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THE CONVERGENCE OF MULTIGRID METHODS FOR NONSYMMETRIC ELLIPTIC VARIATIONAL INEQUALITIES*[1)]

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Abstract

This paper is concerned with the convergence of multigrid methods (MGM) on nonsymmetric elliptic variational inequalities. On the basis of Wang and Zeng's work (1988), we develop the convergence results of the smoothing operator (i.e. PJOR and PSOR). We also extend the multigrid method of J.Mandel (1984) to nonsymmetric varitional inequalities and obtain the convergence of MGM for these problems.

§1. The Multigrid Algorithm

Let us consider the complementary form of elliptic variational inequalities with the domain $\Omega \in R^s$:

$$\begin{cases} Lu(x) \geq f(x), u(x) \geq c(x), & x \in \Omega, \\ (u(x) - c(x))(Lu(x) - f(x)) \geq 0, & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Take the sequence of gridlengths $h_0 > h_1 > \cdots > h_l$, the approximation sequence of $\Omega : G^0 \subset G^1 \subset \cdots \subset G^l$, and the corresponding grid number N_k of G^k. We can discretize (1.1) on G^k as follows:

$$\begin{cases} A^k u^k \geq b^k, u^k \geq c^k, \\ (u^k - c^k)^T (A^k u^k - b^k) = 0. \end{cases} \tag{1.2}$$

Wang and Zeng gave a multigrid algorithm in [5]; the main steps are as follows:

(1) ν_1-smooth: $u_1^{\nu,j} = RELAX^{\nu_1}(u_1^j, A^l, b^l)$,

(2) Correcting: $\tilde{u}_l^{\nu,j} = \tilde{u}_l^{\nu,j} + I_{l-1}^l \tilde{u}_{l-1}$, where $\tilde{u}_{l-1}$ is the $(l-1)$-th grid iterative solution of the defect inequality;

(3) ν_2-smooth: $u_l^{j+1} = RELAX^{\nu_2}(\tilde{u}_l^{\nu,j}, A^l, b^l)$.

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For the sake of convenience, we let $h_1 = h, h_0 = H$ in the case of TWG and rewrite (1.2) and the defect inequality respectively as follows:

$$(\mathrm{P_h}) \quad \begin{cases} A_h u_h \geq b_h, u_h \geq c_h, & (1.3)\\ (u_h - c_h)^T (A_h u_h - b_h) = 0, & (1.4) \end{cases}$$

$$(\mathrm{P_H}) \quad \begin{cases} A_H u_H \geq d_H = (I_H^h)^T (f_h - A_h u_h^{\nu,j}), u_H \geq c_H, & (1.5)\\ (u_H - c_H)^T (A_H u_H - d_H) = 0, & (1.6) \end{cases}$$

where $I_H^h = (p_{ij}) \geq 0$ and, for some positive constant α_h,

$$u_h{}^T A_h u_h \geq \alpha_h u_h^T u_h, \quad \forall u_h \in R^{N_1}, \tag{1.7}$$

$$A_H = (I_H^h)^T A_h I_H^h, \tag{1.8}$$

$$Null(I_H^h) = \{0\}, \tag{1.9}$$

$$c_{HK} = \max\{c_{hi} - u_{hi}^{\nu,j}, p_{ik} \geq 0\}. \tag{1.10}$$

§2. The Convergence of PJOR and PSOR

In this section, we assume that A is a strictly diagonally dominant or irreducible and weakly diagonally dominant matrix, and solve the following problem (P) by relaxation iteration:

$$\begin{cases} Ax \geq b, x \geq c, & (2.1)\\ (x - c)^T (Ax - b) = 0. & (2.2) \end{cases}$$

Let $A = D(I - L - U) = D(I - B)$, where D, L and U are diagonal , strictly lower, and upper triangular matrices respectively. It is not difficult to show that $\rho(|B|) < 1$, where $|B| = (|b_{ij}|)$. Morever we have

Lemma 2.1. *Let x^* be the solution of* (P). *Then*

$$x^* = \max\{c_i, (1 - \omega)x_i^* + \omega(D^{-1}b + Lx^* + Ux^*)_i\}. \tag{2.3}$$

Theorem 2.1. *For $0 < \omega < 2/[1 + \rho(|B|)]$, we have*

$$\|\varepsilon^{k+1}\| \leq \|J_\omega^k\| \cdot \|\varepsilon_0\| \tag{2.4}$$

and

$$J_\omega^k \to 0, \quad as \ k \to \infty \tag{2.5}$$

where ε^k is the iterative error of PJOR *and* $J_\omega = |1 - \omega| I + \omega |B|$, $\|\cdot\| = \|\cdot\|_2$.

Proof. From Lemma 2.1, it is easy to verify that

$$|\varepsilon^{k+1}| \leq J_\omega |\varepsilon^k| \leq J_\omega^{k+1} |\varepsilon^0| \tag{2.6}$$

which implies (2.4). Furthermore, $\rho(J_\omega) = |1 - \omega| + \omega\rho(|B|) < 1$ for $0 < \omega < 2/[1 + \rho(|B|)]$, and this means (2.6) holds.

Theorem 2.2. *For $0 < \omega < 2/[1 + \rho(|B|)]$, we have*

$$\|\varepsilon^{k+1}\| \leq \|L_\omega^K\| \cdot \|\varepsilon_0\|$$

and

$$L_\omega^k \to 0, \quad as \ k \to \infty \tag{2.7}$$

where ε^k *is the iterative error of* PSOR *and*

$$L_\omega = (I - \omega|L|)^{-1}[|1 - \omega|I + \omega|U|].$$

Proof. With the help of Lemma 2.1, $(I - \omega|L|)^{-1} \geq 0$ and $\rho(L_\omega) \leq \rho(J_\omega) < 1$ for $0 < \omega < 2/[1 + rho(|B|)]$, the result becomes obvious.

§3. The Convergence of MGM

In order to set up the convergence , we introduce problem (P_H^*):

$$(P_H^*) \quad \begin{cases} A_H u_H^* \geq d_H = (I_H^h)^T (f_h - A_h u_h^*), U_H^* \geq c_H^*, & (3.1) \\ (u_H^* - c_H^*)^T (A_H u_H^* - d_H^*) = 0, & (3.2) \end{cases}$$

where $c_{HK}^* = \max\{c_{hi} - u_{hi}^*, p_{ik} > 0\}$ .Then we can easily show

Lemma 3.1. $u_H^* = 0$, *and for any* $u_H \in R^{N_1}$,

$$u_H^T A_H u_H \geq \alpha_H u_H^T u_H = \alpha_H \lambda_{MIN}((I_H^h)^T I_H^h) u_H^T u_H$$

where $\alpha_H > 0$.

Lemma 3.2. $\|c_H - c_H^*\| \leq \|u_h^* - u_h^{\nu,j}\|_\infty$ *and* $\|c_H - c_H^*\| \leq \beta \|u_h^* - u_h^{\nu,j}\|$, *where* β *is the largest number of nonzero elements of* I_H^h *in each column.*

Lemma 3.3. *The solution of* (P_H) *and* (P_H^*) *satisfies*

$$\|u_H - u_H^*\| \leq c\|u_h^* - u_h^{\nu,j}\|$$

where $c = \{2[\beta\|(I_H^h)^T A_h\| + (\beta\|A_H\| + \|(I_H^h)^T A_h\|^2)/(2\alpha_H)]/\alpha_H\}^{1/2}$.

Proof. From (1.3)–(1.6) , (3.1)–(3.2) and the above lemmas, we calculate and get

$$\begin{aligned}
&\alpha_H \|u_H - u_H^*\|^2 \leq (u_H^* - u_H)^T A_H (u_H - u_H^*) \\
&\leq \|c_H - c_h^*\|(\|A_H\|\|u_H - u_H^*\| + \|d_H - d_H^*\|) + \|u_H - u_H^*\|\|d_H - d_H^*\| \\
&\leq (\beta\|A_H\| + \|(I_H^{h^T} A_H\|)\|u_H - u_H^*\|\|(u_h^* - u_h^{\nu,j}\| + \beta\|(I_H^h)^T A_H\|\|u_h^* - u_h^{\nu,j}\|^2 \\
&\leq \alpha_H \|u_h - u_h^*\|^2/2 + [(\beta\|A_H\| + \|(I_H^h)^T A_H\|)^2/(2\alpha_H) + \beta\|(I_H^h)^T A_H\|]\|u_h^* - u_h^{\nu,j}\|^2.
\end{aligned}$$

The lemma is then proved.

Theorem 3.1. *If* A_h *is strictly diagonally dominant or irreductible and weakly diagonal dominant, and if* PJOR *or* PSOR *is used as a smoothing operator* $(0 < \omega < 2/[1 + \rho(|L_h + U_h|)])$, *then* TWG *is convergent to* u_h^* *when* $\nu = \nu_1 + \nu_2$ *is sufficiently large.*

Proof. Without loss of generality , we use PJOR as a smoothing operator. Then

$$\|u_h^{j+1} - u_h^*\| \leq \|J_\omega^{\nu_2}\|\|J_\omega^{\nu_1}\|(1 + c\|\bar{I}_H^h\|)\|u_h^j - u_h^*\| \leq r\|u_h^j - u_h^*\|,$$

where $r < 1$ as $\nu \to \infty$. That is, $u_h^j \to u_h^*$ as $j \to \infty$, for sufficiently large ν.

As for MGM , because

$$\|u_h^{j+1} - u_h^*\| \le \|J_\omega^{\nu_2}\|\|u_h^{\nu,j} + \bar{I}_{l-1}^l \tilde{u}_{l-1} - u_h^*\| \le \|J_\omega^{\nu_2}\|(\|u_l^{\nu,j} + \bar{I}_{l-1}^l u_{l-1} - u_h^*\| + \|\bar{I}_{l-1}^l\|\|u_{l-1} - \tilde{u}_{l-1}\|).$$

Theorem 3.2. *If A_l is strictly diagonally dominant or irreducible and weakly diagonal dominant, and if* PJOR *or* PSOR *is used as a smoothing operator* ($0 < \omega < 2/[1 + \rho(|L_l + U_l|)]$ *on the first grid* G^l), *then* TWG *is convergent to u_l^* when ν is sufficiently large.*

Furthermore, if we choose $I_H^h = \bar{I}_H^h$ as an injective extension and denote $u_H^{\nu,j,t}$ as an iterative solution solving (P_H) by t-times PJOR or PSOR iterational using 0 as the initial vector, we then have

Lemma 3.4. *If A_h is strictly diagonal dominant or irreducible and weakly diagonal donminant, then*

$$\|u_h^{\nu,j} + I_H^j u_H^{\nu,j,t} - u_h^*\|_\infty \le \|u_h^{\nu,j} - u_h^*\|_\infty$$

and for $\nu \ge 1$,

$$\|u_h^{j+1} - u_h^*\|_\infty \le \|u_h^{\nu,j} - u_h^*\|_\infty \le \|u_h^j - u_h^*\|_\infty.$$

From the above lemma and by the use of the same techinque as in [5], we can prove the following theorem:

Theorem 3.3. *Under the same conditions as in Theorem* 3.1, *if $I_H^h = \bar{I}_H^h$ is taken as injective extension,* TWG *is convergent to u_h^* for $\nu = \nu_1 + \nu_2 \ge 1$.*

Corallary 3.1. *If u_H is repalaced by $u_H^{\nu,j,t}$, the result of Theorem* 3.3 *still holds.*

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References

[1]. J.X.Wang and F.Brakhagen, Multigrid method for a type of variational inequalities, Proceeding of the Symposium on Differential Geometry and Differential Equations, Beijing, 1985.

[2]. J.X.Wang, Y.X.Huang and J.Y.Gong, Multigrid method for the elasticity, *J. Comput. Math.*, **4** : 2 (1986), 154–163.

[3]. S.Z.Zhou, Variational Inequalities and the Finite Element Methods, Hunan University Press, 1988.

[4]. J.Mandel, A multilevel iterative method for symmetric, positive definite linear complementarity problems, *Appl. Math. Opt.*, **11** (1984), 77–95.

[5]. J.X.Wang and J.P.Zeng, Multigrid method for nonsymmetric elliptic variational problems, *Numer. Math. Sinica Math.*, **10** (1988).