaries $A - F$ in Fig.6 are time-dependent. The location of the point separating boundaries E and F is

$$x_{EF} = \frac{1}{6} + \frac{\sqrt{3}}{3}(1 + 20t).$$

The boundary conditions are

$$\text{A}: \ \mathbf{u}(x, y) = \mathbf{u}_L, \ \text{B: same as } A, \ C: \text{reflecting}$$

$$\text{D}: \ \mathbf{u}(x,y) = \mathbf{u}_R, \ \text{E: same as } A, \ F: \text{same as } D.$$

Fig.7 depicts the density at $t = 0.2$ for the ramp problem with $\Delta x = \Delta y = 1/30$, and Figure 8 the density with $\Delta x = \Delta y = 1/60$, respectively.

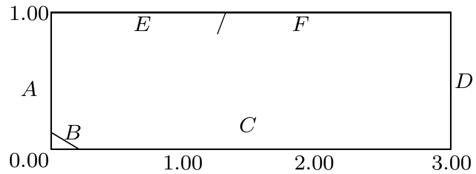

Fig.6. The ramp problem domain

Fig.7. The ramp problem with $\Delta x = \Delta y = 1/30$, final time 0.2, CFL=0.45

Fig.8. The ramp problem with $\Delta x = \Delta y = 1/60$, final time 0.2, CFL=0.45

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