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THE OPTIMAL PRECONDITIONING IN THE DOMAIN DECOMPOSITION METHOD FOR WILSON ELEMENT*

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Abstract

This paper discusses the optimal preconditioning in the domain decomposition method for Wilson element. The process of the preconditioning is composed of the resolution of a small scale global problem based on a coarser grid and a number of independent local subproblems, which can be chosen arbitrarily. The condition number of the preconditioned system is estimated by some characteristic numbers related to global and local subproblems. With a proper selection, the optimal preconditioner can be obtained, while the condition number is independent of the scale of the problem and the number of subproblems.

1. The Construction of Preconditioner

Let Ω be a polygon domain in R^2, $f \in L^2(\Omega)$. Consider the homogeneous Dirichlet boundary value problem of Poisson equation,

$$\begin{cases} -\triangle u = f, & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

Assume that, for domain Ω, there are a coarser subdivision T_H with mesh size H and an another one T_h with mesh size h, which is obtained by refining T_H. The both subdivisions satisfy the quasi-uniformity and the inverse hypothesis.

For a given element T, $P_m(T)$ denotes the space of all polynomials with the degree not greater than m, $Q_m(T)$ denotes the space of all polynomials with the degree corresponding to x or y not greater than m.

Let V_H and V_h be some nonconforming finite element spaces corresponding to T_H and T_h respectively. For problem (1.1), the nodal parameters on the boundary $\partial\Omega$ are all zero. For finite element spaces V_h and V_H, the finite element equations for problem (1.1) are

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{1.2}$$

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$$a_H(u^H, v^H) = (f, v^H), \quad \forall v^H \in V_H, \tag{1.3}$$

respectively. Where $(\cdot,\cdot)$ is $L^2(\Omega)$ inner product and

$$a_h(v,w) = \sum_{T\in \mathsf{T}_h} \int_T \Big(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\Big) dxdy,$$

$$a_H(v,w) = \sum_{T\in \mathsf{T}_H} \int_T \Big(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\Big) dxdy.$$

For $v \in V_h$, denote the vector of its nodal parameters by $C_h(v)$, and for $v \in V_H$, denote the vector of its nodal parameters by $C_H(v)$. Thus, equations (1.2) and (1.3) can be written as

$$A_h C_h(u_h) = F_h \tag{1.4}$$

$$A_H C_H(u^H) = F_H \tag{1.5}$$

where A_h, A_H are the stiffness matrices corresponding to problems (1.2) and (1.3) respectively, and F_h, F_H are the loading vectors.

Now consider how to solve (1.2). The Preconditioned Conjugate Gradient method (PCG) would be used. So the preconditioning matrix Q needs to be constructed.

Let $\{\omega_1, \omega_2, \cdots, \omega_M\}$ be a domain decomposition of Ω, i.e., $\overline{\Omega} = \cup_{k=1}^M \overline{\omega_k}$, and $\omega_m \cap \omega_n = \emptyset (m \neq n)$. For each ω_k, it is extended to Ω_k, such that the boundary of Ω_k is consists of the edges of T_h and

$$\mathrm{dist}\,\{\partial\omega_k, \partial\Omega_k\} \geq L, \tag{1.6}$$

where L is a fixed positive constant. For each element $T \in \mathsf{T}_h$, the number of subdomains $\overline{\Omega}_k$ containing T does not exceed a fixed number.

Corresponding to T_h, a subdivision of Ω_k can be obtained, and the corresponding nonforming finite element space is denoted by $V_{h,k}$. The corresponding finite element equation is

$$a_k(u_k, v_k) = (f, v_k)_k, \quad \forall v_k \in V_{h,k}, \tag{1.7}$$

where $(\cdot,\cdot)_k$ is $L^2(\Omega_k)$ inner product and

$$a_k(u_k, v_k) = \sum_{T\in \mathsf{T}_h, T\subset \overline{\Omega}_k} \int_T \Big(\frac{\partial u_k}{\partial x}\frac{\partial v_k}{\partial x} + \frac{\partial u_k}{\partial y}\frac{\partial v_k}{\partial y}\Big) dxdy.$$

The stiffness matrix is denoted by A_k.

Let E_k be the zero extension operator from $V_{h,k}$ to V_h, i.e., $\forall v_k \in V_{h,k}, \forall T \in \mathsf{T}_h$

$$E_k v_k|_T = \begin{cases} v_k|_T, & T \subset \overline{\Omega}_k \\ 0, & \text{otherwise} \end{cases} \tag{1.8}$$

For $v_k \in V_{h,k}$, its nodal parameter vector is denoted by $C_k(v_k)$. In the sense of nodal parameter vectors, a mapping matrix $\mathbf{E}_k$ is given, that is

$$C_h(E_k v_k) = \mathbf{E}_k C_k(v_k), \quad \forall v_k \in V_{h,k}. \tag{1.9}$$

Let I_H be a linear operator from V_H to V_h. Let $\mathbf{I}_H$ be the matrix such that

$$C_h(I_H v^H) = \mathbf{I}_H C_H(v^H), \quad \forall v^H \in V_H. \tag{1.10}$$

The expression for the inverse Q^{-1} of the preconditioner Q is defined as follows,

$$Q^{-1} = \mathbf{I}_H A_H^{-1} \mathbf{I}_H^\top + \sum_{k=1}^{M} \mathbf{E}_k A_k^{-1} \mathbf{E}_k^\top, \tag{1.11}$$

while Q^{-1} is symmetric and positive.

In the PCG iteration, only Q^{-1} not Q will take part in the operation, the expression for Q is not necessary. The process of Q^{-1} is to solve the finite element equations on coarser subdivision and the subdomains simultaneously. The computing is fully parallel.

The convergence of PCG method is dependent on the condition number of matrix $Q^{-1}A_h$. Smaller the condition number is, faster the convergence is. The condition number of $Q^{-1}A_h$ is bounded by the ratio of the upper bounds of the generalized Rayleich quotient

$$R(v) = \frac{(A_h Q^{-1} A_h C_h(v), C_h(v))}{(A_h C_h(v), C_h(v))}, \quad \forall v \in V_h. \tag{1.12}$$

to the low one.

The remainder of the paper will give the linear operator I_H for Wilson element, and estimate $R(v)$ and get the bound of the condition number.

Throughout the paper, C always denotes the positive constant independent of H, h and the choice of the subdomains.

For a set $G \in R^2$ and an integer m, Sobolev semi-norm is denoted by $|\cdot|_{m,G}$. For subdivisions T_H and T_h, define the following discrete Sobolev norms,

$$|\cdot|_{m,H} = \Big(\sum_{T\in\mathsf{T}_H} |\cdot|_{m,T}^2\Big)^{1/2}, \quad |\cdot|_{m,h} = \Big(\sum_{T\in\mathsf{T}_h} |\cdot|_{m,T}^2\Big)^{1/2}.$$

2. Wilson Element

In the case of Wilson element, the subdivision elements are rectangles. Wilson finite element space $V_h = \{v\,|\,v \in L^2(\Omega),\ v|_T \in P_2(T),\ \forall T \in \mathsf{T}_h$, and v is continuous at vertices of T_h and v vanishes at the vertices on $\partial\Omega$ $\}$. Similarly, spaces V_H and $V_{h,k}$ can be defined. The function v of Wilson space is uniquely determined by its values at the vertices, and the values of $\dfrac{\partial^2}{\partial x^2}v$ and $\dfrac{\partial^2}{\partial y^2}v$ on all elements.

The bilinear interpolation operator using the function values at the vertices, for element T, is denoted by Q^1_T. Q^1_H and Q^1_h are the interpolation operators corresponding to T_H and T_h respectively.

For all $v^H \in V_H$, define $I_H v^H \in V_h$ as follows,

1. $I_H v^H$ equals to $Q^1_H v^H$ at the vertices of T_h.
2. For each element T' of T_h, there exists an element $T \in \mathsf{T}_H$ with $T' \subset T$, then

$$\frac{\partial^2}{\partial x^2} I_H v^H|_{T'} = \frac{H}{h}\frac{\partial^2}{\partial x^2} v^H|_T, \quad \frac{\partial^2}{\partial y^2} I_H v^H|_{T'} = \frac{H}{h}\frac{\partial^2}{\partial y^2} v^H|_T.$$

Before estimating the condition number of $Q^{-1}A_h$, some preparation results will be given.

Lemma 1. *There exists a constant C independent of H, h, such that,*

$$|v^H - I_H v^H|_{m,h} \le C H^{1-m} |v^H|_{1,H}, \quad m = 0,1 \ \forall v^H \in V_H. \tag{2.1}$$

Proof. For a given rectangle T, its four vertices are denoted by $A^i_T (1 \le i \le 4)$. It is easy to show that for arbitrary element T in T_H or in T_h,

$$\frac{1}{C}|p|^2_{0,T} \le \Big\{ \sum_{i=1}^4 |T||p(A^i_T)|^2 + |T|^3\Big(\Big|\frac{\partial^2 p}{\partial x^2}\Big|^2 + \Big|\frac{\partial^2 p}{\partial y^2}\Big|^2\Big)\Big\} \le C|p|^2_{0,T}, \tag{2.2}$$

$$\frac{1}{C}|p|^2_{1,T} \le \Big\{ \sum_{1\le i,j\le 4} |p(A^i_T) - p(A^j_T)|^2 + |T|^2\Big(\Big|\frac{\partial^2 p}{\partial x^2}\Big|^2 + \Big|\frac{\partial^2 p}{\partial y^2}\Big|^2\Big)\Big\} \le C|p|^2_{1,T}, \tag{2.3}$$

are true for all $p \in P_2(T)$, where $|T|$ is the area of T.

Now let $v^H \in V_H$ and $T \in \mathsf{T}_H$, then from the definition of I_H and (2.2),

$$\begin{aligned}
|v^H - I_H v^H|^2_{0,T} &= \sum_{S\in\mathsf{T}_h, S\subset T} |v^H - I_H v^H|^2_{0,S} \le C \sum_{S\in\mathsf{T}_h, S\subset T} \Big\{ \sum_{i=1}^4 h^2 |(v^H - I_H v^H)(A^i_S)|^2 \\
&\quad + h^6\Big(\Big|\frac{\partial^2}{\partial x^2}(v^H - I_H v^H)|_S\Big|^2 + \Big|\frac{\partial^2}{\partial y^2}(v^H - I_H v^H)|_S\Big|^2\Big)\Big\} \\
&= C \sum_{S\in\mathsf{T}_h, S\subset T} \Big\{ \sum_{i=1}^4 h^2 |(Q^1_h - Q^1_H)v^H(A^i_S)|^2 \\
&\quad + h^6\Big(1 - \frac{H}{h}\Big)^2\Big(\Big|\frac{\partial^2}{\partial x^2} v^H|_T\Big|^2 + \Big|\frac{\partial^2}{\partial y^2} v^H|_T\Big|^2\Big)\Big\} \\
&\le C \sum_{S\in\mathsf{T}_h, S\subset T} \Big\{ |(Q^1_h - Q^1_H)v^H|^2_{0,S} + H^2 h^4 \Big(\Big|\frac{\partial^2}{\partial x^2} v^H|_T\Big|^2 + \Big|\frac{\partial^2}{\partial y^2} v^H|_T\Big|^2\Big)\Big\}
\end{aligned}$$

By the interpolation property and the inverse inequality, one gets

$$|v^H - I_H v^H|^2_{0,T} \le C H^2 |v^h|^2_{1,T} \Big\{ 1 + \sum_{S\in\mathsf{T}_h, S\subset T} h^4 H^{-4} \Big\}.$$

Since the number of the elements contained in T is bounded by CH^2/h^2, one has

$$|v^H - I_H v^H|_{0,T}^2 \le CH^2 |v^h|_{1,T}^2. \tag{2.4}$$

From the definition of I_H and (2.3),

$$\begin{aligned}\sum_{S\in \mathsf{T}_h, S\subset T} |v^H - I_H v^H|_{1,S}^2 \le C &\sum_{S\in \mathsf{T}_h, S\subset T}\Big\{\sum_{1\le i,j\le 4} |(v^H - I_H v^H)(A_S^i) - (v^H - I_H v^H)(A_S^j)|^2 \\ &+ h^4\Big(\Big|\frac{\partial^2}{\partial x^2}(v^H - I_H v^H)|_S\Big|^2 + \Big|\frac{\partial^2}{\partial y^2}(v^H - I_H v^H)|_S\Big|^2\Big)\Big\} \\ = C &\sum_{S\in \mathsf{T}_h, S\subset T}\Big\{\sum_{1\le i,j\le 4} \Big|(Q_h^1 - Q_H^1)v^H(A_S^i) - (Q_h^1 - Q_H^1)v^H(A_S^j)\Big|^2 \\ &+ h^4\Big(1 - \frac{H}{h}\Big)^2\Big(\Big|\frac{\partial^2}{\partial x^2}v^H|_T\Big|^2 + \Big|\frac{\partial^2}{\partial y^2}v^H|_T\Big|^2\Big)\Big\} \\ \le C &\sum_{S\in \mathsf{T}_h, S\subset T}\Big\{|(Q_h^1 - Q_H^1)v^H|_{1,S}^2 + H^2h^2\Big(\Big|\frac{\partial^2}{\partial x^2}v^H|_T\Big|^2 + \Big|\frac{\partial^2}{\partial y^2}v^H|_T\Big|^2\Big)\Big\}\end{aligned}$$

It leads to

$$\sum_{S\in \mathsf{T}_h, S\subset T} |v^H - I_H v^H|_{1,S}^2 \le C|v^H|_{1,T}^2. \tag{2.5}$$

Lemma 1 follows from (2.4) and (2.5).

Lemma 2. *There exists a constant C independent of H, h, such that,*

$$|v^H|_{1,H} \le C|I_H v^H|_{1,h}, \quad \forall v^H \in V_H. \tag{2.6}$$

Proof. Let $v^H \in V_h$ and $T \in \mathsf{T}_H$. (2.3) gives

$$\begin{aligned}|v^H|_{1,T}^2 &\le C\Big\{\sum_{1\le i,j\le 4} |v^H(A_T^i) - v^H(A_T^j)|^2 + H^4\Big(\Big|\frac{\partial^2}{\partial x^2}v^H|_T\Big|^2 + \Big|\frac{\partial^2}{\partial y^2}v^H|_T\Big|^2\Big)\Big\} \\ &= C\Big\{\sum_{1\le i,j\le 4} |Q_H^1 v^H(A_T^i) - Q_H^1 v^H(A_T^j)|^2 + H^4\Big(\Big|\frac{\partial^2}{\partial x^2}v^H|_T\Big|^2 + \Big|\frac{\partial^2}{\partial y^2}v^H|_T\Big|^2\Big)\Big\} \\ &\le C\Big\{|Q_H^1 v^H|_{1,T}^2 + H^4\Big(\Big|\frac{\partial^2}{\partial x^2}v^H|_T\Big|^2 + \Big|\frac{\partial^2}{\partial y^2}v^H|_T\Big|^2\Big)\Big\} \\ &= C\Big\{\sum_{S\in \mathsf{T}_h, S\subset T} |Q_H^1 v^H|_{1,S}^2 + H^4\Big(\Big|\frac{\partial^2}{\partial x^2}v^H|_T\Big|^2 + \Big|\frac{\partial^2}{\partial y^2}v^H|_T\Big|^2\Big)\Big\} \\ &\le C\Big\{\sum_{S\in \mathsf{T}_h, S\subset T}\sum_{1\le i,j\le 4} |Q_H^1 v^H(A_S^i) - Q_H^1 v^H(A_S^j)|^2 \\ &\quad + H^4\Big(\Big|\frac{\partial^2}{\partial x^2}v^H|_T\Big|^2 + \Big|\frac{\partial^2}{\partial y^2}v^H|_T\Big|^2\Big)\Big\}\end{aligned}$$

$$= C\Big\{ \sum_{S\in \mathsf{T}_h, S\subset T} \sum_{1\le i,j\le 4} |I_H v^H(A_S^i) - I_H v^H(A_S^j)|^2$$

$$+ H^4\Big(\Big|\frac{\partial^2}{\partial x^2} v^H|_T\Big|^2 + \Big|\frac{\partial^2}{\partial y^2} v^H|_T\Big|^2\Big)\Big\}$$

Noticing that H^2/h^2 is not greater than the number of elements in T_h which are contained in T, one gets, from the definition of I_H,

$$|v^H|_{1,T}^2 \le C \sum_{S\in \mathsf{T}_h, S\subset T} \Big\{ \sum_{1\le i,j\le 4} |I_H v^H(A_S^i) - I_H v^H(A_S^j)|^2$$

$$+ H^2h^2\Big(\Big|\frac{\partial^2}{\partial x^2} v^H|_T\Big|^2 + \Big|\frac{\partial^2}{\partial y^2} v^H|_T\Big|^2\Big)\Big\}$$

$$\le C \sum_{S\in \mathsf{T}_h, S\subset T} \Big\{ \sum_{1\le i,j\le 4} |I_H v^H(A_S^i) - I_H v^H(A_S^j)|^2$$

$$+ h^4\Big(\Big|\frac{\partial^2}{\partial x^2} I_H v^H|_S\Big|^2 + \Big|\frac{\partial^2}{\partial y^2} I_H v^H|_S\Big|^2\Big)\Big\}$$

Combining (2.3) and the above inequality, one has

$$|v^H|_{1,T}^2 \le C \sum_{S\in \mathsf{T}_h, S\subset T} |I_H v^H|_{1,S}^2. \tag{2.7}$$

Lemma 2 follows.

Let $P : L^2(\Omega) \to V_H$ is the orthogonal projection operator in the sense of $L^2(\Omega)$, that is, for $v \in L^2(\Omega)$, $Pv \in V_H$ and

$$(v, v^H) = (Pv, v^H), \quad \forall v^H \in V_H.$$

Lemma 3. *For all $v \in H_0^1(\Omega)$, the following estimates are uniformly true,*

$$|v - Pv|_{m,H} \le CH^{1-m}|v|_{1,H}, \quad m = 0, 1, \tag{2.8}$$

$$|Pv|_{1,H} \le C|v|_{1,H}. \tag{2.9}$$

Lemma 3 can be proved by the similar way used in [2].

3. The Condition Number

Let $P_H : V_h \to I_H V_H$ and $P_k : V_h \to E_k V_{h,k}$ $(k = 1, 2, \cdots, M)$ be the orthogonal projection operators in the sense of inner product $a_h(\cdot,\cdot)$, that is, for $v_h \in V_h$, $P_H v_h \in I_H V_H$ and

$$a_h(P_H v_h, I_H v^H) = a_h(v_h, I_H v^H), \quad \forall v^H \in V_H, \tag{3.1}$$

and $P_k v_h \in E_k V_{h,k}$ and

$$a_h(P_k v_h, E_k v_k) = a_h(v_h, E_k v_k), \quad \forall v_k \in V_{h,k}. \tag{3.2}$$

For all $v \in V_h$, let $u_v^H \in V_H$ be the solution of equation

$$a_H(u_v^H, v^H) = a_h(v, I_H v^H), \quad \forall v^H \in V_H, \tag{3.3}$$

that is,

$$A_H C_H(u_v^H) = \mathbf{I}_H^\top A_h C_h(v).$$

It is easy to show that

$$(A_h Q^{-1} A_h C_h(v), C_h(v)) = a_h(u_v^H, u_v^H) + \sum_{k=1}^{M} a_h(P_k v, v), \quad \forall v \in V_h. \tag{3.4}$$

Lemma 4. *There exists a constant C independent of H, h and the choice of subdomains, such that,*

$$R(v) \le C, \qquad \forall v \in V_h. \tag{3.5}$$

Proof. By the way used in Lemma 2.1 in paper [4], one can prove that

$$\sum_{k=1}^{M} a_h(P_k v, v) \le C a_h(v, v), \quad \forall v \in V_h. \tag{3.6}$$

From (3.3) and (2.1), one gets

$$\begin{aligned} a_H(u_v^H, u_v^H) &= a_h(v, I_H u_v^H) \le a_h(v,v)^{1/2} a_h(I_H u_v^H, I_H u_v^H)^{1/2} \le C a_h(v,v)^{1/2} |I_H u_v^H|_{1,h} \\ &\le C a_h(v,v)^{1/2} |u_v^H|_{1,H} \le C a_h(v,v)^{1/2} a_H(u_v^H, u_v^H)^{1/2}, \\ a_H(u_v^H, u_v^H) &\le C a_h(v,v). \end{aligned} \tag{3.7}$$

Lemma 4 follows from (3.6) and (3.7).

Lemma 5. *For all $v \in V_h$,*

$$a_h(P_H v, v) + \sum_{k=1}^{M} a_h(P_k v, v) \le C a_H(u_v^H, u_v^H) + \sum_{k=1}^{M} a_h(P_k v, v). \tag{3.8}$$

Proof. It is sufficient to show the following inequality

$$a_h(P_H v, v) \le C a_H(u_v^H, u_v^H). \tag{3.9}$$

By (3.1) and (3.3),

$$a_h(P_H v, I_H v^H) = a_h(v, I_H v^H) = a_H(u_v^H, v^H), \quad \forall v^H \in V_H,$$

and

$$a_h(P_Hv,P_Hv)^{1/2}=\sup_{0\neq w\in V_H}\frac{a_h(P_Hv,I_Hw)}{a_h(I_Hw,I_Hw)^{1/2}}=\sup_{0\neq w\in V_H}\frac{a_H(u_v^H,w)}{a_h(I_Hw,I_Hw)^{1/2}}$$
$$\le a_H(u_v^H,u_v^H)^{1/2}\sup_{0\neq w\in V_H}\frac{a_H(w,w)^{1/2}}{a_h(I_Hw,I_Hw)^{1/2}}$$
$$\le Ca_H(u_v^H,u_v^H)^{1/2}\sup_{0\neq w\in V_H}\frac{|w|_{1,H}}{|I_Hw|_{1,h}},$$

(2.6) leads to (3.9).

Lemma 6. *For all $v\in V_h$,*

$$a_h(v,v)\le C\Big(1+\frac{H^2}{L^2}\Big)\Big(1+\frac{h^2}{L^2}\Big)\Big[a_h(P_Hv,v)+\sum_{k=1}^{M}a_h(P_kv,v)\Big]. \tag{3.10}$$

Proof. If there exist $\tilde v^H\in I_HV_H, u_k\in E_kV_{h,k},\ k=1,2,\cdots,M$, such that,

$$\begin{cases} v=\tilde v^H+\sum_{k=1}^{M}u_k\\ a_h(\tilde v^H,\tilde v^H)+\sum_{k=1}^{M}a_h(u_k,u_k)\le\beta a_h(v,v),\end{cases} \tag{3.11}$$

then (see [1])

$$a_h(v,v)\le\beta a_h\Big(P_Hv+\sum_{k=1}^{M}P_kv,v\Big). \tag{3.12}$$

It is necessary to find a decomposition of v which makes (3.11) true for some β. The subdomains $\{\Omega_k,k=1,2,\cdots,M\}$ are an open covering of Ω. There exists a sufficiently smooth partition of unit for the open covering, $\{\varphi_k,\ k=1,2,\cdots,M\}$, such that

1. $\sum_{k=1}^{M}\varphi_k=1$, and $0\le\varphi_k\le1,\ k=1,2,\cdots,M$.
2. $|D\varphi_k|\le CL^{-1}$, $|D^2\varphi_k|\le CL^{-2}$, $k=1,2,\cdots,M$.

Where C is a constant independent of M and the choice of subdomains.

For $v\in V_h$, define $\tilde v^H=I_HPQ_h^1v$, then v can be written by

$$v=\tilde v^H+\sum_{k=1}^{M}\varphi_k(v-\tilde v^H).$$

The interpolation operator of Wilson element, for $T\in\mathsf{T}_h$, is denoted by Π_T, and Π_h is the interpolation operator corresponding to T_h. $\Pi_h(\varphi_k(v-\tilde v^H))$ is well defined because $\varphi_k(v-\tilde v^H)$ is piecewise smooth. On the other hand, φ_k is sufficiently smooth, and $v-\tilde v^H$

is continuous at the vertices. Hence $\Pi_h(\varphi_k(v-\tilde{v}^H)) \in V_h$. $\Pi_h(\varphi_k(v-\tilde{v}^H)) \in E_k V_{h,k}$ since $\Pi_h(\varphi_k(v-\tilde{v}^H))|_{\overline{\Omega}-\Omega_k}=0$. A decomposition of v is obtained by

$$v=\tilde{v}^H+\sum_{k=1}^{M}\Pi_h(\varphi_k(v-\tilde{v}^H)). \tag{3.13}$$

Inequalities (2.1) and (2.9) and the interpolation property of Q_h^1 give

$$a_h(\tilde{v}^H,\tilde{v}^H)\le C|\tilde{v}^H|_{1,h}^2\le C|v|_{1,h}^2. \tag{3.14}$$

Let $k\in\{1,2,\cdots,M\}$, $T\in \mathsf{T}_h$ and $T\subset\overline{\Omega}_k$. Denote the center point of T by A_T^0. By inequality (2.3), one has

$$\begin{aligned}
|\Pi_h[(\varphi_k-\Pi_T^1\varphi_k)(v-\tilde{v}^H)]|_{0,T}^2 \le{}& C\Big\{\sum_{1\le i,j\le 4}|\varphi_k(v-\tilde{v}^H)(A_T^i)-\varphi_k(v-\tilde{v}^H)(A_T^j)|^2\\
&+h^4\Big(\Big|\frac{\partial^2(\varphi_k(v-\tilde{v}^H))}{\partial x^2}(A_T^0)\Big|^2+\Big|\frac{\partial^2(\varphi_k(v-\tilde{v}^H))}{\partial y^2}(A_T^0)\Big|^2\Big)\Big\}\\
\le{}& C\Big\{\sum_{1\le i,j\le 4}|\varphi_k(A_T^i)((v-\tilde{v}^H)(A_T^i)-(v-\tilde{v}^H)(A_T^j))|^2\\
&+\sum_{1\le i,j\le 4}|(\varphi_k(A_T^i)-\varphi(A_T^j))(v-\tilde{v}^H)(A_T^j))|^2\\
&+h^4\Big(\Big|\frac{\partial^2\varphi_k}{\partial x^2}(A_T^0)\Big|^2+\Big|\frac{\partial^2\varphi_k}{\partial y^2}(A_T^0)\Big|^2\Big)|(v-\tilde{v}^H)(A_T^0)|^2\\
&+h^4\Big(\Big|\frac{\partial\varphi_k}{\partial x}(A_T^0)\Big|^2\Big|\frac{\partial(v-\tilde{v}^H)}{\partial x}(A_T^0)\Big|^2+\Big|\frac{\partial\varphi_k}{\partial y}(A_T^0)\Big|^2\Big|\frac{\partial(v-\tilde{v}^H)}{\partial y}(A_T^0)\Big|^2\Big)\\
&+h^4\Big(\Big|\frac{\partial^2(v-\tilde{v}^H)}{\partial x^2}(A_T^0)\Big|^2+\Big|\frac{\partial^2(v-\tilde{v}^H)}{\partial y^2}(A_T^0)\Big|^2\Big)|\varphi(A_T^0)|^2\Big\}
\end{aligned}$$

Denote Sobolev maximum semi-norm by $|\cdot|_{m,\infty,T}$. The property of φ_k leads to

$$\begin{aligned}
|\Pi_h(\varphi_k(v-\tilde{v}^H))|_{1,T}^2 \le{}& C\{(h^2L^{-2}+h^4L^{-4})|v-\tilde{v}^H|_{0,\infty,T}^2\\
&+(h^2+h^4L^{-2})|v-\tilde{v}^H|_{1,\infty,T}^2+h^4|v-\tilde{v}^H|_{2,\infty,T}^2\}.
\end{aligned} \tag{3.15}$$

From the inverse inequality, one gets

$$\begin{aligned}
|\Pi_h(\varphi_k(v-\tilde{v}^H))|_{1,T}^2 \le{}& C\{(L^{-2}+h^2L^{-4})|v-\tilde{v}^H|_{0,T}^2\\
&+(1+h^2L^{-2})|v-\tilde{v}^H|_{1,T}^2\}.
\end{aligned} \tag{3.16}$$

Summing the above inequality for all $T\subset\overline{\Omega}_k$, one gets

$$|\Pi_h(\varphi_k(v-\tilde{v}^H))|_{1,h}^2=\sum_{T\in\mathsf{T}_h,T\subset\overline{\Omega}_k}|\Pi_h(\varphi_k(v-\tilde{v}^H))|_{1,T}^2$$

$$\leq C \sum_{T\in \mathsf{T}_h, T\subset\overline{\Omega}_k} \{(L^{-2}+h^2L^{-4})|v-\tilde{v}^H|_{0,T}^2 + (1+h^2L^{-2})|v-\tilde{v}^H|_{1,T}^2\}.$$

Summing the above inequality for all k, one has

$$\sum_{k=1}^{M} |\Pi_h(\varphi_k(v-\tilde{v}^H))|_{1,h}^2 \leq C\{(L^{-2}+h^2L^{-4})|v-\tilde{v}^H|_{0,h}^2 + (1+h^2L^{-2})|v-\tilde{v}^H|_{1,h}^2\}.$$

where the fact, that the numbers of subdomains containing each elements in T_h are bounded, has been used.

Estimates (2.9) and (2.1) and the interpolation property of Q_h^1 give that

$$\sum_{k=1}^{M} |\Pi_h(\varphi_k(v-\tilde{v}^H))|_{1,h}^2 \leq C\Big(1+\frac{H^2}{L^2}\Big)\Big(1+\frac{h^2}{L^2}\Big)|v|_{1,h}^2.$$

Since the norms $|\cdot|_{1,h}$ and $a_h(\cdot,\cdot)^{1/2}$ are equivalent, the above estimate and (3.14) lead to

$$a_h(\tilde{v}^H,\tilde{v}^H) + \sum_{k=1}^{M} a_h(\Pi_h(\varphi_k(v-\tilde{v}^H)), \Pi_h(\varphi_k(v-\tilde{v}^H))) \leq C\Big(1+\frac{H^2}{L^2}\Big)\Big(1+\frac{h^2}{L^2}\Big)a_h(v,v). \tag{3.17}$$

(3.10) follows.

The main theorem of the paper can be immediately obtained by (3.4), (3.5), (3.8) and (3.10).

Theorem. *For Wilson element and the operator I_H defined in section 2, there exists a constant C independent of H, h and the choice of subdomains, such that the condition number of $Q^{-1}A_h$ satisfies*

$$\mathrm{Cond}\,(Q^{-1}A_h) \leq C\Big(1+\frac{H^2}{L^2}\Big)\Big(1+\frac{h^2}{L^2}\Big) \tag{3.18}$$

Estimate (3.18) leads to that the condition number is independent of the scale of problem and the number of subdomains if $H = L$. So the optimal preconditioning is obtained.

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