Journal of Computational Mathematics, Vol.11, No.4, 1993, 329–338.

SINGULARITY AND QUADRATURE REGULARITY OF $(0,1,\cdots,m-2,m)$-INTERPOLATION ON THE ZEROS OF JACOBI POLYNOMIALS*[1)]

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Abstract

In this paper we show that, if a problem of $(0,1,\cdots,m-2,m)$-interpolation on the zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ $(\alpha,\beta\geq -1)$ has infinite solutions, then the general form of the solutions is $f_0(x)+Cf(x)$ with an arbitrary constant C, where $f_0(x)$ and $f(x)$ are fixed polynomials of degree $\leq mn-1$. Moreover, the explicit form of $f(x)$ is given. A necessary and sufficient condition of quadrature regularity of the interpolation in a manageable form is also established.

1. Introduction and Main Results

Let us consider a system A of nodes

$$1\geq x_1>x_2>\cdots>x_n\geq -1,\quad n\geq 2. \tag{1.1}$$

Let $\mathcal{P}_n$ be the set of polynomials of degree at most n and let $m\geq 2$ be a fixed integer. The problem of $(0,1,\cdots,m-2,m)$-interpolation is, given a set of numbers

$$y_{kj},\quad k\in N:=\{1,2,\cdots,n\},\quad j\in M:=\{0,1,\cdots,m-2,m\}, \tag{1.2}$$

to determine a polynomial $R_{mn-1}(x;A)\in\mathcal{P}_{mn-1}$ (if any) such that

$$R_{mn-1}^{(j)}(x_k;A)=y_{kj},\quad \forall k\in N,\quad \forall j\in M. \tag{1.3}$$

If for an arbitrary set of numbers y_{kj} there exists a unique polynomial $R_{mn-1}(x;A)\in\mathcal{P}_{mn-1}$ satisfying (1.3), then we say that the problem of $(0,1,\cdots,m-2,m)$-interpolation on A is regular (otherwise, is singular) and $R_{mn-1}(x;A)$ can be uniquely written as

$$R_{mn-1}(x;A)=\sum_{\substack{k\in N\\ j\in M}} y_{kj}r_{kj}(x;A) \tag{1.4}$$

where $r_{kj}\in\mathcal{P}_{mn-1}$ satisfy

$$r_{kj}^{(\mu)}(x_\nu)=\delta_{k\nu}\delta_{j\mu},\quad k,\nu\in N,\quad j,\mu\in M. \tag{1.5}$$

* Received April 8, 1992.

[1)] The Project Supported by National Natural Science Foundation of China.

In particular, for convenience of use we set

$$\rho_k(x) := r_{km}(x), \quad k = 1, 2, \cdots, n. \tag{1.6}$$

On the problem of $(0,2)$-interpolation Turán raises in [5] an open problem as follows.

Problem 29. *Find all Jacobi matrices $P(\alpha, \beta)$, $\alpha \neq \beta$, for which $(0,2)$-interpolation problem does have a unique solution.*

By a Jacobi matrix $P(\alpha, \beta)$, Turán means the triangular matrix whose nth row consists of the zeros of the nth Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta \geq -1)$.

Chak, Sharma, and Szabados [1] have given a necessary and sufficient condition of regularity of $(0,2)$-interpolation in a manageable form on all Jacobi matrices $P(\alpha, \beta)$.

Recently, the author generalized in [3] their important result and proved the following theorem, in which

$$\gamma := \frac{1}{2}(m-1)(\alpha+1), \quad \delta := \frac{1}{2}(m-1)(\beta+1), \tag{1.7}$$

$$s_k := 2^{-n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}, \quad k = 0, 1, \cdots, n. \tag{1.8}$$

Theorem A. *The problem of $(0, 1, \cdots, m-2, m)$-interpolation on the zeros of $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta \geq -1)$ is regular if and only if*

$$D_n(\alpha, \beta) \neq 0 \tag{1.9}$$

where

$$D_n(\alpha,\beta) = \begin{cases} \displaystyle\sum_{k=0}^{n} \frac{(-1)^k \binom{\gamma}{k}\binom{\delta}{n-k} s_k}{\binom{n}{k}}, & \alpha, \beta > -1, \\ (m+1)\dbinom{\delta}{n} - (m-1)\dbinom{n+\beta+\delta}{n}, & \alpha = -1, \ \beta > -1, \\ (-1)^n D_n(-1, \alpha), & \alpha > -1, \ \beta = -1, \\ 1 + (-1)^n, & \alpha = \beta = -1. \end{cases} \tag{1.10}$$

In particular, when $\alpha = -1$, $\beta > -1$ or $\alpha > -1$, $\beta = -1$, the problem is always regular; when $\alpha = \beta = -1$, the problem is regular for even n and singular for odd n.

Remark. In [3] we also gave the explicit forms for the fundamental polynomials, which are very complicated and omitted.

If the problem of $(0, 1, \cdots, m-2, m)$-interpolation on A is not regular, then for a given set of numbers y_{kj} either there is no polynomial $R_{mn-1}(x)$ satisfying (1.3) or there is an infinity of polynomials with the property (1.3). The possibility of an infinity of solutions raises the question on the dimensionality of their number. The first result concerning this question, to the best of the author's knowledge, is given by Surányi and Turán in [4, Sections 6 and 11] for $\alpha = \beta = -1$: in the case of infinitely many solutions for $n \geq 3$ the general form of the solutions is

$$f(x) = f_0(x) + C\pi_n(x)[P_{n-1}(x) - 3]$$

where $f_0(x)$ is a fixed polynomial of degree $\leq 2n-1$,

$$\pi_n(x) = (1-x^2)P'_{n-1}(x), \tag{1.11}$$

$P_{n-1}(x)$ stands for the $(n-1)$-th Legendre polynomial with the normalization $P_{n-1}(1) = 1$ and C is an arbitrary number.

The first object of this paper is to generalize this result to all Jacobi matrces $P(\alpha, \beta)$.

Theorem 1. *If the problem of $(0, 1, \cdots, m-2, m)$-interpolation on the zeros of $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta \geq -1)$ is not regular, i.e., $D_n(\alpha, \beta) = 0$, and if for a given set of numbers y_{kj} there is an infinity of polynomials with the property* (1.3), *then the general form of the solutions is*

$$R_{mn-1}(x) = f_0(x) + C[P_n^{(\alpha,\beta)}(x)]^{m-1} q_{n-1}(x) \tag{1.12}$$

with an arbitrary number C, where $f_0(x)$ is a fixed polynomial of degree $\leq mn-1$ and $q_{n-1}(x) \in \mathcal{P}_{n-1}$ is of the form

$$q_{n-1}(x) = \begin{cases} (1-x)^\gamma (1+x)^\delta, \quad \alpha, \beta > -1, \quad n > \gamma + \delta, \\ \qquad\qquad \gamma, \delta = \text{integers} \\ (1-x)^\gamma (1+x)^\delta \Big\{ d + \displaystyle\int_a^x P_n^{(\alpha,\beta)}(t)(1-t)^{-\gamma-1}(1+t)^{-\delta-1} dt \Big\}, \\ \qquad\qquad \text{otherwise}, \end{cases} \tag{1.13}$$

$$a = \begin{cases} 0, & \alpha, \beta > -1, \\ 1, & \alpha = \beta = -1, \end{cases} \tag{1.14}$$

$$d = \begin{cases} \displaystyle\sum_{k=0}^{n-1} (-1)^k \alpha_k, & \alpha, \beta > -1, \\ \dfrac{1}{n(m-1)}, & \alpha = \beta = -1, \end{cases} \tag{1.15}$$

$$\alpha_k = \begin{cases} \displaystyle\sum_{j=0}^{k} \frac{\binom{n-\delta-j-1}{k-j} s_j}{\binom{\gamma-j}{k-j}(\gamma-k)}, & \gamma \neq \text{an integer or } k < \gamma, \\ \displaystyle\sum_{j=k+1}^{n} \frac{\binom{\gamma-k-1}{j-k-1} s_j}{\binom{n-\delta-k-1}{j-k-1}(\delta-n+j)}, & \delta \neq \text{an integer or } k \geq n-\delta. \end{cases} \tag{1.16}$$

Theorem 1 suggests the following plausible.

Conjecture. If the problem of $(0, 1, \cdots, m-2, m)$-interpolation on A is not regular and if for a given set of numbers y_{kj} there is an infinity of polynomials with the property (1.3), then the general form of the solutions is

$$f(x) = f_0(x) + Cf_1(x),$$

where $f_0(x)$ and $f_1(x)$ are fixed polynomials of degree $\leq mn-1$ and C an arbitrary number.

The second object of this paper is the quadrature regularity of $(0, 1, \cdots, m-2, m)$-interpolation. According to the general definition[2], the problem of $(0, 1, \cdots, m-2, m)$-interpolation on A will be called quadrature regular (or q-regular) if there exists a set of numbers

$$c_{\mu\nu}, \quad \mu \in M, \quad \nu \in N \tag{1.17}$$

with the property

$$\int_{-1}^{1} f(x)dx = \sum_{\substack{\mu\in M\\ \nu\in N}} c_{\mu\nu} f^{(\mu)}(x_\nu) \tag{1.18}$$

for all $f \in \mathcal{P}_{mn-1}$. In this paper we also give a necessary and sufficient condition of q-regularity of $(0, 1, \cdots, m-2, m)$-interpolation in a manageable form for all Jacobi matrices $P(\alpha, \beta)$ $(\alpha, \beta \geq -1)$ as follows.

Theorem 2. *If $\alpha, \beta > -1$, then the problem of $(0, 1, \cdots, m-2, m)$-interpolation on the zeros of $P_n^{(\alpha,\beta)}(x)$ is q-regular if and only if (1.9) holds or*

$$Q_n(\alpha, \beta) := \int_{-1}^{1} q_{n-1}(x)[P_n^{(\alpha,\beta)}(x)]^{m-1}dx = 0, \tag{1.19}$$

where $q_{n-1}(x)$ is given by (1.13).

If $\alpha = -1$, $\beta > -1$ or $\alpha > -1$, $\beta = -1$, then the problem is q-regular.

If $\alpha = \beta = -1$, then the problem is not q-regular if and only if both n and m are odd.

In particular, when $\alpha = \beta$, the problem is q-regular if one of the following conditions is valid: (i) *n is odd and m is even;* (ii) *n is even and $\gamma = \delta \neq$ an integer;* (iii) *n is even and $n \leq 2\gamma$. If $\gamma = \delta =$ a positive integer and if $n > 2\gamma$ and m is odd, then the problem is not q-regular.*

The proofs of these two theorems are rather complicated and are put to Sections 3 and 4, respectively. In the next section some useful lemmas of q-regularity for general $(0, 1, \cdots, m-2, m)$-interpolation are given.

2. Auxiliary Lemmas

We first prove some lemmas which are of independent interest. To this end we introduce the fundamental polynomials of $(0, 1, \cdots, m-1)$-interpolation. Let A_{kj}, $B_k \in \mathcal{P}_{mn-1}$ be defined by

$$A_{kj}^{(\mu)}(x_\nu) = \delta_{k\nu}\delta_{j\mu}, \quad k, \nu = 1, 2, \cdots, n, \quad j, \mu = 0, 1, \cdots, m-1 \tag{2.1}$$

and

$$B_k(x) := A_{k,m-1}(x) = \frac{1}{m!}(x - x_k)^{m-1} l_k^m(x), \quad k = 1, 2, \cdots, n, \tag{2.2}$$

where

$$l_k(x) := \frac{\omega_n(x)}{(x - x_k)\omega_n'(x_k)}, \quad \omega_n(x) = c(x - x_1)(x - x_2)\cdots(x - x_n), \quad c \neq 0. \tag{2.3}$$

Then we have

Lemma 1. *The problem of* $(0,1,\cdots,m-2,m)$*-interpolation on* A *is q-regular if and only if*

$$\operatorname{rank}[B_k^{(m)}(x_\nu)]_{k,\nu=1}^n = \operatorname{rank}[B_k^{(m)}(x_\nu), \int_{-1}^1 B_k(x)dx]_{k,\nu=1}^n. \tag{2.4}$$

Proof. The problem of $(0,1,\cdots,m-2,m)$-interpolation on A is q-regular if and only if, by definition, (1.18) with some numbers (1.17) holds for all $f \in \mathcal{P}_{mn-1}$, which is equivalent to that (1.18) holds for all A_{kj}, $k=1,2,..,n$, $j=0,1,\cdots,m-1$. That is,

$$\sum_{\substack{\mu\in M\\ \nu\in N}} c_{\mu\nu}A_{kj}^{(\mu)}(x_\nu) = \int_{-1}^1 A_{kj}(x)dx, \quad k=1,2,\cdots,n, \ \ j=0,1,\cdots,m-1. \tag{2.5}$$

With the help of (2.1) and (2.2), (2.5) becomes

$$c_{jk} + \sum_{\nu=1}^n c_{m\nu}A_{kj}^{(m)}(x_\nu) = \int_{-1}^1 A_{kj}(x)dx,$$

$$k=1,2,\cdots,n, \quad j=0,1,\cdots,m-2, \tag{2.6}$$

$$\sum_{\nu=1}^n c_{m\nu}B_k^{(m)}(x_\nu) = \int_{-1}^1 B_k(x)dx, \quad k=1,2,\cdots,n. \tag{2.7}$$

As we know, the system of linear equations (2.7) has a solution $c_{m\nu}$, $\nu=1,2,\cdots,n$, if and only if (2.4) is true. Then the other c_{jk}, $k=1,2,\cdots,n$, $j=0,1,\cdots,m-2$, can be determined by (2.6).

This completes the proof.

Lemma 2. *The problem of* $(0,1,\cdots,m-2,m)$*-interpolation on* A *is q-regular if and only if for every polynomial* $R_{mn-1}(x;A)\in\mathcal{P}_{mn-1}$ *satisfying* (1.3) *with* $y_{kj}\equiv 0$,

$$\int_{-1}^1 R_{mn-1}(x;A)dx = 0. \tag{2.8}$$

Proof. Let us show the sufficiency only, the necessity being trivial.

Assume that $R(x) := R_{mn-1}(x;A)\in\mathcal{P}_{mn-1}$ satisfies (1.3) with $y_{kj}\equiv 0$. Using the polynomials $A_{kj}(x)$, $R(x)$ can be uniquely written as

$$R(x) = \sum_{\nu=1}^n\sum_{\mu=0}^{m-1} R^{(\mu)}(x_\nu)A_{\nu\mu}(x) = \sum_{\nu=1}^n R^{(m-1)}(x_\nu)B_\nu(x). \tag{2.9}$$

The condition $R^{(m)}(x_k)=0$, $k=1,2,\cdots,n$, yields

$$\sum_{\nu=1}^n R^{(m-1)}(x_\nu)B_\nu^{(m)}(x_k) = 0, \quad k=1,2,\cdots,n. \tag{2.10}$$

Meanwhile, (2.8) can be written by (2.9) as

$$\sum_{\nu=1}^n R^{(m-1)}(x_\nu)\int_{-1}^1 B_\nu(x)dx = 0. \tag{2.11}$$

Thus, every polynomial $R(x)$ satisfying (2.10) implies (2.11). This, in fact, is equivalent to (2.4). Therefore by Lemma 1 this is equivalent to the q-regularity of the problem.

This completes the proof.

3. Proof of Theorem 1

The main idea of the proof can be found in [1].

Theorem 1 will be obvious if we show that in our case every polynomial $R(x) \in \mathcal{P}_{mn-1}$ satisfying (1.3) with $y_{kj} \equiv 0$ is of the form

$$R(x) = C\omega_n^{m-1}(x)q_{n-1}(x), \tag{3.1}$$

where C is an arbitrary constant, $\omega_n(x) := P_n^{(\alpha,\beta)}(x)$ and $q_{n-1}(x) \in \mathcal{P}_{n-1}$. Then, it satisfies all the conditions, except for

$$R^{(m)}(x_k) = 0, \quad k = 1, 2, \cdots, n. \tag{3.2}$$

Using the formula (3.1) the requirement (3.2) yields

$$[\omega_n^{m-1}(x)q_{n-1}(x)]^{(m)}_{x=x_k} = 0, \quad k = 1, 2, \cdots, n. \tag{3.3}$$

It is easy to see that

$$[\omega_n^{m-1}(x)]^{(m)}_{x=x_k} = \frac{1}{2}(m-1)m!\omega_n'(x_k)^{m-2}\omega_n''(x_k)$$

and

$$[\omega_n^{m-1}(x)]^{(m-1)}_{x=x_k} = (m-1)!\omega_n'(x_k)^{m-1}.$$

Then (3.3) becomes

$$\frac{1}{2}(m-1)\omega_n''(x_k)q_{n-1}(x_k) + \omega_n'(x_k)q_{n-1}'(x_k) = 0, \quad k = 1, 2, \cdots, n. \tag{3.4}$$

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, $\alpha, \beta \geq -1$, satisfies the differential equation[1]

$$(1-x^2)y'' + [(\beta-\alpha) - (\beta+\alpha+2)x]y' + n(n+\alpha+\beta+1)y = 0 \tag{3.5}$$

and the normalization

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}. \tag{3.6}$$

It follows from (3.5) that

$$(1-x_k^2)\omega_n''(x_k) = [(\alpha+1)(1+x_k) - (\beta+1)(1-x_k)]\omega_n'(x_k), \quad k = 1, 2, \cdots, n. \tag{3.7}$$

This, coupled with (3.4), gives

$$(1-x_k^2)q_{n-1}'(x_k) + [\gamma(1+x_k) - \delta(1-x_k)]q_{n-1}(x_k) = 0, \quad k = 1, 2, \cdots, n. \tag{3.8}$$

Denote by $\mathcal{D}$ the differential operator

$$\mathcal{D}y := (1-x^2)y' + [\gamma(1+x) - \delta(1-x)]y. \tag{3.9}$$

Then (3.8) implies

$$\mathcal{D}q_{n-1}(x) = C\omega_n(x), \tag{3.10}$$

where C is a constant. Solving this differential equation we get

$$q_{n-1}(x) = (1-x)^\gamma(1+x)^\delta\{d + C\int_a^x \omega_n(t)(1-t)^{-\gamma-1}(1+t)^{-\delta-1}dt\}$$

with the constant d to be determined.

To determine d let us put

$$q_{n-1}(x) = \sum_{k=0}^{n-1}\alpha_k(x-1)^k(x+1)^{n-1-k}. \tag{3.11}$$

Thus

$$d = \begin{cases} q_{n-1}(0) = \sum_{k=0}^{n-1}(-1)^k\alpha_k, & \alpha,\beta > -1, \\ q_{n-1}(1) = 2^{n-1}\alpha_0, & \alpha = \beta = -1. \end{cases} \tag{3.12}$$

We distinguish two cases $\alpha, \beta > -1$ and $\alpha = \beta = -1$, because by Theorem A the problem is always regular for the other cases.

Case I ($\alpha, \beta > -1$). It is known [1, (2.2)] that

$$P_n^{(\alpha,\beta)}(x) = 2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}(x-1)^k(x+1)^{n-k}$$

$$:= \sum_{k=0}^{n} s_k(x-1)^k(x+1)^{n-k}. \tag{3.13}$$

Using (3.11) and (3.13) and comparing the coefficients of $(x-1)^k(x+1)^{n-k}$ on both sides of (3.10) we obtain the system of equations

$$\begin{cases} (\delta - n + k)\alpha_{k-1} + (\gamma - k)\alpha_k = Cs_k, & k = 0, 1, \cdots, n, \\ \alpha_{-1} = \alpha_n = 0. \end{cases} \tag{3.14}$$

Let us calculate α_k. If $\gamma \neq$ an integer or $k < \gamma$, then

$$\alpha_k = \frac{1}{\gamma - k}\{(n-\delta-k)\alpha_{k-1} + Cs_k\} = C\sum_{j=0}^{k}\frac{\binom{n-\delta-j-1}{k-j}s_j}{\binom{\gamma-j}{k-j}(\gamma-k)}, \tag{3.15}$$

and hence we get the first formula of (1.16) with a factor C. Similarly, we can prove the second one of (1.16) with a factor C. This proves the second formula in (1.13) for $\alpha, \beta > -1$.

Let both γ and δ be integers, and let $n > \gamma + \delta$. In this case the equation corresponding to $k = \gamma$ in (3.14) yields

$$\alpha_{\gamma-1} = \frac{Cs_\gamma}{\gamma + \delta - n}. \tag{3.16}$$

On the other hand, it follows from (3.15) that

$$\alpha_{\gamma-1} = C\sum_{j=0}^{\gamma-1}\frac{\binom{n-\delta-j-1}{\gamma-1-j}s_j}{\binom{\gamma-j}{\gamma-1-j}} = C\sum_{j=0}^{\gamma-1}\frac{\binom{n-\delta-j-1}{\gamma-1-j}s_j}{\gamma-j}.$$

We note that $s_\gamma/(\gamma+\delta-n)<0$ and

$$\sum_{j=0}^{\gamma-1}\frac{\binom{n-\delta-j-1}{\gamma-1-j}s_j}{\gamma-j}>0.$$

Hence we conclude $C=0$, which gives the first formula in (1.13).

Case II ($\alpha=\beta=-1$). In this case $D_n(\alpha,\beta)=0$ means that n is odd. Meanwhile, $x_1=1, x_n=-1, \gamma=\delta=s_0=s_n=0$. Then the equations with $k=0$ and $k=n$ in (3.4) become identities. But by [1, p. 441] we have

$$\omega_n(x):=P_n^{(-1,-1)}(x)=c(1-x^2)P'_{n-1}(x)=c\pi(x). \tag{3.17}$$

Thus[2]

$$\frac{\omega''_n(1)}{\omega'_n(1)}=\frac{1}{2}n(n-1). \tag{3.18}$$

It follows from (3.4) that

$$4q'_{n-1}(1)+(m-1)n(n-1)q_{n-1}(1)=0. \tag{3.19}$$

On the other hand, by means of (3.11) we obtain

$$q'_{n-1}(1)=2^{n-2}[(n-1)\alpha_0+\alpha_1] \tag{3.20}$$

and

$$q_{n-1}(1)=2^{n-1}\alpha_0. \tag{3.21}$$

Therefore (3.19) becomes

$$\alpha_1+\frac{1}{2}(n-1)[(m-1)n+2]\alpha_0=0. \tag{3.22}$$

Adding this equation to the one with $k=1$ in (3.14) we get

$$\frac{1}{2}(m-1)(n-1)n\alpha_0=Cs_1=C2^{-n}(n-1)$$

and hence by (3.12)

$$d=q_{n-1}(1)=2^{n-1}\alpha_0=\frac{C}{n(m-1)}.$$

This completes the proof.

4. Proof of Theorem 2

(2.8), coupled with (3.1) and (3.11), yields (1.19).

First we point out that, if the problem is regular, then it must be q-regular[2]. So, according to Theorem A the problem is q-regular if (1.9) holds. In particular, the problem is q-regular if one of the following conditions holds: (i) $\alpha=-1$ and $\beta>-1$, (ii) $\alpha>-1$ and $\beta=-1$, (iii) $\alpha=\beta=-1$ and n is even.

In the case $\alpha=\beta=-1$, according to Theorem A the problem is q-regular for even n. Let n be odd.

First by (3.13) we can determine the constant c in (3.17)

$$c = \lim_{x\to 1} \frac{P_n^{(-1,-1)}(x)}{(1-x^2)P'_{n-1}(x)} = -\frac{1}{2n},$$

i.e.,

$$P_n^{(-1,-1)}(x) = -\frac{1}{2n}\pi_n(x).$$

Hence it follows from (1.13) that

$$q_{n-1}(x) = \frac{1}{n(m-1)} - \frac{1}{2n}\int_1^x P'_{n-1}(t)dt = \frac{m+1}{2n(m-1)} - \frac{1}{2n}P_{n-1}(x).$$

Next, using the formula $\pi'_n(x) = -n(n-1)P_{n-1}(x)$ [2, (12.1.2), p. 170] and by integration by parts we conclude

$$\int_{-1}^1 \pi'_n(x)\pi_n^{m-1}(x)dx = -(m-1)\int_{-1}^1 \pi'_n(x)\pi_n^{m-1}(x)dx.$$

Hence

$$\int_{-1}^1 \pi'_n(x)\pi_n^{m-1}(x)dx = 0$$

and

$$\int_{-1}^1 P_{n-1}(x)\pi_n^{m-1}(x)dx = 0.$$

So, the problem is not q-regular if and only if

$$\int_{-1}^1 R(x)dx = \frac{m+1}{2n(m-1)}\int_{-1}^1 \omega_n^{m-1}(x)dx \neq 0,$$

which is equivalent to that m is odd.

When $\alpha = \beta$ we claim that $q_{n-1}(x)$ must be either an even function or an odd one. In fact, if $R(x)$ in the form (3.1) satisfies (1.3) with $y_{kj} = 0$, then so does

$$R(-x) = C\omega_n^{m-1}(-x)q_{n-1}(-x) = C\omega_n^{m-1}(x)[\pm q_{n-1}(-x)].$$

By Theorem 1 we have, with a constant c,

$$q_{n-1}(-x) = cq_{n-1}(x).$$

But replacing x by $-x$ leads to

$$q_{n-1}(x) = cq_{n-1}(-x) = c^2q_{n-1}(x),$$

i.e., $c = \pm 1$. This proves our claim.

Now let n be odd. Then $\omega_n(x)$ is an odd function and hence by (1.13) $q_{n-1}(x)$ is an even one. Thus for even m, $R(x)$ is an odd function and hence

$$\int_{-1}^1 R(x)dx = 0. \tag{4.1}$$

This by Lemma 2 implies q-regularity of the problem.

Let n be even. Then $\omega_n(x)$ is an even function. In the case $\gamma = \delta \neq$ an integer or $n \leq 2\gamma$ we have

$$q_{n-1}(-x) = (1-x^2)^\gamma \Big\{d + \int_a^0 P_n^{(\alpha,\alpha)}(t)(1-t^2)^{-\gamma-1}dt - \int_0^x P_n^{(\alpha,\alpha)}(t)(1-t^2)^{-\gamma-1}dt\Big\},$$

which implies $q_{n-1}(-x) = -q_{n-1}(x)$, for otherwise this leads to

$$\int_0^x P_n^{(\alpha,\alpha)}(t)(1-t^2)^{-\gamma-1}dt \equiv 0,$$

a contradiction. Then $R(x)$ is always an odd function and hence (4.1) holds. That is, the problem is q-regular.

When $\gamma = \delta =$ a positive integer and $n > 2\gamma$, we have $q_{n-1}(x) = (1-x^2)^\gamma$ and the problem is not q-regular for odd m.

This completes the proof.

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