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LOCAL ARTIFICIAL BOUNDARY CONDITIONS FOR THE INCOMPRESSIBLE VISCOUS FLOW IN A SLIP CHANNEL*1)

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Abstract

In this paper we consider numerical simulation of incompressible viscous flow in an infinite slip channel. Local artificial boundary conditions at an artificial boundary are derived by the continuity of velocity and normal stress at the segment artificial boundary. Then the original problem is reduced to a boundary value problem on a bounded computational domain. Numerical example shows that our artificial boundary conditions are very effective.

1. Introduction

Many boundary value problems of partial differential equations involving unbounded domain occur in many areas of applications, e. g., fluid flow around obstacles, coupling of structures with foundation and so on. For getting the numerical solutions of the problems on unbounded domian, a natural approach is to cut off an unbounded part of the domain by introducing an artificial boundary and set up an appropriate artificial boundary condition on the artificial boundary. Then the original problem is approximated by a problem on bounded domain.

In the last ten years, boundary value problems in an unbounded domain have been studied by many authors. For instance, Goldstein [1], Feng [2], Han and Wu [3,4], Hagstrom and Keller [5,6], Halpern [7], Halpern and Schatzman [8], Nataf [9], Han, Lu and Bao [10], Han and Bao [11,12] and others have studied how to design artificial boudnary conditions for partial differential equations in an unbounded domain. Among their results, two kinds of artificial boundary conditions are designed. One is nonlocal artificial boundary condition, the other is local artificial boundary condition. In engineering, they like to use the second type.

In this paper we design local artificial boundary conditions for Navier-Stokes (N-S) equations in an infinite slip channel. Then the original problem is reduced to a boundary value problem in a bounded domain. Moreover numerical example shows that the artificial boundary conditions given in this paper are very effective.

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2. Navier-Stokes Equations and Oseen Equations

Let Ω_i be an obstruction in a channel defined by $\Re \times (0, L)$ and $\Omega = \Re \times (0, L) \setminus \bar{\Omega}_i$. Consider the following Navier-Stokes equations

$$(u \cdot \nabla)u + \nabla p = \nu \triangle u, \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega, \tag{2.2}$$

with boundary conditions

$$u_2|_{x_2=0,L} = 0, \quad \sigma_{12}|_{x_2=0,L} \equiv \nu\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big)|_{x_2=0,L} = 0, \quad -\infty < x_1 < +\infty. \tag{2.3}$$

$$u|_{\partial\Omega_i} = 0, \tag{2.4}$$

$$u(x) \to u_\infty \equiv (\alpha, 0)^T, \quad \text{when } x_1 \to \pm\infty, \tag{2.5}$$

where $u = (u_1, u_2)^T$ is the velocity, p is the pressure, $\nu > 0$ is the kinematic viscosity, $x = (x_1, x_2)^T$ is coordinate, $\alpha > 0$ is a constant and σ_{12} is the tangential stress on the wall. Obviously condition (2.3) is equivalent to the following condition

$$\frac{\partial u_1}{\partial x_2}|_{x_2=0,L} = u_2|_{x_2=0,L} = 0, \quad -\infty < x_1 < +\infty. \tag{2.6}$$

Taking two constants $b < c$, such that $\Omega_i \subset (b, c) \times (0, L)$, then Ω is divided into three parts Ω_b, Ω_T and Ω_c by the artificial boundary $\Gamma_b = \{x \in \Re^2 \mid x_1 = b, 0 \le x_2 \le L\}$ and $\Gamma_c = \{x \in \Re^2 \mid x_1 = c,\ 0 \le x_2 \le L\}$ with

$$\Omega_b = \{x \in \Re^2 \mid -\infty < x_1 < b, \quad 0 < x_2 < L\},$$
$$\Omega_T = \{x \in \Re^2 \mid b < x_1 < c, \quad 0 < x_2 < L\} \setminus \overline{\Omega}_i,$$
$$\Omega_c = \{x \in \Re^2 \mid c < x_1 < +\infty, \quad 0 < x_2 < L\}.$$

When $|b|$ and c are sufficiently large, in the domain $\Omega_b \cup \Omega_c$ the velocity u is almost constant vector u_∞. So the N-S equations (2.1)–(2.2) can be linearized in domain Ω_c (and Ω_b), namely the solution (u, p) of problem (2.1)–(2.5) approximately satisfies the following problem

$$\alpha\frac{\partial u}{\partial x_1} + \nabla p = \nu \triangle u, \quad \text{in } \Omega_c, \tag{2.7}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega_c, \tag{2.8}$$

$$\frac{\partial u_1}{\partial x_2}|_{x_2=0,L} = u_2|_{x_2=0,L} = 0, \quad c \le x_1 < +\infty, \tag{2.9}$$

$$u(x) \to u_\infty = (\alpha, 0)^T, \quad \text{when } x_1 \to +\infty. \tag{2.10}$$

In [13], the author obtained general solution of the problem (2.7)–(2.10)

$$u_1(x) = \alpha + \sum_{m=1}^{\infty}\Big[a_m e^{-\frac{m\pi}{L}(x_1-c)} - \frac{m\pi}{L\lambda^-(m)} b_m e^{\lambda^-(m)(x_1-c)}\Big] \cos\frac{m\pi x_2}{L}, \tag{2.11}$$

$$u_2(x) = \sum_{m=1}^{\infty}\Big[a_m e^{-\frac{m\pi}{L}(x_1-c)} + b_m e^{\lambda^-(m)(x_1-c)}\Big] \sin\frac{m\pi x_2}{L}, \tag{2.12}$$

$$p(x) = -\alpha \sum_{m=1}^{\infty} a_m e^{-\frac{m\pi}{L}(x_1-c)} \cos\frac{m\pi x_2}{L}, \tag{2.13}$$

where

$$\lambda^-(m) = \frac{\alpha - \sqrt{\alpha^2 + 4\nu^2 m^2\pi^2/L^2}}{2\nu}, \quad m = 1, 2, \cdots$$

and $a_1,\ b_1,\ a_2,\ b_2,\ \cdots$ are any constants.

3. Local Artificial Boundary Conditions at Γ_c

Let $\varepsilon(u) = (\varepsilon_{ij}(u))$ and $\sigma(u,p) = (\sigma_{ij}(u,p))$ denote the rate of strain and stress tensors respectively. We have

$$\varepsilon_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i,j = 1,2, \tag{3.1}$$

and

$$\sigma_{ij}(u,p) = -p\delta_{ij} + 2\nu\varepsilon_{ij}(u), \quad i,j = 1,2, \tag{3.2}$$

where δ_{ij} is the Kronecker Delta whose properties are:

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

$\sigma_n = (\sigma_{n_1}, \sigma_{n_2})^T$ denote the normal stress on the artificial boundary Γ_c, then

$$\sigma_{n_1} = n_1\sigma_{11} + n_2\sigma_{12} = \sigma_{11} = \Big(-p + 2\nu\frac{\partial u_1}{\partial x_1}\Big)|_{\Gamma_c}, \tag{3.3}$$

$$\sigma_{n_2} = n_1\sigma_{21} + n_2\sigma_{22} = \sigma_{21} = \nu\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big)|_{\Gamma_c}, \tag{3.4}$$

where $n = (n_1, n_2)^T = (1, 0)^T$ is the outward normal vector on Γ_c.

We now use the transmission conditions

$$u(c^-, x_2) = u(c^+, x_2), \tag{3.5}$$

$$\sigma_n(c^-, x_2) = \sigma_n(c^+, x_2), \tag{3.6}$$

to obtain the local artificial boundary conditions at the artificial boundary Γ_c. Let $x_1 = c$ in (2.11)–(2.12), we obtain

$$u_1(x)|_{\Gamma_c} = \alpha + \sum_{m=1}^{\infty}\Big[a_m - \frac{m\pi}{L\lambda^-(m)}b_m\Big]\cos\frac{m\pi x_2}{L}, \tag{3.7}$$

$$u_2(x)|_{\Gamma_c} = \sum_{m=1}^{\infty}[a_m + b_m]\sin\frac{m\pi x_2}{L}. \tag{3.8}$$

Substituting (3.7)–(3.8) into (3.3)–(3.4), we have

$$\sigma_{n_1} = \sum_{m=1}^{\infty}\Big[\Big(\alpha - \frac{2\nu m\pi}{L}\Big)a_m - \frac{2\nu m\pi}{L}b_m\Big]\cos\frac{m\pi x_2}{L}, \tag{3.9}$$

$$\sigma_{n_2} = \nu\sum_{m=1}^{\infty}\Big[-\frac{2m\pi}{L}a_m + \Big(\lambda^-(m) + \frac{m^2\pi^2}{L^2\lambda^-(m)}\Big)b_m\Big]\sin\frac{m\pi x_2}{L}. \tag{3.10}$$

From the form of equalities (3.7)–(3.10), we assume σ_{n_1} and σ_{n_2} have the following form:

$$\sigma_{n_1} = \sum_{n=1}^{\infty} \Big[c_n \frac{\partial^{2(n-1)}(u_1(c,x_2)-\alpha)}{\partial x_2^{2(n-1)}} + d_n \frac{\partial^{2n-1} u_2(c,x_2)}{\partial x_2^{2n-1}}\Big], \tag{3.11}$$

$$\sigma_{n_2} = \sum_{n=1}^{\infty} \Big[e_n \frac{\partial^{2n-1}(u_1(c,x_2)-\alpha)}{\partial x_2^{2n-1}} + f_n \frac{\partial^{2(n-1)} u_2(c,x_2)}{\partial x_2^{2(n-1)}}\Big]. \tag{3.12}$$

Inserting (3.7)–(3.8) into (3.11)–(3.12), we obtain

$$\begin{aligned}
\sigma_{n_1} &= \sum_{n=1}^{\infty} \Big[(-1)^{n-1} c_n \sum_{m=1}^{\infty} \Big(a_m - \frac{m\pi}{L\lambda^-(m)} b_m\Big)\Big(\frac{m\pi}{L}\Big)^{2(n-1)} \\
&\quad + (-1)^{n-1} d_n \sum_{m=1}^{\infty} (a_m + b_m)\Big(\frac{m\pi}{L}\Big)^{2n-1}\Big] \cos\frac{m\pi x_2}{L} \\
&= \sum_{m=1}^{\infty} \Big[a_m \sum_{n=1}^{\infty} \Big(c_n + \frac{m\pi}{L} d_n\Big)(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)} \\
&\quad + b_m \sum_{n=1}^{\infty} \Big(\frac{m\pi}{L} d_n - \frac{m\pi}{L\lambda^-(m)} c_n\Big)(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)}\Big] \cos\frac{m\pi x_2}{L} \\
&= \sum_{m=1}^{\infty} \Big[\Big(\alpha - \frac{2\nu m\pi}{L}\Big) a_m - \frac{2\nu m\pi}{L} b_m\Big] \cos\frac{m\pi x_2}{L}, \\
\sigma_{n_2} &= \sum_{n=1}^{\infty} \Big[(-1)^{n} e_n \sum_{m=1}^{\infty} \Big(a_m - \frac{m\pi}{L\lambda^-(m)} b_m\Big)\Big(\frac{m\pi}{L}\Big)^{2n-1} \\
&\quad + (-1)^{n-1} f_n \sum_{m=1}^{\infty} (a_m + b_m)\Big(\frac{m\pi}{L}\Big)^{2(n-1)}\Big] \sin\frac{m\pi x_2}{L} \\
&= \sum_{m=1}^{\infty} \Big[a_m \sum_{n=1}^{\infty} \Big(f_n - \frac{m\pi}{L} e_n\Big)(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)} \\
&\quad + b_m \sum_{n=1}^{\infty} \Big(\frac{m^2\pi^2}{L^2\lambda^-(m)} e_n + f_n\Big)(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)}\Big] \sin\frac{m\pi x_2}{L} \\
&= \sum_{m=1}^{\infty} \Big[-\frac{2\nu m\pi}{L} a_m + \nu\Big(\lambda^-(m) + \frac{m^2\pi^2}{L^2\lambda^-(m)}\Big) b_m\Big] \sin\frac{m\pi x_2}{L}.
\end{aligned}$$

Thus, we have

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty} \Big(c_n + \frac{m\pi}{L} d_n\Big)(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)} = \alpha - \frac{2\nu m\pi}{L}, \\ \displaystyle\sum_{n=1}^{\infty} \Big(\frac{m\pi}{L} d_n - \frac{m\pi}{L\lambda^-(m)} c_n\Big)(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)} = -\frac{2\nu m\pi}{L}, \end{cases} \quad m = 1, 2, \cdots \tag{3.13}$$

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty} \Big(f_n - \frac{m\pi}{L} e_n\Big)(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)} = -\frac{2\nu m\pi}{L}, \\ \displaystyle\sum_{n=1}^{\infty} \Big(\frac{m^2\pi^2}{L^2\lambda^-(m)} e_n + f_n\Big)(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)} = \nu\Big(\lambda^-(m) + \frac{m^2\pi^2}{L^2\lambda^-(m)}\Big), \\ m = 1, 2, \cdots \end{cases}$$

(3.14)

We can derive c_n, d_n, e_n, f_n $(n = 1, 2, \cdots)$ from equalities (3.13)–(3.14). Then we obtain a local artificial boundary condition (3.11)–(3.12) at the artificial boundary Γ_c for the problem (2.1)–(2.5).

In the following, we consider the approximations of the local artificial boundary condition (3.11)–(3.12). Assume that

$$\sigma_{n_1} = \sum_{n=1}^{N} \Big[\tilde{c}_n^N \frac{\partial^{2(n-1)}(u_1(c,x_2) - \alpha)}{\partial x_2^{2(n-1)}} + \tilde{d}_n^N \frac{\partial^{2n-1} u_2(c,x_2)}{\partial x_2^{2n-1}}\Big] \equiv \sigma_{n_1}^N, \tag{3.15}$$

$$\sigma_{n_2} = \sum_{n=1}^{N} \Big[\tilde{e}_n^N \frac{\partial^{2n-1}(u_1(c,x_2) - \alpha)}{\partial x_2^{2n-1}} + \tilde{f}_n^N \frac{\partial^{2(n-1)} u_2(c,x_2)}{\partial x_2^{2(n-1)}}\Big] \equiv \sigma_{n_1}^N. \tag{3.16}$$

A computation shows that

$$\begin{cases} \displaystyle\sum_{n=1}^{N} \Big(\tilde{c}_n^N + \frac{m\pi}{L}\tilde{d}_n^N\Big)(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)} = \alpha - \frac{2\nu m\pi}{L}, \\ \displaystyle\sum_{n=1}^{N} \Big(\frac{m\pi}{L}\tilde{d}_n^N - \frac{m\pi}{L\lambda^-(m)}\tilde{c}_n^N\Big)(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)} = -\frac{2\nu m\pi}{L}, \quad m = 1, 2, \cdots, N \end{cases} \tag{3.17}$$

and

$$\begin{cases} \displaystyle\sum_{n=1}^{N} \Big(\tilde{f}_n^N - \frac{m\pi}{L}\tilde{e}_n^N\Big)(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)} = -\frac{2\nu m\pi}{L}, \\ \displaystyle\sum_{n=1}^{N} \Big(\frac{m^2\pi^2}{L^2\lambda^-(m)}\tilde{e}_n^N + \tilde{f}_n^N\Big)(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)} = \nu\Big(\lambda^-(m) + \frac{m^2\pi^2}{L^2\lambda^-(m)}\Big), \\ m = 1, 2, \cdots, N. \end{cases} \tag{3.18}$$

The equalities (3.17)–(3.18) are equivalent to the following

$$\sum_{n=1}^{N}(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)}\tilde{c}_n^N = \frac{\nu(L\lambda^-(m) - m\pi)}{L}, \quad m = 1, 2, \cdots, N, \tag{3.19}$$

$$\sum_{n=1}^{N}(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)}\tilde{d}_n^N = -\nu - \frac{\nu m\pi}{L\lambda^-(m)}, \quad m = 1, 2, \cdots, N, \tag{3.20}$$

$$\sum_{n=1}^{N}(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)}\tilde{e}_n^N = \nu + \frac{\nu L\lambda^-(m)}{m\pi}, \quad m = 1, 2, \cdots, N, \tag{3.21}$$

$$\sum_{n=1}^{N}(-1)^{n-1}\Big(\frac{m\pi}{L}\Big)^{2(n-1)}\tilde{f}_n^N = \frac{\nu(L\lambda^-(m) - m\pi)}{L}, \quad m = 1, 2, \cdots, N. \tag{3.22}$$

Obviously, equalities (3.19)–(3.22) have a unique solution. Thus the equalities (3.17) and (3.18) have a unique solution.

For $N = 1, 2, 3$, we have the following approximate local artificial boundary conditions

$$N = 1$$

$$\sigma_{n_1}^1 = \Big[\nu\lambda^-(1) - \frac{\nu\pi}{L}\Big](u_1(c, x_2) - \alpha) + \Big[-\nu - \frac{\nu\pi}{L\lambda^-(1)}\Big]\frac{\partial u_2(c, x_2)}{\partial x_2}, \tag{3.23}$$

$$\sigma_{n_2}^1 = \Big[\nu + \frac{\nu L\lambda^-(1)}{\pi}\Big]\frac{\partial u_1(c, x_2)}{\partial x_2} + \Big[\nu\lambda^-(1) - \frac{\nu\pi}{L}\Big]u_2(c, x_2), \tag{3.24}$$

$$N = 2$$

$$\begin{aligned}\sigma_{n_1}^2 =& \Big[\frac{4\nu\lambda^-(1)}{3} - \frac{\nu\lambda^-(2)}{3} - \frac{2\nu\pi}{3L}\Big](u_1(c, x_2) - \alpha)\\ &+ \Big[\frac{\nu L^2\lambda^-(1)}{3\pi^2} - \frac{\nu L^2\lambda^-(2)}{3\pi^2} + \frac{\nu L}{3\pi}\Big]\frac{\partial^2 u_1(c, x_2)}{\partial {x_2}^2}\\ &+ \Big[-\nu - \frac{4\nu\pi}{3L\lambda^-(1)} + \frac{2\nu\pi}{3L\lambda^-(2)}\Big]\frac{\partial u_2(c, x_2)}{\partial x_2} + \Big[\frac{2\nu L}{3\pi\lambda^-(2)} - \frac{\nu L}{3\pi\lambda^-(1)}\Big]\frac{\partial^3 u_2(c, x_2)}{\partial {x_2}^3},\end{aligned} \tag{3.25}$$

$$\begin{aligned}\sigma_{n_2}^2 =& \Big[\nu + \frac{4\nu L\lambda^-(1)}{3\pi} - \frac{\nu L\lambda^-(2)}{6\pi}\Big]\frac{\partial u_1(c, x_2)}{\partial x_2} + \Big[\frac{\nu L^3\lambda^-(1)}{3\pi^2} - \frac{\nu L^3\lambda^-(2)}{6\pi^3}\Big]\frac{\partial^3 u_1(c, x_2)}{\partial {x_2}^3}\\ &+ \Big[\frac{4\nu\lambda^-(1)}{3} - \frac{\nu\lambda^-(2)}{3} - \frac{2\nu\pi}{3L}\Big]u_2(c, x_2)\\ &+ \Big[\frac{\nu L^2\lambda^-(1)}{3\pi^2} - \frac{\nu L^2\lambda^-(2)}{3\pi^2} + \frac{\nu L}{3\pi}\Big]\frac{\partial^2 u_2(c, x_2)}{\partial {x_2}^2}.\end{aligned} \tag{3.26}$$

$$N = 3$$

$$\begin{aligned}\sigma_{n_1}^3 =& \Big[\frac{3\nu\lambda^-(1)}{2} - \frac{3\nu\lambda^-(2)}{5} + \frac{\nu\lambda^-(3)}{10} - \frac{3\nu\pi}{5L}\Big](u_1(c, x_2) - \alpha)\\ &+ \Big[\frac{13\nu L^2\lambda^-(1)}{24\pi^2} - \frac{2\nu L^2\lambda^-(2)}{3\pi^2} + \frac{\nu L^2\lambda^-(3)}{8\pi^2} + \frac{5\nu L}{12\pi}\Big]\frac{\partial^2 u_1(c, x_2)}{\partial {x_2}^2}\\ &+ \Big[\frac{\nu L^4\lambda^-(1)}{24\pi^4} - \frac{\nu L^4\lambda^-(2)}{15\pi^4} + \frac{\nu L^4\lambda^-(3)}{40\pi^4} + \frac{\nu L^3}{60\pi^3}\Big]\frac{\partial^4 u_1(c, x_2)}{\partial {x_2}^4}\\ &+ \Big[-\nu - \frac{3\nu\pi}{2L\lambda^-(1)} + \frac{6\nu\pi}{5L\lambda^-(2)} - \frac{3\nu\pi}{10L\lambda^-(3)}\Big]\frac{\partial u_2(c, x_2)}{\partial x_2}\\ &+ \Big[-\frac{13\nu L}{24\pi\lambda^-(1)} + \frac{4\nu L}{3\pi\lambda^-(2)} - \frac{3\nu L}{8\pi\lambda^-(3)}\Big]\frac{\partial^3 u_2(c, x_2)}{\partial {x_2}^3}\\ &+ \Big[-\frac{\nu L^3}{24\pi^3\lambda^-(1)} + \frac{2\nu L^3}{15\pi^3\lambda^-(2)} - \frac{3\nu L^3}{40\pi^3\lambda^-(3)}\Big]\frac{\partial^5 u_2(c, x_2)}{\partial {x_2}^5},\end{aligned} \tag{3.27}$$

$$\begin{aligned}\sigma_{n_2}^3 =& \Big[\nu + \frac{3\nu L\lambda^-(1)}{2\pi} - \frac{3\nu L\lambda^-(2)}{10\pi} + \frac{\nu L\lambda^-(3)}{30\pi}\Big]\frac{\partial u_1(c, x_2)}{\partial x_2}\\ &+ \Big[\frac{13\nu L^3\lambda^-(1)}{24\pi^2} - \frac{\nu L^3\lambda^-(2)}{3\pi^3} + \frac{\nu L^3\lambda^-(3)}{24\pi^3}\Big]\frac{\partial^3 u_1(c, x_2)}{\partial {x_2}^3}\\ &+ \Big[\frac{\nu L^5\lambda^-(1)}{24\pi^5} - \frac{\nu L^5\lambda^-(2)}{30\pi^5} + \frac{\nu L^5\lambda^-(3)}{120\pi^5}\Big]\frac{\partial^5 u_1(c, x_2)}{\partial {x_2}^5}\end{aligned}$$

$$+\Big[\frac{3\nu\lambda^-(1)}{2}-\frac{3\nu\lambda^-(2)}{5}+\frac{\nu\lambda^-(3)}{10}-\frac{3\nu\pi}{5L}\Big]u_2(c,x_2)$$
$$+\Big[\frac{13\nu L^2\lambda^-(1)}{24\pi^2}-\frac{2\nu L^2\lambda^-(2)}{3\pi^2}+\frac{\nu L^2\lambda^-(3)}{8\pi^2}+\frac{5\nu L}{12\pi}\Big]\frac{\partial^2 u_2(c,x_2)}{\partial {x_2}^2}$$
$$+\Big[\frac{\nu L^4\lambda^-(1)}{24\pi^4}-\frac{\nu L^4\lambda^-(2)}{15\pi^4}+\frac{\nu L^4\lambda^-(3)}{40\pi^4}+\frac{\nu L^3}{60\pi^3}\Big]\frac{\partial^4 u_2(c,x_2)}{\partial {x_2}^4}. \tag{3.28}$$

In a similar way, we can design approximate local artificial boundary conditions at the artificial boundary Γ_b.

Therefore the problem (2.1)–(2.5) can be approximated by the following problem for different N.

$$(u\cdot\nabla)u+\nabla p=\nu\triangle u,\quad \text{in }\Omega_T, \tag{3.29}$$
$$\nabla\cdot u=0,\quad \text{in }\Omega_T, \tag{3.30}$$
$$\frac{\partial u_1}{\partial x_2}\Big|_{x_2=0,L}=u_2|_{x_2=0,L}=0,\quad b\le x_1\le c, \tag{3.31}$$
$$u|_{\partial\Omega_i}=0, \tag{3.32}$$
$$u|_{\Gamma_b}=u_\infty, \tag{3.33}$$
$$\sigma_n=\begin{pmatrix}\sigma_{n_1}\\ \sigma_{n_2}\end{pmatrix}=\begin{pmatrix}\sigma^N_{n_1}\\ \sigma^N_{n_2}\end{pmatrix},\quad \text{on }\Gamma_c. \tag{3.34}$$

where $\sigma^0_{n_1}=\sigma^0_{n_2}=0$.

4. Numerical Implementation and Example

In this section we use finite element method to solve the problem (3.29)–(3.34). Let $H^m(\Omega_T)$ denote the usual sobolev space on the domain Ω_T with integer m. Furthermore let $\Gamma_1=\{x\in\Re^2\mid x_2=0,\ b\le x_1\le c\}\cup\{x\in\Re^2|\ x_2=L,\ b\le x_1\le c\}$, $\Gamma_i=\partial\Omega_i$, $V=\{u\in H^1(\Omega_T)\times H^1(\Omega_T)\mid u|_{\Gamma_b\cup\Gamma_i}=0,\ u_2|_{\Gamma_1}=0\}$ with norm $\|u\|_V^2=\|u_1\|_{1,2,\Omega_T}+\|u_2\|_{1,2,\Omega_T}$, $W=L^2(\Omega_T)$ with norm $\|q\|_W=\|q\|_{L^2(\Omega_T)}$ and $M=\{u\in H^1(\Omega_T)\times H^1(\Omega_T)\mid u|_{\Gamma_i}=0,\ u|_{\Gamma_b}=u_\infty,\ u_2|_{\Gamma_2}=0\}$.

For the sake of simplicity, Let Π_h be a rectangle partition of Ω_T, with $\Omega_T=\cup_{K\in\Pi_h}K$, where K is a rectangle.

For each rectangle $K\in\Pi_h$, connect the mid-points of the opposite sides of K, then each rectangle K is divided into four smaller rectangles. Let $\Pi_{\hat h}$ denote this new partition. Let $V_h=\{v\in V\mid v|_K$ is a bilinear polynomial, $\forall K\in\Pi_{\hat h}\}$, $W_h=\{p\in W\mid p|_K$ is constant, $\forall K\in\Pi_h\}$, $M_h=\{v\in M\mid v|_K$ is a bilinear polynomial, $\forall K\in\Pi_{\hat h}\}$. Then V_h and W_h satisfy the Babuŝka-Brezzi (B-B) condition [14] and the following approximation property [14] $\inf_{v\in V_h}\|u-v\|_V\le Ch|u|_{2,2,\Omega_T}$, $\inf_{q\in W_h}\|p-q\|_V\le Ch|p|_{1,2,\Omega_T}$. We use this finite element method to solve the following example.

Example. Consider the fluid flow in a horizontal channel with a rectangle cylinder obstacle.

The obstacle Ω_i is defined by the domain

$$\Omega_i=\Big\{x\in\Re^2\mid 0.8<x_1<1.2,\quad \frac{2L}{5}<x_2<\frac{3L}{5}\Big\}.$$

Then the bounded computational domain Ω_T is given by

$$\Omega_T = \{x \in \Re^2 \mid b < x_1 < c, \quad 0 < x_2 < L\} \setminus \bar{\Omega}_i.$$

We take $b = 0$, $c = 2.8$, $L = 1.0$, $\alpha = 1.0$. The nonlinear term $(u \cdot \nabla)u$ is linearized by Newton method. At every iterative step, we use the finite element method to solve a linear problem. Two meshes are used in computation. The partition Π_h for mesh A is the same as that used in [13, p57]. Mesh B is generated by divided each rectangle in mesh A into four equal smaller rectangles. Let (u^h_∞, p^h_∞) denote the solution, which is under the mesh Π_h, of the problem (3.29)-(3.33) and the exact artificial boundary condition at Γ_c derived in [13]. Tables 1-6 show the maximum errors of $u^h_\infty - u^h_N$ and $p^h_\infty - p^h_N$ for mesh A and B with different kinematic viscosity ν.

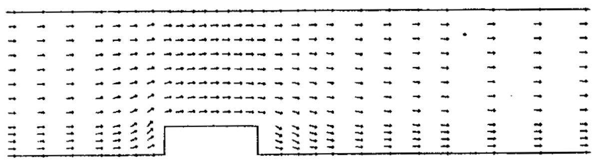

Fig. 1. Velocity field ($Re = 20$)

Table 1. $\nu = 0.05$, Mesh A

errors	$\max\|u^h_{1\infty} - u^h_{1N}\|$	$\max\|u^h_{2\infty} - u^h_{2N}\|$	$\max\|p^h_\infty - p^h_N\|$
N=0	1.2671E-2	1.1950E-2	4.2973E-3
N=1	2.5927E-3	1.2078E-3	1.9079E-3

Table 2. $\nu = 0.02$, Mesh A

errors	$\max\|u^h_{1\infty} - u^h_{1N}\|$	$\max\|u^h_{2\infty} - u^h_{2N}\|$	$\max\|p^h_\infty - p^h_N\|$
N=0	2.8978E-2	2.8099E-2	7.8807E-3
N=1	4.7097E-3	2.3229E-3	3.0410E-3

Table 3. $\nu = 0.01$, Mesh A

errors	$\max\|u^h_{1\infty} - u^h_{1N}\|$	$\max\|u^h_{2\infty} - u^h_{2N}\|$	$\max\|p^h_\infty - p^h_N\|$
N=0	3.2707E-2	2.6642E-2	8.8190E-3
N=1	6.2699E-3	1.4701E-3	2.0011E-3

Table 4. $\nu = 0.05$, Mesh B

errors	$\max\|u^h_{1\infty} - u^h_{1N}\|$	$\max\|u^h_{2\infty} - u^h_{2N}\|$	$\max\|p^h_\infty - p^h_N\|$
N=0	1.0687E-2	1.1811E-2	5.5930E-3
N=1	2.5041E-4	1.0233E-3	4.1667E-3

Table 5. $\nu = 0.02$, Mesh B

errors	$\max\|u^h_{1\infty} - u^h_{1N}\|$	$\max\|u^h_{2\infty} - u^h_{2N}\|$	$\max\|p^h_\infty - p^h_N\|$
N=0	1.2586E-2	2.5316E-2	6.3328E-3
N=1	6.6883E-4	3.1055E-4	6.2051E-4

Table 6. $\nu = 0.01$, Mesh B

errors	$\max\|u^h_{1\infty} - u^h_{1N}\|$	$\max\|u^h_{2\infty} - u^h_{2N}\|$	$\max\|p^h_\infty - p^h_N\|$
N=0	1.3709E-2	2.5785E-2	6.1071E-3
N=1	3.8312E-3	9.7455E-4	2.9904E-4

Furthermore Figure 1 shows the velocity field for mesh B with $\nu = 0.05$. Figures 2-3 show the velocity for mesh B with $\nu = 0.05$. Figures 4-6 show the errors $u^h_\infty - u^h_N$ and $p^h_\infty - p^h_N$ at outflow boundary Γ_c for mesh B with $\nu = 0.01$.

The example shows that the local artificial boundary conditions presented in this paper are very effective.

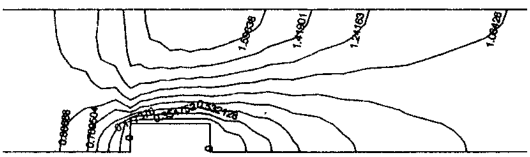

Fig. 2. $u_1(Re = 20)$

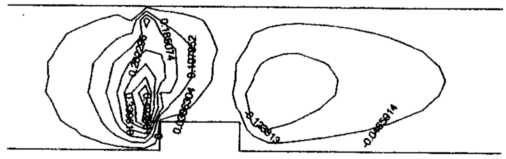

Fig. 3. $u_2(Re = 20)$

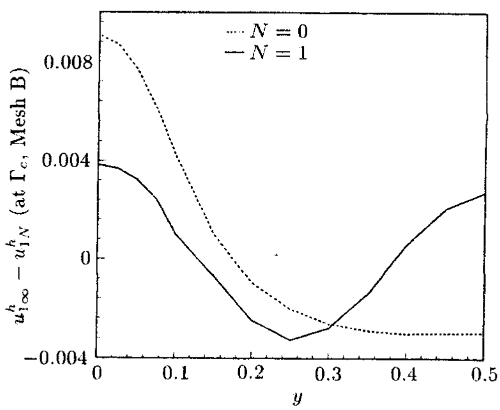

Fig. 4 $Re = 100, c = 2.8$

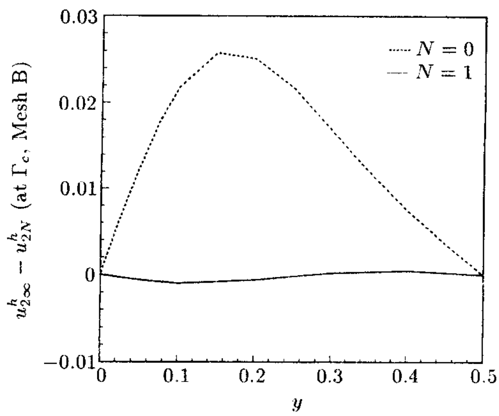

Fig. 5 $Re = 100, c = 2.8$

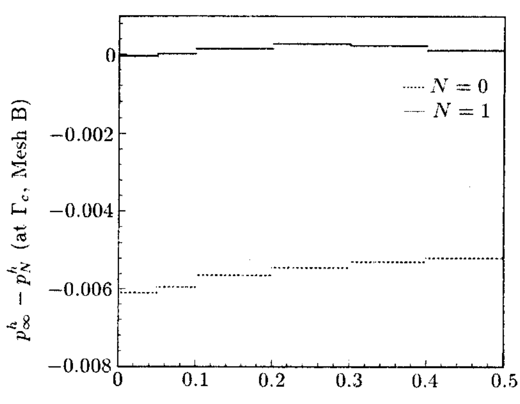

Fig. 6. $Re = 100, c = 2.8$

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