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NUMERICAL ANALYSIS OF NONSTATIONARY THERMISTOR PROBLEM*1)

Yue Xing-ye
(*Suzhou University, Suzhou, Jiangsu, China*)

Abstract

The thermistor problem is a coupled system of nonlinear PDEs which consists of the heat equation with the Joule heating as a source, and the current conservation equation with temperature dependent electrical conductivity. In this paper we make a numerical analysis of the nonsteady thermistor problem. $L^\infty(\Omega)$, $W^{1,\infty}(\Omega)$ stability and error bounds for a piecewise linear finite element approximation are given.

1. A Mathematical Model and a Discrete Scheme

The model of a nonstationary thermistor problem is derived from the conservation laws of current and energy (see [1] [2] [3]):

Find a pair $\{\varphi, u\}$ such that

$$\nabla\cdot(\sigma(u)\nabla\varphi)=0 \qquad \text{in} \quad Q_T=\Omega\times(0,T), \tag{1.1}$$

$$\varphi=\varphi_\partial \qquad \text{on} \quad \partial\Omega\times(0,T), \tag{1.2}$$

$$u_t-\Delta u=\sigma(u)\mid\nabla\varphi\mid^2 \quad \text{in} \quad Q_T, \tag{1.3}$$

$$u=0 \qquad \text{on} \quad \partial\Omega\times(0,T), \tag{1.5}$$

$$u(x,0)=u_0(x) \qquad \text{in} \quad \Omega \tag{1.5}$$

where $\Omega\subset\mathbf{R}^N(N\geq 1)$ is a bounded domain, occupied by the thermistor; $\varphi=\varphi(x,t)$, $u=u(x,t)$ are distributions of the electrical potential and the temperature in Ω, respectively; $\sigma(u)$ is the temperature dependent electrical conductivity; $\sigma(u)\mid\nabla\varphi\mid^2$ is the Joule heating. Throughout this paper, we assume that $0<\sigma_1\leq\sigma(s)\leq\sigma_2<+\infty \quad \forall s, \in\mathbf{R}^1$.

There has been interest in the problem mathematically (see [1] [2] [3]) and references therein recent mathematical. Yuan [3] proved the following result.

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Theorem 1. *If* $\varphi_\partial \in L^\infty(0,T;C^{1+\alpha}(\bar\Omega))$, $u_0 \in C^\alpha(\bar\Omega) \cap H_0^1(\Omega)$, $0 < \alpha < 1$, $\sigma(s) \in C^\alpha(\mathbf{R}^1)$, *then problem* (1.1)–(1.5) *has a uniqe solution* (φ, u) *satisfying*

$$u \in C^{\beta,\frac{\beta}{2}}(\bar Q_T), \varphi \in L^\infty(0,T;C^{1+\beta}(\bar\Omega))$$

and

$$\|u\|_{C^{\beta,\frac{\beta}{2}}(\bar Q_T)} \le C, \|\varphi\|_{L^\infty(0,T;C^{1+\beta}(\bar\Omega))} \le C$$

where $\beta \in (0,\alpha)$, *and* C *depends only on the given data.*

As a corollary, we have

Theorem 2. *Under the conditions of Theorem 1 and* $\sigma(s) \in C^1(\mathbf{R}^1)$, $\varphi_\partial \in L^\infty(0,T;H^2(\Omega))$,

(1) *If* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, *then*

$$u \in W_P^{2,1}(Q_T), \quad \forall\, 2 \le P < +\infty; \quad \varphi \in L^\infty(0,T;H^2(\Omega)) \tag{1.6}$$

(2) *If* $u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$, *then*

$$u_t \in W_P^{1,0}(Q_T), \ \forall 2 \le P < +\infty; \quad u \in L^\infty(0,T;H^2(\Omega)) \tag{1.7}$$

Proof. (1) From Theorem 1, $\sigma(u) \mid \nabla\varphi \mid^2 \in L^\infty(Q_T)$. By the standard parabolic estimate[7], (1.3) gives

$$u \in W_P^{2,1}(Q_T), \quad \forall 2 \le P < +\infty.$$

Furthermore, by the Corollary in [7]

$$\exists \gamma \in (0,1), \quad u_{x_i} \in C^\gamma(\bar Q_T), \quad i = 1,2,\cdots,N$$

Therefore, $\sigma(u) \in C^0(0,T;C^1(\bar\Omega))$.

By the standard elliptic estimate, from (1.1) we get

$$\varphi \in L^\infty(0,T;H^2(\Omega))$$

(2) From (1.3), we gain

$$u_{x_i,t} - \triangle u_{x_i} = \sigma'(u)u_{x_i} \mid \nabla\varphi \mid^2 + 2\sigma(u)\nabla\varphi \cdot \nabla\varphi_i \in L^\infty(Q_T).$$

It follows that

$$u_{x_i} \in W_P^{2,1}(Q_T), \ \forall\, 2 \le P < +\infty. \tag{1.8}$$

Hence, $u_t \in W_P^{1,0}(Q_T)$, $\forall\, 2 \le P < +\infty$.

On the other hand, by the embedding theory, we again have

$$\exists \gamma' \in (0,1), u_{x_ix_j} \in C^{\gamma'}(\bar Q_T), \ i,j = 1,2,\cdots,N.$$

Now the theorem is proved.

Problem (1.1)–(1.5) has a weak form as follows, Find $u \in H_0^1(\Omega)$, $\varphi \in \varphi_\partial + H_0^1(\Omega)$, such that

$$(\sigma(u)\nabla\varphi, \nabla\psi) = 0, \quad t \in (0,T), \quad \forall \psi \in H_0^1(\Omega), \tag{1.9}$$

$$(u_t, v) + (\nabla u, \nabla v) = (\sigma(u) \mid \nabla\varphi \mid^2, v) \quad t \in (0,T), \quad \forall v \in H_0^1(\Omega), \tag{1.10}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.11}$$

From (1.1), (1.3) can be changed into

$$u_t - \Delta u = \nabla \cdot (\sigma(u)\varphi\nabla\varphi).$$

Therefore, we can define another weak form as follows: Find $u \in H_0^1(\Omega)$, $\varphi \in \varphi_\partial + H_0^1(\Omega)$, such that

$$(\sigma(u)\nabla\varphi, \nabla\psi), \quad = \quad 0 \qquad \forall\psi \in H_0^1(\Omega), \tag{1.12}$$

$$(u_t, v) + (\nabla u, \nabla v) + (\sigma(u)\varphi\nabla\varphi, \nabla v) = 0, \quad \forall v \in H_0^1(\Omega), \tag{1.13}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega. \tag{1.14}$$

Now we consider the finite element approximation to problem (1.9)-(1.11). Assume that Ω is a polygonal domain in $\mathbf{R}^2$ only for simplicity. A triangle is employed as the element in the discretization mesh.

The semi-discrete scheme is defined as follows: Find $u_h \in S_h^0$, $\varphi_h \in \varphi_\partial^h + S_h^0$, such that

$$(\sigma(u_h)\nabla\varphi_h, \nabla\psi) = 0, \qquad \forall\psi \in S_h^0, \tag{1.15}$$

$$(u_{h,t}, v) + (\nabla u_h, \nabla v) = (\sigma(u_h) \mid \nabla\varphi_h \mid^2, v), \quad \forall v \in S_h^0, \tag{1.16}$$

$$u_h(x,0) = R_h u_0(x), \tag{1.17}$$

where $\varphi_\partial^h = I_h\varphi_\partial$, with I_h the linear interpolation operator from $H^1(\Omega)$ to S_h, and R_h is the Ritz projection $H_0^1(\Omega) \to S_h^0$. For $u \in H_0^1(\Omega)$, $R_h u \in S_h^0$ satisfies

$$(\nabla(R_h u - u), \nabla v) = 0, \quad \forall v \in S_h^0 \tag{1.18}$$

where

$$S_h = \{v(x) \in H^1(\Omega) : v(x) \text{ is a continuous piecewise linear function}\},$$

$$S_h^0 = \{v(x) \in S_h : v(x) = 0 \ \text{ on } \ \partial\Omega\}.$$

As to the approximation problem (1.15)-(1.17), we are ready to have

Lemma 1. *If (φ_h, u_h) is a solution of problem* (1.15)–(1.17), *then* $\|\varphi_h\|_1 \le C$, *where C is independent of h and t.*

We also have

Theorem 3. *There exists a unique pair (φ_h, u_h) satisfying* (1.15)–(1.17).

The existence is obtained from Schauder's Fixed Point Theorem and the equivalence of any norms in the finite dimensional space S_h^0.

As a consequence of $W^{1,\infty}(\Omega)$ stability of φ_h, the uniqueness will be given in Section 3.

In this paper, $\|\cdot\|$, $\|\cdot\|_k$ and $\|\cdot\|_{k,p}$ denote the $L^2(\Omega)$-, $H^k(\Omega)$- and $W^{k,p}(\Omega)$-norm respectively.

2. $L^\infty(\Omega)$ and $W^{1,\infty}(\Omega)$ Stability of φ_h

Just as in the continuous case, it is difficult to deal with the nonlinear term $\sigma(u_h) \mid \nabla\varphi_h \mid^2$, which is why we have to know more information on φ_h than Lemma 1 can show. In this section, we examine the $L^\infty(\Omega)$ and $W^{1,\infty}(\Omega)$ stability of φ_h, which plays an essential role in the error estimation. The $L^\infty(\Omega)$ stability comes from the discrete maximum principle of (1.15). Just as for the simple Laplacian, we have to place some restrictions on the regularity of the mesh.

Here we assume

(A1): All triangles in the mesh are acute-angled or right-angled. (2.1)

Lemma 2[5]. *Let T be a triangular element; j, k be its two vertices, h_j, h_k be the normal distance of vertexes j, k to their opposite edges respectively; γ_{jk} be the angle between the inward normal vectors of the two edges which are opposite to the nodes j, k respectively. Then*

$$\int_T \sigma(u_h)\nabla v_j \cdot \nabla v_k = \cos(\gamma_{jk})\frac{1}{h_j h_k}\int_T \sigma(u_h) \tag{2.2}$$

where v_j, v_k are the node bases of S_h at nodes j, k.

Now we can prove the following discrete maximum principle.

Theorem 4. *Under assumption* (A1), *if* (φ_h, u_h) *solves* (1.15)–(1.17), *then*

$$\min_{x\in\partial\Omega} \varphi_\partial^h \le \varphi_h \le \max_{x\in\partial\Omega}\varphi_\partial^h, \qquad \forall x \in \Omega. \tag{2.3}$$

Therefore $\|\varphi_h\|_{L^\infty(0,T;L^\infty(\Omega))} \le C$, where C is independent of h.

Proof. We will check only the right-hand inequality.

We will prove that the maximum must occur at the boundary. Otherwise, the maximum takes place only at some inner node k, so we have

$$\max_{x\in\Omega}\varphi_h = \varphi_h^k$$

where φ_h^i is a value of φ_h at node i.

Now in (1.15), replacing ψ with the node basis v_k yields

$$(\sigma(u_h)\nabla\varphi_h, \nabla v_k) = 0, \tag{2.4}$$

for φ_h can be formed as $\varphi_h = \sum_j \varphi_h^j v_j(x)$.

(2.4) can be rewritten as follows:

$$\int_\Omega \sigma(u_h)\sum_j \varphi_h^j \nabla v_j \cdot \nabla v_k = 0.$$

Employing the equivalent formula: $\sum_j v_j(x) \equiv 1$, we have

$$\int_\Omega \sigma(u_h)\sum_j (\varphi_h^j - \varphi_h^k)\nabla v_j \cdot \nabla v_k = 0.$$

Noting the properties of linear node basises, it follows that

$$\sum_{j \text{ adj } k} \alpha_{kj}(\varphi_h^j - \varphi_h^k) = 0, \tag{2.5}$$

where 'a adj b' means 'a is adjacent to b',

$$\alpha_{kj} = \sum_{T \text{ adj } kj} \int_T \sigma(u_h)\nabla v_j \cdot \nabla v_k,$$

and kj is the edge connecting vertices j, k.

Under (A1), using Lemma 2, we can see that

$$\alpha_{kj} \le 0, \quad \forall\, j \text{ adjacent to } k. \tag{2.6}$$

Therefore, $\alpha_{kj} < 0$ implies $\varphi_h^j = \varphi_h^k$.

But $\alpha_{kj} = 0$ if and only if both of the two inner angles, which belong respectively to the two triangles adjacent to edge kj and opposite to the common edge, are right-angled.

See Figure 1, where $\alpha_{kj} = 0$, and $\alpha_{km} < 0$, so we have $\varphi_h^m = \varphi_h^k$. That is to say, m is also a node at which the maximum occurs. If m lies on the boundary, then we get a contradiction, which proves the theorem. If m is also an inner node, repeating the preceding procedure, we certainly have $\varphi_h^j = \varphi_h^m$, and hence, $\varphi_h^j = \varphi_h^k$. Now we come to

$$\varphi_h^n = \varphi_h^k \qquad \forall\, n \text{ adjacent to } k \tag{2.7}$$

The last formula implies that $\varphi_h \equiv \varphi_h^k$, $\forall\, x \in \Omega$. This is also a contradiction. So (2.3) is again proved.

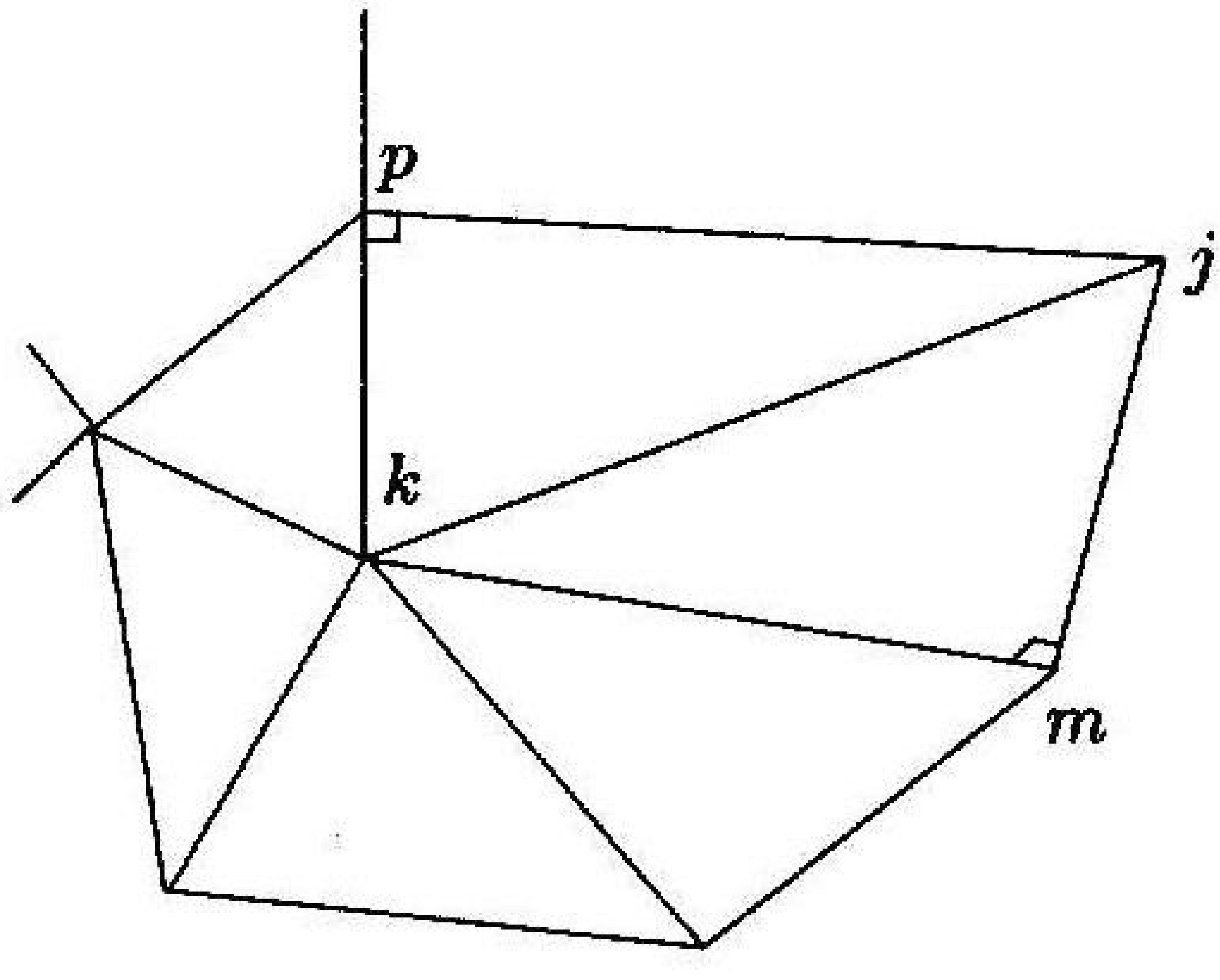

Fig.1

Now we begin to examine the $W^{1,\infty}(\Omega)$ stability of φ_h. First, we give some further restrictions on the grid.

(A2)[6] The mesh is *regular*, and satisfies the inverse assumption.

Set

$$e = u - u_h = (u - R_h u) + (R_h u - u_h) = \rho + \theta, \tag{2.8}$$

$$\tilde{e} = \varphi - \varphi_h = (\varphi - r_h \varphi) + (r_h \varphi - \varphi_h) = \eta + \xi, \tag{2.9}$$

where R_h is the Ritz-projection defined in (1.18), and r_h is an operator from $\varphi_\partial + H_0^1(\Omega)$ to $\varphi_\partial^h + S_h^0$, defined as follows: For $\varphi \in \varphi_\partial + H_0^1(\Omega)$, $r_h \varphi \in \varphi_\partial^h + S_h^0$, such that

$$(\sigma(u)\nabla(\varphi - r_h\varphi), \nabla v) = 0, \qquad \forall v \in S_h^0. \tag{2.10}$$

Obviously, $\theta, \xi \in S_h^0$, and it is well known that

$$\|\rho\|_s \le c\, h^{2-s}, \quad \|\eta\|_s \le c\, h^{2-s} \qquad s = 0, 1. \tag{2.11}$$

Using (1.15), we can rewrite the right-hand term of (1.16) as follows:

$$\begin{aligned}(\sigma(u_h) \mid \nabla\varphi_h \mid^2, v) &= (\sigma(u_h)\nabla\varphi_h, v\nabla\varphi_h) = -(\sigma(u_h)\varphi_h\nabla\varphi_h, \nabla v) \\ &+ (\sigma(u_h)\nabla\varphi_h, \nabla(\varphi_h v)) = -(\sigma(u_h)\varphi_h\nabla\varphi_h, \nabla v) \\ &+ (\sigma(u_h)\nabla\varphi_h, \nabla(\varphi_h v - I_h(\varphi_h v)), \quad \forall\, v \in S_h^0 \end{aligned} \tag{2.12}$$

where $\varphi_h v \in H_0^1(\Omega)$.

By (1.13) and (1.16),

$$\begin{aligned}&(e_t, v) + (\nabla(u - u_h), \nabla v) = -(\sigma(u)\varphi\nabla\varphi - \sigma(u_h)\varphi_h\nabla\varphi_h, \nabla v) \\ &\quad - (\sigma(u_h)\nabla\varphi_h, \nabla(\varphi_h v - I_h(\varphi_h v))), \qquad \forall v \in S_h^0 \\ &(\theta_t, v) + (\nabla\theta, \nabla v) = -((\sigma(u) - \sigma(u_h))\varphi\nabla\varphi, \nabla v) - (\sigma(u_h)(\varphi - \varphi_h)\nabla\varphi, \nabla v) \\ &\quad - (\sigma(u_h)\varphi_h\nabla(\varphi - \varphi_h), \nabla v) - (\sigma(u_h)\nabla\varphi_h, \nabla(\varphi_h v - I_h(\varphi_h v))) \\ &\quad - (\rho_t, v), \quad \forall v \in S_h^0, \\ &\theta(0) = 0\end{aligned}$$

Observe that φ_h and v are both piecewise linear functions, and

$$\begin{aligned}\int_\Omega \mid \nabla(\varphi_h v - I_h(\varphi_h v)) \mid &= \sum_T \int_T \mid \nabla(\varphi_h v - I_h(\varphi_h v)) \mid \le ch \sum_T \int_T \mid \partial_{i,j}(\varphi_h v) \mid \\ &= ch \sum_T \int_T \mid \partial_i\varphi_h\ \partial_j v \mid = ch \int_\Omega \mid \partial_i\varphi_h\partial_j v \mid \\ &\le ch\|\nabla\varphi_h\|\|\nabla v\| \ \le\ ch\|\nabla v\|. \end{aligned} \tag{2.13}$$

From Theorem 4 and (2.13), using Poincare's inequality, we see that

$$\begin{aligned}(\theta_t, v) + (\nabla\theta, \nabla v) &\le c\,(\|e\| + \|\tilde{e}\| + \|\nabla\tilde{e}\|)\|\nabla v\| + c\,\|\nabla\varphi_h\|_{0,\infty}\, h\|\nabla v\| \\ &+ \|\rho_t\|_{-1}\|v\|_1 \le c(\|\rho\| + \|\theta\| + \|\eta\| + \|\nabla\eta\| + \|\nabla\xi\|)\|\nabla v\| \\ &+ c\, h(\|\nabla\varphi_h\|_{0,\infty} + \|u_t\|)\|\nabla v\| \ \ \forall v \in S_h^0. \end{aligned} \tag{2.14}$$

But, from (1.9), (1.15) and (2.10),

$$\|\nabla\xi\|^2 \le c\ (\sigma(u_h)\nabla\xi, \nabla\xi) = c\ (\sigma(u_h)\nabla(r_h\varphi - \varphi_h), \nabla\xi)$$
$$= c\ ((\sigma(u_h) - \sigma(u))\nabla r_h\varphi, \nabla\xi)$$
$$= c\ \|e\|\|\nabla\xi\|\|\nabla r_h\varphi\|_{0,\infty} \le c\ \|e\|\|\nabla\xi\|$$

i.e.

$$\|\nabla\xi\| \le c\ \|e\|. \tag{2.15}$$

From (2.14) (2.15), we see that

$$(\theta_t, v) + (\nabla\theta, \nabla v) \le c\ h\|\nabla\varphi_h\|_{0,\infty}\|\nabla v\| + c\ \|\theta\|\|\nabla v\| + c\ h\|u_t\|\|\nabla v\| \qquad \forall v \in S_h^0. \tag{2.16}$$

Replacing v by θ in (2.16), it follows that

$$\frac{d}{dt}\|\theta\|^2 + \|\nabla\theta\|^2 \le c\ (h^2(\|\nabla\varphi_h\|_{0,\infty}^2 + \|u_t\|^2) + \|\theta\|^2).$$

By Gronwall's inequality, noting that $\theta(0) = 0$, we get

$$\|\theta\|^2 + \int_0^s \|\nabla\theta\|^2 \le c\ h^2 \int_0^s (\|\nabla\varphi_h\|_{0,\infty}^2 + \|u_t\|^2)$$
$$\le c\ h^2(\|\nabla\varphi_h\|_{L^2(0,T;L^\infty(\Omega))} + \|u_t\|_{L^2(Q_T)}^2), \quad \forall\ 0 < s < T.$$

From Theorem 2 (1), $u_t \in L^p(Q_T), \forall\ 2 \le p < +\infty$, we see that

$$\|\theta\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla\theta\|_{L^2(0,T;L^2(\Omega))} \le c\ h\|\nabla\varphi_h\|_{L^2(0,T;L^\infty(\Omega))}. \tag{2.17}$$

Noting (2.15), we have

$$\|\nabla\xi\| \le c\ h\|\nabla\varphi_h\|_{L^2(0,T;L^\infty(\Omega))}$$

i.e.

$$\|\nabla\tilde{e}\|_{L^\infty(0,T;L^2(\Omega))} \le c\ h\|\nabla\varphi_h\|_{L^2(0,T;L^\infty(\Omega))}. \tag{2.18}$$

Now we state the $L^2(0,T;W^{1,\infty}(\Omega))$ stability of φ_h in the following theorem.

Theorem 5. *Under* (A1)–(A2), *and if* (φ_h, u_h) *is the solution of problem* (1.15)–(1.17), *then*

$$\|\varphi_h\|_{L^2(0,T;W^{1,\infty}(\Omega))} \le C \quad \textit{for } h \textit{ sufficiently small}$$

where C *is independent of* h.

Proof. For the $W^{1,\infty}(\Omega)$ estimate of φ_h, we introduce the derivative Green's function in [8].

Pick any point $z \in \Omega$ contained in the interior of some triangular element T, and denote by ∂ any of the operators $\frac{\partial}{\partial x_i}(i = 1,2)$. There is a function $\delta_z \in C_0^\infty(T)$ such that

$$\int_\Omega \delta_z dx = 1, \quad |\nabla_s\delta_z| \le c\ h^{-2-s}, \quad s = 0,1,\cdots. \tag{2.19}$$

Green's function $g_z \in H_0^1(\Omega)$ is defined by

$$(\sigma(u_h)\nabla g_z, \nabla v) = (\frac{\partial \delta_z}{\partial x_i}, v), \quad i = 1,2, \quad \forall v \in H_0^1(\Omega). \tag{2.20}$$

We are ready to get the estimate

$$\begin{aligned} \|\nabla g_z\|_{0,p} &\le c\|\frac{\partial \delta_z}{\partial x_i}\|_{-1,P} \le c \sup_{\phi\in W_0^{1,q}} \frac{|\langle \frac{\partial \delta_z}{\partial x_i}, \phi\rangle|}{\|\phi\|_{1,q}} = c \sup_{\phi\in W_0^{1,q}} \frac{|\langle \delta_z, \partial\phi\rangle|}{\|\phi\|_{1,q}} \\ &\le c\|\delta_z\|_{0,p} \quad \forall 1 < p < +\infty, \end{aligned} \tag{2.21}$$

where $1/p + 1/q = 1$.

The finite element approximation to g_z is defined by

Find $g_h \in S_h^0$, such that

$$(\sigma(u_h)\nabla(g_z - g_h)\nabla v) = 0, \qquad \forall v \in S_h^0. \tag{2.22}$$

It is obvious that

$$\|\nabla g_h\|_{0,p} \le c\, \|\nabla g_z\|_{0,p}, \qquad \forall 1 < p < +\infty. \tag{2.23}$$

From (2.19), (2.20), (1.15) and (2.10), we have

$$\begin{aligned} \partial\xi &= (\partial\xi, \delta_z) = -(\xi, \partial\delta_z) \\ &= -(\sigma(u_h)\nabla g_z, \nabla\xi) = -(\sigma(u_h)\nabla g_h, \nabla(r_h\varphi - \nabla\varphi_h)) \\ &= -(\sigma(u_h)\nabla g_h, \nabla r_h\varphi) \;=\; ((\sigma(u) - \sigma(u_h))\nabla g_h, \nabla r_h\varphi) \end{aligned}$$

Therefore, by Hölder's inequality,

$$|\,\partial\xi\,| \le c\|u - u_h\|_{0,6}\|\nabla g_h\|_{0,\frac{6}{5}}\|\nabla r_h\varphi\|_{0,\infty} \tag{2.24}$$

By Poincare's inequality, $\|u - u_h\|_{0,6} \le c\|\nabla(u - u_h)\| \;\le\; c(\|\nabla\rho\| + \|\nabla\theta\|)$.

From (2.23), (2.21), (2.19), we get

$$\|\nabla g_h\|_{0,p} \le \|\delta_z\|_{0,p} \le ch^{-2+\frac{N}{p}}, \quad 1 < p < +\infty. \tag{2.25}$$

Therefore, taking p as $6/5$ in (2.25), we can turn (2.24) into

$$|\,\partial\xi\,| \le c\,(h\|u\|_2 + \|\nabla\theta\|)\; h^{-\frac{1}{3}}. \tag{2.26}$$

Hence,

$$\|\nabla\xi\|_{L^2(0,T;L^\infty(\Omega))} \le c\; h^{-\frac{1}{3}}(h\|u\|_{L^2(0,T;H^2(\Omega))} + \|\nabla\theta\|_{L^2(0,T;L^2(\Omega))}). \tag{2.27}$$

From (1.6) and (2.17), (2.27) becomes

$$\|\nabla\xi\|_{L^2(0,T;L^\infty(\Omega))} \le c\; h^{\frac{2}{3}} + ch^{\frac{2}{3}}\|\nabla\varphi_h\|_{L^2(0,T;L^\infty(\Omega))}. \tag{2.28}$$

Now, we get

$$\begin{aligned} \|\nabla\varphi_h\|_{L^2(0,T;L^\infty(\Omega))} &\le \|r_h\varphi\|_{L^2(0,T;L^\infty(\Omega))} + \|\nabla\xi\|_{L^2(0,T;L^\infty(\Omega))} \\ &\le c + ch^{\frac{2}{3}}\|\nabla\varphi_h\|_{L^2(0,T;L^\infty(\Omega))} \le C. \text{ for } h \text{ sufficiently small,} \end{aligned} \tag{2.29}$$

where C is independent of h.

(2.29) and Theorem 4 implies that the theorem 5 is true.

Remark. Though Green's function in (2.20) cannot reach higher regularity as $g_z \in W^{2,p}(\Omega)$, that would not cause any sevese olifficulty. We do not require so high a regularity in the former procedure.

3. Error Estimates

On the basis of $L^\infty(0,T;L^\infty(\Omega))$ and $L^2(0,T;W^{1,\infty}(\Omega))$ stability of φ_h, we are ready now to handle the nonlinear term $\sigma(u_h) \mid \nabla\varphi_h \mid^2$, and to derive the error bounds.

From (1.10), (1.16) and (2.15), we have

$$\begin{aligned}(\theta_t, v) + (\nabla\theta, \nabla v) &= (\sigma(u) \mid \nabla\varphi \mid^2 - \sigma(u_h) \mid \nabla\varphi_h \mid^2, v) - (\rho_t, v)\\ &= ((\sigma(u) - \sigma(u_h)) \mid \nabla\varphi \mid^2, v) + (\sigma(u_h)(\mid \nabla\varphi \mid^2 - \mid \nabla\varphi_h \mid^2), v) - (\rho_t, v)\\ &\le c\, \|u - u_h\| \|v\| + c\, (\|\nabla\varphi\|_{0,\infty} + \|\nabla\varphi_h\|_{0,\infty}) \|\nabla(\varphi - \varphi_h)\| \|v\| + \|\rho_t\|_{-1} \|v\|_1\\ &\le c\, (\|\rho\| + \|\theta\| + \|\nabla\eta\|) \|v\| + c(\|\nabla\eta\| + \|\rho\| + \|\theta\|) \|\nabla\varphi_h\|_{0,\infty} \|v\| + c\, \|\rho_t\|_{-1} \|\nabla v\|\end{aligned} \tag{3.1}$$

subject to

$$\theta(0) = 0.$$

Replacing v by θ, yields

$$\frac{d}{dt}\|\theta\|^2 + \|\nabla\theta\|^2 \le c\, h^2 + c\, \|\theta\|^2 + c\, h^2(\|\nabla\varphi\|_{0,\infty}^2 + \|u_t\|^2) \tag{3.2}$$

By Gronwall's inequality,

$$\|\theta\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\theta\|_{L^2(0,T;H^1(\Omega))}^2 \le c\, h^2 + c\, h^2(\|\nabla\varphi\|_{L^2(0,T;L^\infty(\Omega))} + \|u_t\|_{L^2(Q_T)}).$$

From Theorem 2, (1) and Theorem 5, we obtain

$$\|\theta\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta\|_{L^2(0,T;H^1(\Omega))} \le c\, h. \tag{3.3}$$

Theorem 6. *Under assumptions* (A1)–(A2) *and conditions of Theorem* 2 (1), *if* (φ_h, u_h) *is a solution of problem* (1.15)–(1.17), *then*

$$\|u - u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|u - u_h\|_{L^2(0,T;H^1(\Omega))} \le c\, h, \tag{3.4}$$

and

$$\|\varphi - \varphi_h\|_{L^\infty(0,T;H^1(\Omega))} \le c\, h, \tag{3.5}$$

where (3.5) *comes from* (3.4) *and* (2.15).

Furthermore, we can give the following error estimate.

Theorem 7. *Under the conditions of Theorem* 6 *and Theorem* 2. (2), *if* (φ_h, u_h) *is a solution of problem* (1.15)–(1.17), *then*